Dear Editors and Referee,

We were happy to read that the referee thought a "new and more readable presentation of [our results] would make an interesting paper in the EJC." We hope that our revised submission does this and sufficiently addresses all of the referee's concerns. We were particularly happy that the referee finds only one mathematical error (using "free monoid" when we should have stated "free commutative monoid") and one typo (resulting from using a single dash instead of a double-dash in our bibliography file). We think this is indicative of the care with which we have prepared our paper.

Detailed responses to the referee's report follow below. We would especially like to follow a *variation* of one of the options suggested by the referee for handling the non-even cases: namely, to keep the statements but also include extremely abbreviated proofs (containing only those details that would save the reader considerable effort). Even so, our revised paper is 4 pages shorter than the original.

We thank all of you for your help.

Response to "**Detailed remarks on some points in the paper**":

- *The introduction is too long, the description of the sandpile model in details is not necessary here, many other texts which introduce the model are available.*

  The introduction is now two-thirds its original size. We give a minimal rough description of the sandpile model solely to set terminology necessary to communicate our results in the introduction.

- *Is there a written text for the question of Irena Swanson? If not, mention "oral communication" in a footnote.*

  Done.

- *I do not like the notation $\widetilde{V}$ for $V \setminus \{s\}$, I would prefer V' for instance.*

  We tried changing $\widetilde{V}$ to V'. However, this entails the corresponding change from $\widetilde{\Delta}$ to Δ' for the reduced Laplacian. The problem comes when a group is acting. For instance, in Proposition 2.10,

  $$\mathbb{Z}\mathcal{O}/\text{image}(\widetilde{\Delta}^G)$$

  would become

  $$\mathbb{Z}\mathcal{O}/\text{image}({\Delta'}^G)$$

  or

  $$\mathbb{Z}\mathcal{O}/\text{image}((\Delta')^G),$$

  both of which seem undesirable to us.

- *Remove the long outline in page 5 and put the Acknowledgments at the end of the paper.*

  Done.

- *In page 6 line 4, notice that a free monoid is not commutative and its elements are words, so that $\mathbb{N}V$ is not a free monoid.*

  We changed "free monoid" to "free commutative monoid".

- *You often give too much details of your calculations. This is the case in pages 18 and 19. You should skip some formulas.*

  Done. For instance, we removed about a page of calculations from the proof of Theorem 4.2.

- *In page 29 the proof that the all 2 configuration is recurrent is immediate and should not be included.*

  Done.

- *Add the references below to those of your paper.* [A list of six references is given.]

  Done.

- *Your references 5 and 6 are the same paper. In reference 27 the pages are 2221-2226 and not 22212226.*

  Fixed.

Response to general comments:

- *I especially like the proof of the second one which shows that an integer a divides b by building a finite group of order b and a subgroup of it of order a.*

  Thanks!

- *However the notation used in the paper is complicated and heavy.*

  We have devoted an enormous number of hours over the course of several years trying to refine our results and the presentation to make them as natural, clear, precise, and easily accessible as possible. We especially hope that the effort we put into the examples and figures has made the material easier to approach. Hopefully, the editing we have done in response to the referee report has helped.

- *The authors use a generalization of the sandpile model to directed graphs with multiple edges, ... It seemed very strange to me, that this complicated definition was introduced, since the question considered deals with the very simple simple model of a (non directed) grid. I had to wait until page 22 to understand (after a long effort) why ...*

    At the beginning of section 2.1, which summarizes the sandpile model, we have added an explanation as to why we must consider the model for directed multigraphs and include a pointer to where directed graphs become relevant later in the paper.

- *... this very complicated model had to be defined with a notation different from any other peer dealing with sandpiles.*

    The model we use is from *Chip-firing and rotor-routing on directed graphs*, by Holroyd, Levine, Mészáros, Peres, Propp, and Wilson. These authors are some of the most active and respected in the theory of sandpiles. In our opinion (and several others working in the area), this is the gold-standard for the exposition of Dhar's sandpile model. Their notation is used by the Microsoft theory group and was common parlance at the international workshop *Generalizations of chip-firing and the critical group* conducted at the American Institute of Mathematics in July, 2013 (for which the third author served as one of the organizers). We expect it to be the standard for next year's BIRS conference on sandpiles.

    In the revision of our introduction and of Section 2.1, we now have clear pointers to this paper.

    There are two minor differences between our model and that presented by Holroyd, et al. The first is that we prefer to have a *set* of edges and a weight function on edges rather than a *multiset* of edges. So we think of a single edge of weight 3 rather than 3 separate edges with the same endpoints, for example. The second difference is that our configurations are formal sums of vertices, $\mathbb{N}\widetilde{V}$ rather than, dually, functions on the vertices, $\mathbb{N}^{\widetilde{V}}$. We prefer this for functorial reasons and to conform with Baker's notion of divisors on graphs.

    As now emphasized more clearly in the paper, our results for grid graphs of any dimension *require* the use of multigraphs since corner vertices are connected with edges of multiplicity 2 to the sink. The only change in our summary of the sandpile model that would change by restricting to undirected multigraphs would be to change the word "directed" to "undirected". In the end, *any* version of the sandpile model amounts to interpreting columns of the reduced Laplacian matrix as firing rules, and the reduced Laplacian is minimally more complex for undirected graphs than for directed multigraphs. (Of course, from this point of view, one may even replace the Laplacian by a larger class of matrices, as is done in the work on arithmetical graphs by Lorenzini or in the theory of M-matrices as developed by Gabrielov and by Guzmán and Klivans.)

- *I suggest to the authors to limit their paper to that case and not to introduce their complicated model in the main part of the paper. An added appendix could consider the odd cases and possibly the model needed for enumerating them if presented in a more comprehensive text. My opinion is that the claim of the results for the odd cases would be sufficient, the proof of them being not very interesting.*

  The fact the there is a connection between symmetric recurrents on an even×even grid graph and domino tilings of a checkerboard was surprising to us. It was even more surprising that changing to an even×odd grid would have anything to do with domino tilings on a Möbius strip. We consider this second result an essential part of our paper, and hence, should not be relegated to an appendix. It is especially interesting since it leads to counting tilings of Möbius strips by counting tilings of ordinary grids with weighted squares (cf. Part (2) of Theorem 4.5 and Example 4.8). We think that the open question—the first in the list at the end of the paper—asking for a combinatorial explanation of this fact is compelling and would be of general interest. For these reasons and for completeness, we would very much like to present the results for grids of arbitrary dimensions.

  As mentioned earlier, we have considerably shortened the proof of the even×even case. The proofs for grids of other dimensions have been reduced to a bare minimum. For instance, we feel it is necessary to point out the trick of taking the transpose of the symmetrized Laplacian. So these later proofs only include the set-up and leave out all details of the calculations (which are, of course, similar to those for the even×even case).