A Bijective Proof of the Alladi-Andrews-Gordon Partition Theorem

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Submitted: Jul 28, 2014; Accepted: Mar 1, 2015; Published: Mar 13, 2015
Mathematics Subject Classifications: 05A17, 05A19

Abstract

Based on the combinatorial proof of Schur's partition theorem given by Bressoud, and the combinatorial proof of Alladi's partition theorem given by Padmavathamma, Raghavendra and Chandrashekara, we obtain a bijective proof of a partition theorem of Alladi, Andrews and Gordon.

Keywords: bijection; partition; Schur's partition theorem; Göllnitz's partition theorem; the Alladi-Andrews-Gordon partition theorem

1 Introduction

In 1926, Schur [15] proved one of the most profound results in the theory of partitions, which can be stated as follows.

Theorem 1.1 (Schur). *The number of partitions of n into distinct parts $\equiv 1, 2 \pmod 3$ is equal to the number of partitions of n into distinct parts $\lambda_1 > \lambda_2 > \lambda_3 > \cdots$ where $\lambda_i - \lambda_{i+1} \geqslant 3$ with strict inequality if $\lambda_i \equiv 3 \pmod 3$.*

Throughout this paper $x \equiv y \pmod M$ means that $x = y + kM$ for a nonnegative integer k, where $x \geqslant y$ and $x > 0$. Theorem 1.1 is usually called *Schur's celebrated partition theorem of 1926*. It was extended by Göllnitz [13] in 1967.

Theorem 1.2 (Göllnitz). *Let $B(n)$ be the number of partitions of n into distinct parts $\equiv 2, 4, 5 \pmod 6$. Let $C(n)$ be the number of partitions of n into distinct parts $\lambda_1 > \lambda_2 > \lambda_3 > \cdots$ where no part equals 1 or 3, and $\lambda_i - \lambda_{i+1} \geqslant 6$ with strict inequality if $\lambda_i \equiv 6, 7$ or $9 \pmod 6$. Then $B(n) = C(n)$.*

Theorem 1.2 is one of the most striking extensions of Theorem 1.1. It is not a priori evident that $B(n) = C(n)$. Göllnitz's proof is quite involved. Andrews gave two simpler proofs of Theorem 1.2, one by generating functions [8], and the other by computer algebra [9, §10]. Göllnitz [13] also gave the following refinement of Theorem 1.2:

$$\mathcal{B}(n; s) = \mathcal{C}(n; s), \tag{1.1}$$

where $\mathcal{B}(n; s)$ and $\mathcal{C}(n; s)$ denote, respectively, the number of partitions enumerated by $B(n)$ and $C(n)$ with exactly s parts and the parts $\equiv 6, 7$ or $9 \pmod 6$ are counted twice. Andrews [9] asked for a proof which would offer more insights into the refinement (1.1) of Göllnitz's theorem.

There has been a lot of progress towards this direction, see [1, 5, 14]. The first combinatorial approach to Theorem 1.2 was provided by Alladi [1]. Precisely, Alladi constructed a bijection to prove a three-parameter q-identity [1, Eq. (1.2)], which first appeared in [5] and is a deep refinement of Theorem 1.2. However, as mentioned by Alladi [1], his construction can not be used to give a bijection between the sets of partitions of n counted by $B(n)$ and $C(n)$. Padmavathamma, Raghavendra and Chandrashekara [14] presented a bijective proof of another partition theorem due to Alladi [2, Theorem 1], and remarked that their bijection also implies Theorem 1.2. They also noted that their method is very similar in spirit to Bressoud's [11] combinatorial proof of Schur's partition theorem.

By using weighted words introduced by Alladi and Gordon [6, 7], Alladi, Andrews and Gordon [5] obtained a more general partition theorem.

Theorem 1.3 (Alladi-Andrews-Gordon). *Let $M \geqslant 6$ and let r_1, r_2, r_3 be residues satisfying the following conditions:*

$$0 < r_1 < r_2 < r_3 < M \leqslant r_1 + r_2 \quad \text{and} \quad r_1 + M < r_2 + r_3. \tag{1.2}$$

Let $B(n; s)$ denote the number of partitions of n into s distinct parts congruent to r_1, r_2 or $r_3 \pmod M$. Let $C(n; s)$ denote the number of partitions of n into s distinct parts $\lambda_1 > \lambda_2 > \lambda_3 > \cdots$ such that

(i) *each part λ_i is $\equiv r_1, r_2, r_3, r_1 + r_2, r_1 + r_3$ or $r_2 + r_3 \pmod M$,*

(ii) *$\lambda_i - \lambda_{i+1} \geqslant M$ with strict inequality if $\lambda_i \equiv r_1 + r_2, r_1 + r_3$ or $r_2 + r_3 \pmod M$,*

(iii) *the parts $\equiv r_1 + r_2, r_1 + r_3$ or $r_2 + r_3 \pmod M$ are counted twice.*

Then $B(n; s) = C(n; s)$.

Clearly, Theorem 1.3 reduces to (1.1) by setting $M = 6$, $r_1 = 2$, $r_2 = 4$, and $r_3 = 5$. As remarked by Alladi, Andrews and Gordon [5, §1], Theorem 1.3 also generalizes two extensions of (1.1) given by Göllnitz [13, Sätze 4.8 and 4.10]. In fact, Alladi, Andrews and Gordon established a three-parameter key identity [5, Eq. (1.4)] which implies Theorem 1.3. Alladi [1, §6] noticed that Jacobi's triple product identity [12, p. 12] can be derived from a special case of this key identity.

Alladi, Andrews and Berkovich [4] found an interpretation of Theorem 1.3 in terms of partitions into six colored integers, and they obtained a more general theorem on partitions into eleven colored integers. Moreover, they showed that the partition theorem involving eleven colored integers is combinatorially equivalent to a four-parameter key-identity [4, Eq. (1.7)]. Further studies related to Theorem 1.2 and Theorem 1.3 can be found in Alladi and Andrews [3] and Andrews, Bringmann and K. Mahlburg [10].

The objective of this paper is to provide a bijective proof of Theorem 1.3. Our proof is in the spirit of the combinatorial proof of Alladi's partition theorem [2, Theorem 1] given by Padmavathamma, Raghavendra and Chandrashekara [14].

2 A Bijective Proof of Theorem 1.3

In this section, we present a bijective proof of Theorem 1.3. Let $\mathbb{B}(n;s)$ and $\mathbb{C}(n;s)$ denote the sets of partitions counted by $B(n;s)$ and $C(n;s)$, respectively. We define a map from $\mathbb{B}(n;s)$ to $\mathbb{C}(n;s)$, then we show that it is a bijection. We need Lemma 2.1 to transform the congruence condition for integers congruent to r_i+r_j modulo M $(1 \leqslant i < j \leqslant 3)$ into difference conditions for consecutive integers congruent to r_i and r_j modulo M.

By the conditions in (1.2), we see that

$$0 \leqslant r_1+r_2-M < r_1+r_3-M < r_1 < r_2+r_3-M < r_2 < r_3 < M. \tag{2.1}$$

This implies that r_1, r_2, r_3, r_1+r_2, r_1+r_3 and r_2+r_3 are distinct modulo M. For a partition μ in $\mathbb{C}(n,s)$, if a part μ_k is congruent to r_i+r_j modulo M, where $1 \leqslant i < j \leqslant 3$, we can represent μ_k as a sum of two positive integers congruent to r_i and r_j modulo M subject to a difference condition. This property also holds for $\mu_k - tM$, where t is an integer such that $\mu_k - tM \geqslant r_i + r_j$.

Lemma 2.1. *Let r_1, r_2 and r_3 be integers satisfying the conditions in* (1.2). *Let u be a positive integer congruent to r_i+r_j modulo M and $u \geqslant r_i+r_j$, where $1 \leqslant i < j \leqslant 3$. Let $w = (u-r_i-r_j)/M$. Then for integer $0 \leqslant t \leqslant w$, $u-tM$ can be uniquely expressed as*

$$u - tM = a_t + b_t, \tag{2.2}$$

where a_t and b_t are positive integers such that

$$a_t, b_t \equiv r_i \text{ or } r_j \pmod M \quad \text{and} \quad a_t \not\equiv b_t \pmod M, \tag{2.3}$$

and

$$0 < a_t - b_t < M. \tag{2.4}$$

More precisely,

$$a_t = \ell M + r_j, \quad b_t = \ell M + r_i, \tag{2.5}$$

if $u - tM = 2\ell M + r_i + r_j$, and

$$a_t = (\ell+1)M + r_i, \quad b_t = \ell M + r_j, \tag{2.6}$$

if $u - tM = (2\ell+1)M + r_i + r_j$, where ℓ is a nonnegative integer.

Proof. Clearly, $u - tM \equiv r_i + r_j \pmod{M}$ can be deduced from (2.2) and (2.3). To determine a_t and b_t from (2.2), (2.3) and (2.4), we may represent $u - tM$ by $2\ell M + r_i + r_j$ or $(2\ell+1)M + r_i + r_j$, where ℓ is a nonnegative integer. First consider the case $u - tM = 2\ell M + r_i + r_j$. There are two possibilities. Subcase 1: $a_t = \ell' M + r_i$ and $b_t = \ell'' M + r_j$, where ℓ' and ℓ'' are nonnegative integers such that $\ell' + \ell'' = 2\ell$. Hence we have

$$a_t - b_t = (\ell' - \ell'')M + r_i - r_j = 2(\ell' - \ell)M + r_i - r_j. \tag{2.7}$$

Since $0 < r_1 < r_2 < r_3 < M$ as given in (1.2), we have

$$-M < r_i - r_j < 0. \tag{2.8}$$

Under the condition $a_t - b_t > 0$, it follows from (2.7) and (2.8) that $2(\ell' - \ell) \geqslant 1$. Moreover, since $a_t - b_t < M$, by (2.7) and (2.8) we get $2(\ell' - \ell) \leqslant 1$. So we deduce that $2(\ell' - \ell) = 1$. But this is impossible since ℓ' and ℓ are integers. This means that Subcase 1 cannot happen.

We now consider Subcase 2: $a_t = \ell' M + r_j$ and $b_t = \ell'' M + r_i$, where ℓ' and ℓ'' are nonnegative integers such that $\ell' + \ell'' = 2\ell$. In this case, we have

$$a_t - b_t = (\ell' - \ell'')M + r_j - r_i = 2(\ell' - \ell)M + r_j - r_i. \tag{2.9}$$

Under the condition $a_t - b_t > 0$, it follows from (2.9) and (2.8) that $2(\ell' - \ell) \geqslant 0$. Moreover, since $a_t - b_t < M$, by (2.9) and (2.8) we get $2(\ell' - \ell) \leqslant 0$. So we deduce that $\ell' = \ell'' = \ell$, which yields (2.5).

For the case $u - tM = (2\ell + 1)M + r_i + r_j$, we also consider two subcases. Subcase 1: $a_t = \ell' M + r_j$ and $b_t = \ell'' M + r_i$, where ℓ' and ℓ'' are nonnegative integers such that $\ell' + \ell'' = 2\ell + 1$. Subcase 2: $a_t = \ell' M + r_i$ and $b_t = \ell'' M + r_j$, where ℓ' and ℓ'' are nonnegative integers such that $\ell' + \ell'' = 2\ell + 1$. In Subcase 1, there is no solution for ℓ'. In Subcase 2, there is only one solution, that is, $\ell' = \ell + 1$ and $\ell'' = \ell$. So we arrive at (2.6). The detailed proof is similar to the argument for the first case and hence it is omitted. □

We are now ready to give a bijective proof of Theorem 1.3.

Proof of Theorem 1.3. Define a map $\Phi\colon \mathbb{B}(n;s) \longrightarrow \mathbb{C}(n;s)$ by the following procedure. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ be a partition in $\mathbb{B}(n;s)$. We aim to construct a partition μ such that $\mu_k - \mu_{k+1} \geqslant M$ with strictly inequality if $\mu_k \equiv r_i + r_j \pmod{M}$ $(1 \leqslant i < j \leqslant 3)$. Assume that λ has only positive parts. For notational convenience, set $\lambda_0 = +\infty$. Consider the following two cases.

Case 1: Condition (ii) in Theorem 1.3 holds for all consecutive parts of λ, that is, for any $1 \leqslant i \leqslant s-1$, we have $\lambda_i - \lambda_{i+1} \geqslant M$ with strict inequality if λ_i is congruent to $r_1 + r_2$, $r_1 + r_3$ or $r_2 + r_3$ modulo M. In this case, we see that $\lambda \in \mathbb{C}(n;s)$, and we set $\mu = \lambda$.

Case 2: Condition (ii) in Theorem 1.3 does not hold, that is, there exists an integer i such that $\lambda_i - \lambda_{i+1} < M$. We choose i_1 to be the minimum integer subject to this condition. We aim to construct a partition, denoted $\alpha^{(1)}$, such that the condition (ii) holds for the first i_1 parts of $\alpha^{(1)}$. If this can be achieved, then one can iterate this process to find a desired partition in $\mathbb{C}(n, s)$. Here are two subcases.

Subcase 2.1: $\lambda_{i_1-1} - (\lambda_{i_1} + \lambda_{i_1+1}) \geqslant M$. Let

$$\alpha^{(1)} = (\lambda_1, \ldots, \lambda_{i_1-2}, \lambda_{i_1-1}, \lambda_{i_1} + \lambda_{i_1+1}, \lambda_{i_1+2} \ldots).$$

It is easily checked that the condition (ii) holds for the first i_1 parts of $\alpha^{(1)}$, that is, for any $1 \leqslant j \leqslant i_1 - 1$ we have $\alpha_j^{(1)} - \alpha_{j+1}^{(1)} \geqslant M$ with strict inequality if $\alpha_j^{(1)} \equiv r_1 + r_2$, $r_1 + r_3$ or $r_2 + r_3 \pmod M$.

Since $0 < \lambda_i - \lambda_{i+1} < M$, we get $\lambda_i \not\equiv \lambda_{i+1} \pmod M$. This means that $\lambda_{i_1} + \lambda_{i_1+1} \equiv r_i + r_j \pmod M$. So we need also show that $\lambda_{i_1} + \lambda_{i_1+1} - \lambda_{i_1+2} > M$ when $s \geqslant i_1 + 2$. This relation is obvious when $\lambda_{i_1} - \lambda_{i_1+2} \geqslant M$. We now assume that $\lambda_{i_1} - \lambda_{i_1+2} < M$. Note that λ_{i_1}, λ_{i_1+1} and λ_{i_1+2} are positive integers congruent to r_1, r_2 or r_3 modulo M. By the condition $0 < r_1 < r_2 < r_3 < M$ as given in (1.2) and the assumption $\lambda_{i_1} - \lambda_{i_1+2} < M$, we see that $(\lambda_{i_1}, \lambda_{i_1+1}, \lambda_{i_1+2})$ can be expressed in one of the three forms $(\ell M + r_3, \ell M + r_2, \ell M + r_1)$, $((\ell + 1)M + r_1, \ell M + r_3, \ell M + r_2)$ and $((\ell + 1)M + r_2, (\ell + 1)M + r_1, \ell M + r_3)$, where ℓ is a nonnegative integer. Using the conditions $0 < r_1 < r_2 < r_3 < M$, $r_1 + M < r_2 + r_3$ as given in (1.2) and the condition $\lambda_{i_1} - \lambda_{i_1+2} < M$, one can check that $\lambda_{i_1} + \lambda_{i_1+1} - \lambda_{i_1+2} > M$ holds in any of the above three cases. So we have shown that $\alpha^{(1)}$ is a desired partition in Subcase 2.1.

Subcase 2.2: $\lambda_{i_1-1} - (\lambda_{i_1} + \lambda_{i_1+1}) < M$. There is a unique integer $1 \leqslant k_1 \leqslant i_1 - 1$ such that

$$\lambda_{i_1-1-t} - (\lambda_{i_1} + \lambda_{i_1+1} + tM) < M \tag{2.10}$$

for $0 \leqslant t \leqslant k_1 - 1$, and

$$\lambda_{i_1-1-k_1} - (\lambda_{i_1} + \lambda_{i_1+1} + k_1 M) \geqslant M. \tag{2.11}$$

Let

$$\alpha^{(1)} = (\lambda_1, \ldots, \lambda_{i_1-1-k_1}, \lambda_{i_1} + \lambda_{i_1+1} + k_1 M, \lambda_{i_1-k_1} - M, \ldots, \lambda_{i_1-1} - M, \lambda_{i_1+2}, \ldots).$$

As i_1 is chosen to be the minimum integer i such that $\lambda_i - \lambda_{i+1} < M$, for any $1 \leqslant j \leqslant i_1 - 1$, we have $\lambda_j - \lambda_{j+1} \geqslant M$. This implies that for $i_1 - k_1 \leqslant j \leqslant i_1 - 2$, $(\lambda_j - M) - (\lambda_{j+1} - M) \geqslant M$. By (2.11), $\lambda_{i_1-1-k_1} - (\lambda_{i_1} + \lambda_{i_1+1} + k_1 M) \geqslant M$. To verify the condition (ii) for the first i_1 parts of $\alpha^{(1)}$, it remains to show that

$$(\lambda_{i_1} + \lambda_{i_1+1} + k_1 M) - (\lambda_{i_1-k_1} - M) > M, \tag{2.12}$$

since the part $\lambda_{i_1} + \lambda_{i_1+1} + k_1 M$ is congruent to $r_i + r_j$ modulo M. Notice that (2.12) can be deduced from (2.10) by setting $t = k_1 - 1$. This completes the proof in Subcase 2.2.

For the partition $\alpha^{(1)}$, if condition (ii) holds for all consecutive parts, then we set $\mu = \alpha^{(1)}$. Otherwise, we can find a minimum integer i_2 such that $i_2 \geqslant i_1$ and $\alpha_{i_2}^{(1)} - \alpha_{i_2+1}^{(1)} < M$. Then we may repeat the above process in Case 2. Finally, we obtain a partition μ for which condition (ii) holds for all consecutive parts.

We observe that each part of μ is congruent to $r_1, r_2, r_3, r_1+r_2, r_1+r_3$ or r_2+r_3 modulo M, and the number of parts of λ is equal to the number of parts of μ if the number of parts congruent to r_1+r_2, r_1+r_3 or r_2+r_3 modulo M are counted twice. Hence conditions (i) and (iii) in Theorem 1.3 also hold for μ. So we have $\mu \in \mathbb{C}(n;s)$.

To prove that Φ is a bijection, we now describe the inverse map Φ^{-1}. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_v)$ be a partition in $\mathbb{C}(n;s)$. Assume that $\mu_1 > \mu_2 > \cdots > \mu_v > 0$. We aim to construct a partition λ such that $\Phi(\lambda) = \mu$ by transforming the congruence condition for parts congruent to $r_i + r_j$ modulo M into difference conditions for consecutive parts congruent to r_i and r_j modulo M. For notational convenience, set $\mu_{t+1} = 0$ if μ_t is the last positive part of μ. Consider the following two cases.

Case 1: There is no part of μ that is congruent to $r_1 + r_2$, $r_1 + r_3$ or $r_2 + r_3$ modulo M. In this case, we see that $\mu \in \mathbb{B}(n;s)$, and we set $\lambda = \mu$.

Case 2: There exists an integer j such that μ_j is congruent to $r_1 + r_2$, $r_1 + r_3$ or $r_2 + r_3$ modulo M. We choose j_1 to be the maximum integer subject to this condition. Using Lemma 2.1 for $u = \mu_{j_1}$ and $t = 0$, we get $\mu_{j_1} = a_0 + b_0$, where a_0 and b_0 are given by (2.5) or (2.6). We can transform μ into a partition, denoted $\beta^{(1)}$, such that the number of parts congruent to $r_i + r_j$ modulo M in $\beta^{(1)}$ is one less than the number of parts congruent to $r_i + r_j$ modulo M in μ. There are two cases.

Case (i): $0 \leqslant \mu_{j_1+1} < b_0$. Let

$$\beta^{(1)} = (\mu_1, \ldots, \mu_{j_1-1}, a_0, b_0, \mu_{j_1+1}, \ldots, \mu_v).$$

We claim that $\beta^{(1)}$ is a partition. Let $\beta^{(1)} = (\beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_{v+1}^{(1)})$. Since $\mu_1 > \mu_2 > \cdots > \mu_v > 0$, by (2.5) and (2.6), we see that $\mu_1 > \mu_2 > \cdots > \mu_{j_1-1} > a_0 > b_0 > 0$ if $\mu_{j_1+1} = 0$, and $\mu_1 > \mu_2 > \cdots > \mu_{j_1-1} > a_0 > b_0 > \mu_{j_1+1} > \cdots > \mu_v > 0$ if $\mu_{j_1+1} > 0$. It follows that $\beta_1^{(1)} > \beta_2^{(1)} > \cdots > \beta_{v+1}^{(1)} > 0$.

As j_1 is the maximum integer such that μ_{j_1} is congruent to $r_1 + r_2$, $r_1 + r_3$ or $r_2 + r_3$ modulo M, for $j_1 \leqslant t \leqslant v+1$, we have $\beta_t^{(1)} \equiv r_1$, r_2 or $r_3 \pmod{M}$ since all parts of μ are congruent to r_1, r_2, r_3, $r_1 + r_2, r_1 + r_3$ or $r_2 + r_3$ modulo M. So the number of parts congruent to $r_i + r_j$ modulo M in $\beta^{(1)}$ is one less than the number of parts congruent to $r_i + r_j$ modulo M in μ.

Case (ii): $\mu_{j_1+1} \geqslant b_0$. The following procedure generates a partition $\beta^{(1)}$ from μ. Using Lemma 2.1 for $u = \mu_{j_1}$ and $t \geqslant 1$ with $\mu_{j_1} - tM \geqslant r_i + r_j$, we obtain a unique expression $\mu_{j_1} - tM = a_t + b_t$, where a_t and b_t are given by (2.5) or (2.6). Since $\mu_{j_1+1} \geqslant b_0$, there is a unique integer $1 \leqslant k_1 \leqslant v - j_1$ such that

$$\mu_{j_1+t+1} \geqslant b_t \tag{2.13}$$

for $0 \leqslant t \leqslant k_1 - 1$, and

$$0 \leqslant \mu_{j_1+k_1+1} < b_{k_1}. \tag{2.14}$$

Let

$$\beta^{(1)} = (\mu_1, \ldots, \mu_{j_1-1}, \mu_{j_1+1} + M, \ldots, \mu_{j_1+k_1} + M, a_{k_1}, b_{k_1}, \mu_{j_1+k_1+1}, \ldots),$$

and denote $\beta^{(1)}$ by $(\beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_{v+1}^{(1)})$. Note that a_{k_1} and b_{k_1} are congruent to r_1, r_2 or r_3 modulo M. Recall that j_1 is the maximum integer such that μ_{j_1} is congruent to $r_1 + r_2$,

r_1+r_3 or r_2+r_3 modulo M. Since all parts of μ are congruent to r_1, r_2, r_3, r_1+r_2, r_1+r_3 or r_2+r_3 modulo M, for $j_1 \leqslant t \leqslant v+1$, we have $\beta_t^{(1)} \equiv r_1$, r_2 or $r_3 \pmod{M}$. Hence the number of parts congruent to r_i+r_j modulo M in $\beta^{(1)}$ is one less than the number of parts congruent to r_i+r_j modulo M in μ.

It remains to show that $\beta^{(1)}$ is a partition. First, if $j_1 \geqslant 2$, we need to verify that

$$\mu_{j_1-1} > \mu_{j_1+1} + M. \tag{2.15}$$

Since $\mu=(\mu_1,\mu_2,\ldots,\mu_v)$ is a partition in $\mathbb{C}(n;s)$, we have $\mu_i-\mu_{i+1} \geqslant M$ for $1 \leqslant i \leqslant v-1$. It follows that

$$\mu_{j_1-1} - (\mu_{j_1+1}+M) = (\mu_{j_1-1}-\mu_{j_1}) + (\mu_{j_1}-\mu_{j_1+1}) - M > 0,$$

which yields (2.15). Next, we prove that

$$\mu_{j_1+k_1} + M > a_{k_1}. \tag{2.16}$$

We claim that

$$a_{k_1} = b_{k_1-1}. \tag{2.17}$$

To derive (2.17), we note that $\mu_{j_1}-k_1M = a_{k_1}+b_{k_1}$ and $\mu_{j_1}-(k_1-1)M = a_{k_1-1}+b_{k_1-1}$, where a_{k_1}, b_{k_1}, a_{k_1-1} and b_{k_1-1} are given by (2.5) or (2.6). If $\mu_{j_1}-k_1M$ can be represented by $2\ell M+r_i+r_j$, where ℓ is a nonnegative integer, then we have $\mu_{j_1}-(k_1-1)M = (2\ell+1)M+r_i+r_j$. By (2.5) and (2.6) we deduce that

$$a_{k_1} = \ell M + r_j \quad \text{and} \quad b_{k_1-1} = \ell M + r_j,$$

as required. Similarly, it can be shown that (2.17) also holds if $\mu_{j_1}-k_1M$ can be represented by $(2\ell+1)M+r_i+r_j$ for a nonnegative integer ℓ. So (2.17) is confirmed.

Setting $t=k_1-1$ in (2.13) gives

$$\mu_{j_1+k_1} \geqslant b_{k_1-1}. \tag{2.18}$$

Combining (2.18) and (2.17), we find that $\mu_{j_1+k_1} \geqslant a_{k_1}$. It follows that

$$\mu_{j_1+k_1} + M \geqslant a_{k_1} + M > a_{k_1}.$$

This proves (2.16). So we have shown that $\beta^{(1)}$ is a partition. Since μ is a partition in $\mathbb{C}(n;s)$, it has distinct parts. Thus we have reached the conclusion that $\beta^{(1)}$ has distinct parts. This completes the proof in Case (ii).

For either case (i) or case (ii), if each part of $\beta^{(1)}$ is congruent to r_1, r_2 or r_3 modulo M, then we set $\lambda=\beta^{(1)}$. Otherwise, we can find a maximum integer j_2 such that $j_2<j_1$ and $\beta_{j_2}^{(1)}$ is congruent to r_1+r_2, r_1+r_3 or r_2+r_3 modulo M. Then we may iterate the above process until we obtain a partition λ with all parts congruent to r_1, r_2 or r_3 modulo M.

Moreover, it can be seen that the number of parts of λ is equal to the number of parts of μ with the convention that the parts congruent to r_1+r_2, r_1+r_3 or r_2+r_3 modulo M are counted twice. Thus we have $\lambda \in \mathbb{B}(n;s)$, and so Φ is surjective.

Due to the uniqueness of the expression of a positive integer congruent to $r_i + r_j$ modulo M in Lemma 2.1, we see that every step of Φ is reversible. Hence Φ is a bijection between $\mathbb{B}(n, s)$ and $\mathbb{C}(n, s)$. So we have $B(n, s) = C(n, s)$. This completes the proof. $\square$

The following example gives an illustration of the map Φ. Let $M = 6$, $r_1 = 2$, $r_2 = 4$ and $r_3 = 5$, for which the conditions in (1.2) are satisfied. Let

$$\lambda = (92, 70, 64, 53, 52, 46, 38, 35, 23, 17, 4, 2),$$

which is a partition in $\mathbb{B}(496; 12)$. In the construction of $\Phi(\lambda)$, the intermediate partitions $\alpha^{(1)}$, $\alpha^{(2)}$ and $\alpha^{(3)}$ are given below:

$$\begin{aligned}
\alpha^{(1)} &= (123, 86, 64, 58, 46, 38, 35, 23, 17, 4, 2),\\
\alpha^{(2)} &= (123, 97, 80, 58, 52, 40, 23, 17, 4, 2),\\
\alpha^{(3)} &= (123, 97, 80, 58, 52, 40, 23, 17, 6).
\end{aligned}$$

Note that condition (ii) in Theorem 1.3 holds for all consecutive parts of $\alpha^{(3)}$, that is, for $1 \leqslant i \leqslant 8$, we have $\alpha_i^{(3)} - \alpha_{i+1}^{(3)} \geqslant M$ with strict inequality if $\alpha_i^{(3)}$ is congruent to $r_1 + r_2$, $r_1 + r_3$ or $r_2 + r_3$ modulo M. Moreover, there are only three parts, 123, 97 and 6, which are congruent to $r_1 + r_2$, $r_1 + r_3$ or $r_2 + r_3$ modulo M, and therefore should be counted twice. Hence

$$\mu = \alpha^{(3)} = (123, 97, 80, 58, 52, 40, 23, 17, 6),$$

which belongs to $\mathbb{C}(496; 12)$.

The following example gives an illustration of the inverse map Φ^{-1}. Let $M = 6$, $r_1 = 2$, $r_2 = 4$ and $r_3 = 5$, for which the conditions in (1.2) are satisfied. Let

$$\mu = (123, 97, 80, 58, 52, 40, 23, 17, 6),$$

which is a partition in $\mathbb{C}(496; 12)$. The intermediate partitions $\beta^{(1)}$, $\beta^{(2)}$ and $\beta^{(3)}$ are given below:

$$\begin{aligned}
\beta^{(1)} &= (123, 97, 80, 58, 52, 40, 23, 17, 4, 2),\\
\beta^{(2)} &= (123, 86, 64, 58, 46, 38, 35, 23, 17, 4, 2),\\
\beta^{(3)} &= (92, 70, 64, 53, 52, 46, 38, 35, 23, 17, 4, 2).
\end{aligned}$$

Clearly, all the parts of $\beta^{(3)}$ are congruent to 2, 4 or 5 modulo M. Hence

$$\lambda = \beta^{(3)} = (92, 70, 64, 53, 52, 46, 38, 35, 23, 17, 4, 2),$$

which belongs to $\mathbb{B}(496; 12)$.

Acknowledgments

I am grateful to Arthur L.B. Yang for valuable suggestions. I also wish to thank the referee for helpful comments. This work was supported by the 973 Project and the National Science Foundation of China.

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