Lai's conditions for spanning and dominating closed trails

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Submitted: Jul 5, 2014; Accepted: Feb 19, 2015; Published: Mar 6, 2015
Mathematics Subject Classifications: 05C45, 05C38, 05C35

Abstract

A graph is supereulerian if it has a spanning closed trail. For an integer r, let $\mathcal{Q}_0(r)$ be the family of 3-edge-connected nonsupereulerian graphs of order at most r. For a graph G, define $\delta_L(G) = \min\{\max\{d(u), d(v)\}|$ for any $uv \in E(G)\}$. For a given integer $p \geqslant 2$ and a given real number ϵ, a graph G of order n is said to satisfy a Lai's condition if $\delta_L(G) \geqslant \frac{n}{p} - \epsilon$. In this paper, we show that if G is a 3-edge-connected graph of order n with $\delta_L(G) \geqslant \frac{n}{p} - \epsilon$, then there is an integer $N(p, \epsilon)$ such that when $n > N(p, \epsilon)$, G is supereulerian if and only if G is not a graph obtained from a graph G_p in the finite family $\mathcal{Q}_0(3p-5)$ by replacing some vertices in G_p with nontrivial graphs. Results on the best possible Lai's conditions for Hamiltonian line graphs of 3-edge-connected graphs or 3-edge-connected supereulerian graphs are given, which are improvements of the results in [J. Graph Theory 42(2003) 308-319] and in [Discrete Mathematics, 310(2010) 2455-2459].

Keywords: Degree conditions, Spanning and dominating closed trails, Hamiltonian line graphs, Collapsible graphs

*Research is supported by Butler University Academic Grant (2014).

1 Introduction

We follow Bondy and Murty [3] for terms and notations, unless otherwise stated. Graphs considered in this paper are finite and loopless, but multiple edges are allowed. As in [3], $\kappa'(G)$ and $d_G(v)$ (or $d(v)$) denote the edge-connectivity of G and the degree of a vertex v in G, respectively. The maximum size of a matching in G is denoted by $\alpha'(G)$. Let $O(G)$ be the set of vertices of odd degree in G. A connected graph G is *Eulerian* if $O(G) = \emptyset$. An Eulerian subgraph H in a graph G is called a **closed trail**; and is called a **spanning closed trail** of G if $V(H) = V(G)$; and is called a **dominating closed trail** of G if $G - V(H)$ is edgeless. A graph is *supereulerian* if it has a spanning closed trail. We use $\mathcal{SL}$ denote the family of supereulerian graphs. A graph G is *collapsible* if for any even subset $R \subseteq V(G)$ or $R = \emptyset$, G has a spanning connected subgraph H_R with $O(H_R) = R$. We use $\mathcal{CL}$ denote the family of collapsible graphs. Thus, $\mathcal{CL} \subset \mathcal{SL}$.

Catlin's reduction method

Let G be a graph and let $X \subseteq E(G)$. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. A graph is **trivial** if it is edgeless. If H is a subgraph of G, then we write G/H for $G/E(H)$. If H is a connected subgraph of G, and if v_H denotes the vertex in G/H to which H is contracted, then H is called the **preimage** of v_H. A vertex v in a contraction of G is **nontrivial** if v has a nontrivial preimage. If $G_0 = G/X$ and if every vertex of G_0 is a nontrivial vertex, then G_0 is a **nontrivial contraction** of G.

In [6], Catlin showed that every graph G has a unique collection of pairwise disjoint maximal collapsible subgraphs $H_1, H_2, \cdots, H_c$ such that $V(G) = \bigcup_{i=1}^{c} V(H_i)$. The **reduction** of G is $G' = G/(\cup_{i=1}^{c} H_i)$, the graph obtained from G by contracting all nontrivial maximal collapsible subgraphs of G. A graph G is **reduced** if $G' = G$.

For an integer r, let $\mathcal{Q}_0(r)$ be the set of 3-edge-connected reduced nonsupereulerian graphs of order at most r. In this paper, we use P for the Petersen graph and use P_{14} for the graph in Figure 1.1. It is known that $Q_0(13) = \{P\}$ and $Q_0(14) = \{P, P_{14}\}$ [13].

P_{14}

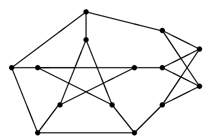

Figure 1.1

For a graph G, the line graph $L(G)$ has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. The following theorem relates dominating closed trails and Hamiltonian line graphs.

Theorem A (Harary and Nash-Williams [17]). Let G be a graph with at least three edges. Then $L(G)$ is Hamiltonian if and only if G has a dominating closed trail.

Graphs with spanning or dominating closed trails have been studied by many authors. The subject is closely related to the study on Hamiltonian graphs [23], Chinese Postman problem [2] and Traveling Salesman problem [4]. Pulleyblank [22] showed that the problem of determining if a graph G is supereulerian is NP-complete. Like the studies on many NP-complete problems in graph theory, various degree conditions have been studied for problems related to supereulerian graphs. For a graph G, we define

$\delta(G) = \min\{d(v) \mid \text{for any } \ v \in V(G)\}$;
$\sigma_2(G) = \min\{d(u) + d(v) \mid \text{for any } uv \notin E(G)\}$;
$\sigma_t(G) = \min\{\Sigma_{i=1}^{t} d(v_i) \mid \{v_1, v_2, \cdots, v_t\} \text{ is independent in } G \ (t \geqslant 2)\ \}$;
$\delta_F(G) = \min\{\max\{d(u),\ d(v)\} \mid \ \text{for any } u, v \in V(G) \text{ with } dist(u, v) = 2\}$;
$\overline{\sigma}_2(G) = \min\{d(u) + d(v) \mid \text{for every edge } uv \in E(G)\}$;
$\delta_L(G) = \min\{\max\{d(u), d(v)\} \mid \ \text{for every edge } uv \in E(G)\}$.

These are all the degree parameters that have been studied for graphs with spanning or dominating closed trails (see [5, 6, 8, 10, 11, 18, 19, 24]). Let

$$\Omega(G) = \{\delta(G), \sigma_2(G), \sigma_t(G), \delta_F(G), \overline{\sigma}_2(G), \delta_L(G)\}.$$

For a given integer p and a given real number ϵ, a graph G of order n is said to satisfy a **Lai's degree condition** or **Lai's condition** if

$$\delta_L(G) \geqslant \frac{n}{p} - \epsilon. \tag{1}$$

Such degree condition was first considered by Lai [18] in the study of Hamiltonian line graphs. Obviously, if $\overline{\sigma}_2(G) \geqslant 2(\frac{n}{p} - \epsilon)$, then $\delta_L(G) \geqslant \frac{n}{p} - \epsilon$.

Here are some prior results related to Lai's conditions.

Settling a conjecture posted in [1], Veldman [24] proved the following.

Theorem B (Veldman [24]). Let G be a 2-edge-connected simple graph on n vertices. If

$$\overline{\sigma}_2(G) > \frac{2n}{5} - 2, \tag{2}$$

then, for n sufficiently large, $L(G)$ is Hamiltonian.

Condition (1) with $p = 5$ and $\epsilon = 1$ is a relaxation of (2). Lai [18] proved the following.

Theorem C (Lai [18]). Let G be a 2-edge-connected simple graph on n vertices. If $\delta_L(G) \geqslant \frac{n}{5} - 1$, then, for n sufficiently large, either $L(G)$ is Hamiltonian, or (2) is violated and G can be contracted to one of seven specified graphs.

For a 3-edge-connected graph, the Lai's condition in Theorem C can be lowered.

Theorem D (Chen, et al. [12]). Let G be a 3-edge-connected simple graph on n vertices and let $\epsilon \geqslant 1$ be a constant. If $\delta_L(G) \geqslant \frac{n}{12} - \epsilon$, then, for n sufficiently large, $L(G)$ is Hamiltonian if and only if G does not have the Petersen graph as a nontrivial contraction.

Adding $\delta(G) \geqslant 4$ to Theorem D, Li et al. [20] proved the following.

Theorem E (Li, et al. [20]). Let G be a 3-edge-connected simple graph on n vertices. If $\delta(G) \geqslant 4$ and if $\delta_L(G) \geqslant \frac{n-13}{12}$, then either $G \in \mathcal{SL}$ or $G' = P$.

For a 3-edge-connected graph with $\overline{\sigma}_2(G)$ condition, the following was proved:

Theorem F (Chen and Lai [11]). Let $p > 0$ be a given integer and let G be a 3-edge-connected simple graph of order $n > 12p(p-1)$. Let G' be the reduction of G. If $\overline{\sigma}_2(G) \geqslant \frac{n}{p} - 2$, then either $G \in \mathcal{CL}$ or $G' \neq K_1$ with $|V(G')| \leqslant 3p - 4$ and $\alpha'(G') \leqslant p$.

In this paper, we prove the following theorem analogous to Theorem F, which unifies Theorems D, E and their improvements given in Section 4.

Theorem 1.1. Let G be a 3-edge-connected simple graph of order n. Let G' be the reduction of G. Let S_0 be the set of nontrivial vertices in G'. Let $Y = V(G') - S_0$. Let $N(p,\epsilon) = \max\{(\epsilon-5)p(p+1), 12p(p+1), (6p+\epsilon-4)p, (\epsilon-1)p(p-1)\}$, where $p > 1$ is a given integer and ϵ is a given real number. If

$$\delta_L(G) \geqslant \frac{n}{p} - \epsilon, \tag{3}$$

then, for $n > N(p,\epsilon)$, either $G \in \mathcal{CL}$, or $G' \in \mathcal{Q}_0(3p-5)$ with $\alpha'(G') \leqslant |S_0| \leqslant p$, and $|V(G')| \leqslant 3|S_0| - 5 \leqslant 3p - 5$. Furthermore,

(i) if G' has a closed trail containing S_0, then G has a dominating closed trail;
(ii) if $|S_0| = p$, then $\epsilon \geqslant 1$ and $|Y| \leqslant (\epsilon-1)p$ and $|V(G')| \leqslant \epsilon p$;
(iii) if $\delta(G) \geqslant 4$, then $|V(G')| \leqslant \max\{|S_0|, 2|S_0| - 3\} \leqslant \max\{p, 2p-3\}$.

Combining Theorem 1.1 with the recently proved result on Fan-type condition [15], Theorem F and the prior results in [8, 10, 11], we have the following:

Theorem 1.2. Let G be a 3-edge-connected graph of order n. Let $p > 1$ be a given integer and let $\epsilon \geqslant 0$ be a given real number. For any $d(G) \in \Omega(G)$, if $d(G) \geqslant \frac{n}{p} - \epsilon$, then, for n sufficiently large, $G \in \mathcal{SL}$ if and only if $G' \notin \mathcal{Q}_0(cp)$, where c depends on $d(G) \in \Omega(G)$ and $c \leqslant 5$ for all $d(G)$.

Thus, a 3-edge-connected graph G of order n with $d(G) \geqslant \frac{n}{p} - \epsilon$ where $d(G) \in \Omega(G)$ is supereulerian unless G is a graph obtained from a graph G_p in $\mathcal{Q}_0(cp)$ by replacing some (or all) vertices in G_p with nontrivial subgraphs. From a computational point of view, for given p and c, the number of graphs in $\mathcal{Q}_0(cp)$ is fixed and so it can be determined in a constant time. Like the characterizations of planar graphs, people view that K_5 and $K_{3,3}$ are the only non-planar graphs. Thus, in some sense, only a finite number of 3-edge-connected graphs G with $d(G) \geqslant \frac{n}{p} - \epsilon$ are nonsupereulerian.

With Ryjáček's closure concept on claw-free graphs [23], the techniques used in this paper can be applied to solve degree condition problems of Hamiltonian claw-free graphs.

In Section 2, we present some prior results related to Catlin's reduction method, which are the needed mechanism in our proofs in this paper. The proof of Theorem 1.1 is given in Section 3. Applications of Theorem 1.1 are presented in Section 4.

2 Prior results related to Catlin's reduction method

For a graph G, let $F(G)$ be the minimum number of extra edges that must be added to G to obtain a spanning supergraph having two edge-disjoint spanning trees. For an integer $i > 0$, the set of vertices of degree i in G is denoted by $D_i(G)$.

Theorem G. Let G be a connected graph and let G' be the reduction of G. Then each of the following holds:

(a) (Catlin [6]) $G \in \mathcal{CL}$ if and only if $G' = K_1$; $G \in \mathcal{SL}$ if and only if $G' \in \mathcal{SL}$; and G has a dominating closed trail if and only if G' has a dominating closed trail containing all the nontrivial vertices of G'.

(b) (Catlin, et al.[9]) G' is simple and K_3-free with $\delta(G') \leqslant 3$, and either $F(G') \geqslant 3$ or $G' \in \{K_1, K_2, K_{2,t}(t \geqslant 2)\}$.

(c) (Catlin [7]) $F(G') = 2|V(G')| - |E(G')| - 2$.

Theorem H. Let G be a 3-edge-connected graph of order n. Let G' be the reduction of G. Then each of the following holds:

(a) (Chen [13]). If $n \leqslant 14$, then either $G \in \mathcal{SL}$ or $G' \in \{P, P_{14}\}$.

(b) (Chen, et al. [16]). If $\alpha'(G) \leqslant 7$, then G is supereulerian if and only if $G' \notin \{P, P_{14}\}$. If $n \leqslant 15$, then G is supereulerian if and only if $G' \notin \{P, P_{14}\}$.

Theorem I (Chen, et al. [12]). Let G be a 3-edge-connected graph and let $S \subseteq V(G)$ be a vertex subset such that $|S| \leqslant 12$. Then either G has a closed trail H such that $S \subseteq V(H)$, or G can be contracted to the Petersen graph in such a way that the preimage of each vertex of the Petersen graph contains at least one vertex in S.

Theorem J (Li, et al. [21]). Let G be a 3-edge-connected reduced graph and $G \notin \mathcal{SL}$.

(a) (Lemma 2.6 [21]). If $F(G) = 3$, then $|D_3(G)| =$ the number of edge-cuts of size 3.

(b) (Theorem 1.3 [21]). If G has at most 11 edge-cuts of size 3, then $G = P$.

Lemma 2.1. If G is a reduced graph with $\kappa'(G) \geqslant 3$ and $G \notin \mathcal{SL} \cup \{P\}$, then $|D_3(G)| \geqslant 12$.

Proof. Since $G \notin \mathcal{SL}$ and $\kappa'(G) \geqslant 3$, $G' \notin \{K_1, K_2, K_{2,t}\}$. By Theorem G(b), $F(G) \geqslant 3$.

If $F(G) = 3$, then since $G \notin \mathcal{SL} \cup \{P\}$, by Theorem J(b), G has at least 12 edge-cuts of size 3 and so by Theorem J(a) $|D_3(G)| \geqslant 12$.

If $F(G) \geqslant 4$, then by Theorem G(c), $4 \leqslant F(G) = 2|V(G)| - |E(G)| - 2$, and so $|E(G)| \leqslant 2|V(G)| - 6$. Since $2|E(G)| = \sum_{v \in V(G)} d_G(v) = \sum_{i=3} i|D_i(G)|$ and $|V(G)| = \sum_{i=3} |D_i(G)|$,

$$\begin{aligned} 2|E(G)| \leqslant 4|V(G)| - 12 &= 4|D_3(G)| + 4|D_4(G)| + \cdots - 12; \\ 3|D_3(G)| + 4|D_4(G)| + 5|D_5(G)| + \cdots &\leqslant 4|D_3(G)| + 4|D_4(G)| + \cdots - 12; \\ 12 + |D_5(G)| + \cdots &\leqslant |D_3(G)|. \end{aligned}$$

Thus, $|D_3(G)| \geqslant 12$. $\square$

3 Proof of Theorem 1.1

We prove the following lemma first.

Lemma 3.1. Let G be a graph of order n with the properties stated in Theorem 1.1. Let $G' \neq K_1$ be the reduction of G. For a vertex v in $V(G')$, let $H(v)$ be the preimage of v in G. Let $S_0 = \{v \in V(G') \mid |V(H(v))| > 1\}$. Let $Y = V(G') - S_0$. Then, for $n > N(p, \epsilon)$ (as required in Theorem 1.1), $|V(G')| \leqslant 3p - 1$. Furthermore, each of the following holds:

(a) If $|V(H(v))| > 1$ then $|V(H(v))| \geqslant \frac{n}{p} - \epsilon + 1$.

(b) $|S_0| \leqslant p$. If $\epsilon < 1$, $|S_0| < p$.

(c) Y is an independent set and so for any $v \in Y$, $N_{G'}(v) \subseteq S_0$.

Proof. Let $c = |V(G')|$ and let $p_1 = p + 1$. When $n > (\epsilon - 5)p(p + 1)$, $\frac{n}{p} - \epsilon + 3 \geqslant \frac{n}{p_1} - 2$. Since $\kappa'(G) \geqslant 3$, $d(u) \geqslant 3$ for any $u \in V(G)$. Thus, for any $xy \in E(G)$, by (3)

$$d(x) + d(y) \geqslant \max\{d(x), d(y)\} + 3 \geqslant \delta_L(G) + 3 \geqslant \frac{n}{p} - \epsilon + 3 \geqslant \frac{n}{p_1} - 2.$$

Thus, $\overline{\sigma}_2(G) \geqslant \frac{n}{p_1} - 2$. By $n > 12p(p + 1) = 12p_1(p_1 - 1)$ and Theorem F, $c \leqslant 3p_1 - 4 = 3p - 1$.

(a) Since $|H(v)| > 1$, $H(v)$ has an edge xy. We may assume that $d(x) \geqslant d(y)$. By (3)

$$d(x) = \max\{d(x), d(y)\} \geqslant \delta_L(G) \geqslant \frac{n}{p} - \epsilon. \tag{4}$$

Let $i(x)$ be the number of edges in $E(G')$ incident with x. Then by (4)

$$\begin{aligned} \frac{n}{p} - \epsilon \leqslant d(x) &\leqslant i(x) + |V(H(v))| - 1; \\ \frac{n}{p} - \epsilon + 1 - i(x) &\leqslant |V(H(v))|. \end{aligned} \tag{5}$$

Since $i(x) \leqslant d_{G'}(v) \leqslant c - 1 \leqslant 3p - 2$, by (5) and $n > (6p + \epsilon - 4)p$, $|V(H(v))| > 3p - 1 \geqslant c$. Hence, $H(v)$ has edges that are not incident with any edges in $E(G')$. We may assume xy is such an edge and so $i(x) = 0$. Therefore, by (5), $|V(H(v))| \geqslant \frac{n}{p} - \epsilon + 1$.

(b) Since $\cup_{v \in S_0} V(H(v)) \subset V(G)$, $|S_0|(\frac{n}{p} - \epsilon + 1) \leqslant n$ and so $|S_0| \leqslant \frac{np}{n - (\epsilon - 1)p}$. If $\epsilon < 1$, $n - (\epsilon - 1)p > n$, and so $|S_0| < p$. Otherwise, for $n > (\epsilon - 1)p(p - 1)$, $|S_0| \leqslant p$.

(c) Suppose that there are two vertices y_1 and y_2 in Y such that $y_1 y_2 \in E(G')$. Since y_1 and y_2 are trivial vertices in G', $d_G(y_i) = d_{G'}(y_i) \leqslant c - 1$ $(i = 1, 2)$. By (3) and $n > (6p + \epsilon - 4)p$,

$$3p - 2 \geqslant c - 1 \geqslant \max\{d_G(y_1), d_G(y_2)\} \geqslant \frac{n}{p} - \epsilon > 2(3p - 2),$$

a contradiction. Lemma 3.1 is proved. □

Proof of Theorem 1.1. Let G' be the reduction of G. If $G' = K_1$, Theorem 1.1 is true trivially. We may assume that $G' \neq K_1$. Since $\kappa'(G') \geqslant 3$, $G' \notin \{K_1, K_2, K_{2,t}\ (t \geqslant 2)\}$.

By Lemma 3.1, $|S_0| \leqslant p$. For each vertex $v \in Y$, $N_{G'}(v) \subseteq S_0$. Thus, for any edge e in $E(G')$, at least one of the ends of e is in S_0. Therefore, $\alpha'(G') \leqslant |S_0| \leqslant p$.

Since $d_{G'}(v) \geqslant 3$ for any $v \in Y$, $|E(G')| \geqslant 3|Y|$. By Theorem G(b), $F(G') \geqslant 3$. By Theorem G(c) and $|V(G')| = |S_0| + |Y|$,

$$3 \leqslant F(G') = 2|V(G')| - |E(G')| - 2 \leqslant 2(|S_0| + |Y|) - 3|Y| - 2.$$

It follows that $|Y| \leqslant 2|S_0| - 5$. Therefore, $|V(G')| = |S_0| + |Y| \leqslant 3|S_0| - 5 \leqslant 3p - 5$.

(i) G' has a closed trail H containing S_0. By Lemma 3.1 (c), for any $y \in Y$, $N_{G'}(y) \subseteq S_0$. Thus, H is a dominating closed trail containing all the nontrivial vertices in S_0. By Theorem G(a), G has a dominating closed trail. (i) is proved.

(ii) $|S_0| = p$. By Lemma 3.1 (a), for any $v \in S_0$, $|V(H(v))| \geqslant \frac{n}{p} - \epsilon + 1$. Then

$$n = |V(G)| = \sum_{v \in S_0} |V(H(v))| + \sum_{u \in Y} |V(H(u))| \geqslant |S_0|(\frac{n}{p} - \epsilon + 1) + |Y| = n - p\epsilon + p + |Y|.$$

Thus, $|Y| \leqslant (\epsilon - 1)p$, and so $|V(G')| = |S_0| + |Y| \leqslant p + (\epsilon - 1)p = \epsilon p$. (ii) is proved.

(iii) $\delta(G) \geqslant 4$. If $Y = \emptyset$, then $|V(G')| = |S_0| \leqslant p$. We are done for this case.

Next, $Y \neq \emptyset$. By Lemma 3.1(c), for any $u \in Y$, $N_{G'}(u) \subseteq S_0$. Since for every $u \in Y$, u is a trivial vertex in G' and $\delta(G) \geqslant 4$, $d_{G'}(u) = d_G(u) \geqslant 4$. Hence, $|E(G')| \geqslant 4|Y|$.

By Theorem G(b), $|E(G')| \leqslant 2|V(G')| - 5$. Since $|V(G')| = |Y| + |S_0|$, we have

$$\begin{aligned} 4|Y| &\leqslant |E(G')| \leqslant 2|V(G')| - 5 = 2(|Y| + |S_0|) - 5; \\ 2|Y| &\leqslant 2|S_0| - 5. \end{aligned}$$

Since $|Y|$ is an integer, $|Y| \leqslant |S_0| - 3$, and so $|V(G')| = |Y| + |S_0| \leqslant 2|S_0| - 3 \leqslant 2p - 3$. $\square$

4 Applications of Theorem 1.1

Using Theorem 1.1, we obtain some new results on Lai's degree conditions for supereulerian graphs and hamiltonian line graphs.

For Hamiltonian line graphs, the following theorem is an improvement of Theorem D.

Theorem 4.1. Let G be a 3-edge-connected simple graph of order n. For any given $\epsilon < \frac{16}{13}$, if $\delta_L(G) \geqslant \frac{n}{13} - \epsilon$, then, for n sufficiently large, G has a dominating closed trail, i.e., $L(G)$ is Hamiltonian, if and only if G does not have the Petersen graph P as a nontrivial contraction.

Proof. This is the special case of Theorem 1.1 with $p = 13$ and $\epsilon < \frac{16}{13}$. Let G' be the reduction of G. If $G \in \mathcal{SL}$, then we are done. Thus, we may assume that $G \notin \mathcal{SL}$. Let S_0 and Y be the sets defined in Theorem 1.1. Then $|S_0| \leqslant 13$.

Case 1. $|S_0| = 13$. By Theorem 1.1(ii), $p = 13$ and $\epsilon < \frac{16}{13}$, $|V(G')| \leqslant \epsilon p < \frac{16}{13} \cdot 13 = 16$. Thus, $13 \leqslant |V(G')| \leqslant 15$. By Theorem H(b), $G' = P_{14}$. Then $Y = V(P_{14}) - S_0$ and $|Y| = 1$.

Let u be the only vertex in Y. Note that P_{14} is obtained by replacing a vertex in the Petersen graph P with a $K_{2,3}$ such that $P_{14}/K_{2,3} = P$. Let v_0 be the contraction image of $K_{2,3}$ in P. Then v_0 is a non-trivial contraction. If u is a vertex in the $K_{2,3}$ subgraph in P_{14}, then G has the Petersen graph P as a nontrivial contraction. We are done for this case.

Next, u is a vertex in $V(P_{14}) - V(K_{2,3})$. Then P has a closed trail containing $V(P) - \{u\}$. Thus, $G' = P_{14}$ has a dominating closed trail containing S_0. By Theorem 1.1(i), G has a dominating closed trail. Theorem 4.1 is proved for Case 1.

Case 2. $|S_0| \leqslant 12$. By Theorem I, one of the following holds:
Subcase 1. G' has a closed trail H containing S_0. By Lemma 3.1(c) for any $v \in Y = V(G') - S_0$, $N_{G'}(v) \subseteq S_0$ and so the closed trail H is a dominating closed trail containing all the nontrivial vertices of G'. By Theorem 1.1(i), G has a dominating closed trail.

Subcase 2. G' can be contracted to P in such a way that the preimage of each vertex of P contains at least one vertex in S_0. Then G has P as a nontrivial contraction. □

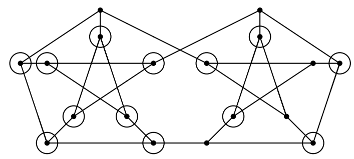

Figure 4.1

Remark. Let G_a be the graph shown in Figure 4.1. Let S be the set of the 13 vertices marked by $\odot$. Then G_a does not have a closed trail containing S. Let G be the graph obtained from G_a by replacing each vertex with $\odot$ by a complete graph K_s where $s = \frac{n-5}{13}$. Then $\delta_L(G) \geqslant s - 1 = \frac{n-18}{13}$. But G does not have a dominating closed trail. By Theorem A, $L(G)$ is non-Hamiltonian. Thus, $p = 13$ in Theorem 4.1 cannot be replaced by $p = 14$ and ϵ cannot be reduced to $\frac{18}{13}$ with $p = 13$.

For supereulerian graphs, the following theorem is an improvement of Theorem E.

Theorem 4.2. Let G be a 3-edge-connected graph of order n with $\delta(G) \geqslant 4$. For any given $\epsilon < \frac{4}{3}$, if $\delta_L(G) \geqslant \frac{n}{12} - \epsilon$, then, for n sufficiently large, $G \in \mathcal{SL}$ if and only if G does not have the Petersen graph P as a nontrivial contraction.

Proof. This is the special case of Theorem 1.1 with $p = 12$ and $\epsilon < \frac{4}{3}$. Let G' be the reduction of G. If $G' \in \mathcal{SL}$, then by Theorem G(a), graph $G \in \mathcal{SL}$, and we are done. Thus, we may assume that $G' \notin \mathcal{SL}$ and so $G' \neq K_1$. Since $\kappa'(G) \geqslant 3$, $\kappa'(G') \geqslant 3$.

Let S_0 and Y be the two sets defined in Theorem 1.1. Since $\delta(G) \geqslant 4$, $D_3(G') \subseteq S_0$. By Theorem 1.1, $|D_3(G')| \leqslant |S_0| \leqslant 12$. By Lemma 2.1, either $G' = P$ or $|D_3(G')| = 12$.

Case 1. $|D_3(G')| = |S_0| = 12$. By Theorem 1.1(ii), $|V(G')| \leqslant \epsilon p < \frac{4}{3} \cdot 12 = 16$, and so $|V(G')| \leqslant 15$. By Theorem H(b), since $G' \notin \mathcal{SL}$, $G' \in \{P, P_{14}\}$, contrary to $|D_3(G')| = 12$.

Case 2. $G' = P$. Since $|D_3(G')| = |S_0| = 10 = |V(G')|$, each vertex of $G' = P$ is a non-trivial contraction. Theorem 4.2 is proved. $\square$

If $\delta(G) \geqslant 4$ is dropped from Theorem 4.2, then the best Lai's degree condition for 3-edge-connected supereulerian graphs is the following:

Theorem 4.3. Let G be a 3-edge-connected simple graph of order n. If $\delta_L(G) \geqslant \frac{n}{8} - \frac{13}{8}$, then, for n sufficiently large, $G \in \mathcal{SL}$ if and only if $G' \neq P$.

Proof. This is the special case of Theorem 1.1 with $p = 8$ and $\epsilon = \frac{13}{8}$. Using some prior results on 3-edge-connected reduced graphs in [14], one can prove Theorem 4.3 in the same way as the proof of Theorem 4.2. The details are omitted here. $\square$

Remark. Let G_1 be the graph shown in Figure 4.2, where each $\odot$ represents a $K_{\frac{n-6}{8}}$ subgraph. Then G_1 is a 3-edge-connected graph of order n with $\delta_L(G) \geqslant \frac{n-6}{8} - 1 = \frac{n}{8} - \frac{14}{8}$. However, $G_1' = P_{14}$. Thus, the Lai's degree condition in Theorem 4.3 is the best possible.

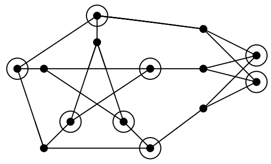

Figure 4.2

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