A De Bruijn–Erdős theorem for chordal graphs

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Abstract

A special case of a combinatorial theorem of De Bruijn and Erdős asserts that every noncollinear set of n points in the plane determines at least n distinct lines. Chen and Chvátal suggested a possible generalization of this assertion in metric spaces with appropriately defined lines. We prove this generalization in all metric spaces induced by connected chordal graphs.

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1 Introduction

It is well known that

(i) *every noncollinear set of n points in the plane determines at least n distinct lines.*

As noted by Erdős [12], theorem (i) is a corollary of the Sylvester–Gallai theorem (asserting that, for every noncollinear set S of finitely many points in the plane, some line goes through precisely two points of S); it is also a special case of a combinatorial theorem proved later by De Bruijn and Erdős [11].

Theorem (i) involves neither measurement of distances nor measurement of angles: the only notion employed here is incidence of points and lines. Such theorems are a part of *ordered geometry* [7], which is built around the ternary relation of *betweenness*: point b is said to lie between points a and c if b is an interior point of the line segment with endpoints a and c. It is customary to write $[abc]$ for the statement that b lies between a and c. In this notation, a *line* $\overline{uv}$ is defined — for any two distinct points u and v — as

$$\{u, v\} \cup \{p : [puv] \vee [upv] \vee [uvp]\}. \tag{1}$$

In terms of the Euclidean metric dist, we have

$$[abc] \Leftrightarrow a, b, c \text{ are three distinct points and } \mathrm{dist}(a, b) + \mathrm{dist}(b, c) = \mathrm{dist}(a, c). \tag{2}$$

In an arbitrary metric space, equivalence (2) defines the ternary relation of *metric betweenness* introduced in [14] and further studied in [1, 3, 8]; in turn, (1) defines the line $\overline{uv}$ for any two distinct points u and v in the metric space. The resulting family of lines may have strange properties. For instance, a line can be a proper subset of another: in the metric space with points u, v, x, y, z and

$$\mathrm{dist}(u, v) = \mathrm{dist}(v, x) = \mathrm{dist}(x, y) = \mathrm{dist}(y, z) = \mathrm{dist}(z, u) = 1,$$
$$\mathrm{dist}(u, x) = \mathrm{dist}(v, y) = \mathrm{dist}(x, z) = \mathrm{dist}(y, u) = \mathrm{dist}(z, v) = 2,$$

we have

$$\overline{vy} = \{v, x, y\} \quad \text{and} \quad \overline{xy} = \{v, x, y, z\}.$$

Chen [4] proved, using a definition of $\overline{uv}$ different from (1), that the Sylvester–Gallai theorem generalizes in the framework of metric spaces. Chen and Chvátal [5] suggested that theorem (i), too, might generalize in this framework:

(ii) *True or false? Every metric space on n points, where $n \geqslant 2$, either has at least n distinct lines or else has a line that consists of all n points.*

They proved that

- every metric space on n points either has at least $\lg n$ distinct lines or else has a line that consists of all n points and noted that the lower bound $\lg n$ can be improved to $\lg n + \frac{1}{2}\lg\lg n + \frac{1}{2}\lg\frac{\pi}{2} - o(1)$. (Here, as usual, $\lg x$ stands for $\log_2 x$.)

Every connected undirected graph induces a metric space on its vertex set, where $\mathrm{dist}(u, v)$ is the familiar graph-theoretic distance between vertices u and v, defined as the smallest number of edges in a path from u to v. (Some people call this the 'hop distance'.) Chiniforooshan and Chvátal [6] proved that

- every metric space induced by a connected graph on n vertices either has $\Omega(n^{2/7})$ distinct lines or else has a line that consists of all n vertices;

we will prove that the answer to (ii) is 'true' for all metric spaces induced by connected chordal graphs. (We follow the graph-theoretic terminology of Bondy and Murty [2]. In particular, a *chordal graph* is a graph that contains no induced cycle of length four or more.)

Theorem 1. *Every metric space induced by a connected chordal graph on n vertices, where $n \geqslant 2$, either has at least n distinct lines or else has a line that consists of all n vertices.*

2 The proof

Given an undirected graph, let us write $[abc]$ to mean that a, b, c are three distinct vertices such that $\mathrm{dist}(a, b) + \mathrm{dist}(b, c) = \mathrm{dist}(a, c)$; this is equivalent to saying that b is an interior vertex of a shortest path from a to c.

Lemma 2. *Let s, x, y be vertices in a finite chordal graph such that $[sxy]$. If $\overline{sx} = \overline{sy}$, then x is a cut vertex separating s and y.*

Proof. The set of all vertices u such that $\mathrm{dist}(s, u) = \mathrm{dist}(s, x)$ separates s and y. Among all its subsets that separate s and y, choose a minimal one and call it C. Since x is an interior vertex of a shortest path from s to y, it belongs to C. To prove that C includes no other vertex, assume, to the contrary, that C includes a vertex u other than x.

Our graph with C removed has distinct connected components S and Y such that $s \in S$ and $y \in Y$; the minimality of C guarantees that each of its vertices has at least one neighbour in S and at least one neighbour in Y. Since each of u and x has at least one neighbour in S, there is a path from u to x with at least one interior vertex and with all interior vertices in S. Let P be a shortest such path; note that P has no chords except possibly the chord ux. Similarly, there is a path Q from u to x with at least one interior vertex, and with all interior vertices in Y, that has no chords except possibly the chord ux. The union of P and Q is a cycle of length at least four; since this cycle must have a chord, vertices u and x must be adjacent. In turn, the union of Q and ux is a chordless cycle, and so Q has precisely two edges. This means that some vertex v in Y is adjacent to both u and x. (Similarly, some vertex in S is adjacent to both u and x; however, this fact is irrelevant to our argument.)

Write $i = \text{dist}(s, x)$ and $j = \text{dist}(x, y)$. Since all vertices t with $\text{dist}(s, t) < i$ belong to S and since v has no neighbours in S, we must have $\text{dist}(s, v) > i$; since $\text{dist}(x, v) = 1$, we conclude that $\text{dist}(s, v) = i + 1$ and that $v \in \overline{sx}$. Since $\overline{sx} = \overline{sy}$, it follows that $v \in \overline{sy}$. Since $\text{dist}(v, x) = 1$ and $\text{dist}(x, y) = j$, we have $\text{dist}(v, y) \leqslant j + 1$. From $\text{dist}(s, v) = i+1$, $\text{dist}(s, y) = i+j$, $\text{dist}(v, y) \leqslant j+1$, $i \geqslant 1$, $j \geqslant 1$, and $v \in \overline{sy}$, we deduce that $\text{dist}(v, y) = j - 1$.

Since $\text{dist}(u, v) = 1$, it follows that $\text{dist}(u, y) \leqslant j$; since $\text{dist}(s, u) = i$ and $\text{dist}(s, y) = i + j$, we conclude that $\text{dist}(u, y) = j$ and $u \in \overline{sy}$. Since $\text{dist}(s, u) = i$, $\text{dist}(s, x) = i$, and $\text{dist}(u, x) = 1$, we have $u \notin \overline{sx}$. But then $\overline{sx} \neq \overline{sy}$, a contradiction. □

A vertex of a graph is called *simplicial* if its neighbours are pairwise adjacent.

Lemma 3. *Let s, x, y be three distinct vertices in a finite connected chordal graph. If s is simplicial and $\overline{sx} = \overline{sy}$, then $\overline{xy}$ consists of all the vertices of the graph.*

Proof. Since $\overline{sx} = \overline{sy}$, we have $y \in \overline{sx}$, and so $[ysx]$ or $[syx]$ or $[sxy]$; since s is simplicial, $[ysx]$ is excluded; switching x and y if necessary, we may assume that $[sxy]$. Given an arbitrary vertex u, we have to prove that $u \in \overline{xy}$. Let P be a shortest path from s to u and let Q be a shortest path from u to y. Lemma 2 guarantees that x is a cut vertex separating s and y, and so the concatenation of P and Q must pass through x. This means that $[sxu]$ or $[uxy]$ (or both). If $[uxy]$, then $u \in \overline{xy}$; to complete the proof, we may assume that $[sxu]$, and so $u \in \overline{sx}$.

Since $\overline{sx} = \overline{sy}$, we have $[usy]$ or $[suy]$ or $[syu]$; since s is simplicial, $[usy]$ is excluded. If $[suy]$, then $[sxu]$ implies $[xuy]$; if $[syu]$, then $[sxy]$ implies $[xyu]$; in either case, $u \in \overline{xy}$. □

Proof of Theorem 1. Consider a connected chordal graph on n vertices where $n \geqslant 2$. By a theorem of Dirac [10, Theorem 4], this graph has at least two simplicial vertices; choose one of them and call it s. We may assume that the lines $\overline{sz}$ with $z \neq s$ are pairwise distinct (else some line consists of all n vertices by Lemma 3). Since the graph is connected and has at least two vertices, s has at least one neighbour; choose one and call it u. If u is the only neighbour of s, then every path from s to another vertex must pass through u, and so $\overline{su}$ consists of all n vertices. If s has a neighbour v other than u, then line $\overline{uv}$ is distinct from all of the $n - 1$ lines $\overline{sz}$ with $z \neq s$: since s, u, v are pairwise adjacent, we have $s \notin \overline{uv}$. □

3 Related theorems

In Theorem 1, 'connected chordal graph' can be replaced by 'connected bipartite graph':

- every metric space induced by a connected bipartite graph on n vertices, where $n \geqslant 2$, has a line that consists of all n vertices.

In fact, $\overline{xy}$ consists of all n vertices whenever x and y are adjacent. To prove this, consider an arbitrary vertex u. Since the graph is bipartite, $\mathrm{dist}(u, x)$ and $\mathrm{dist}(u, y)$ have distinct parities; since $\mathrm{dist}(x, y) = 1$, they differ by at most one. We conclude that $\mathrm{dist}(u, x)$ and $\mathrm{dist}(u, y)$ differ by precisely one, and so $u \in \overline{xy}$.

In Theorem 1, 'connected chordal graph' can be also replaced by 'graph of diameter two': Chvátal [9] proved that

- every metric space on n points where $n \geqslant 2$ and each nonzero distance equals 1 or 2 either has at least n distinct lines or else has a line that consists of all n vertices.

Kantor and Patkós [13] proved that

- if no two of n points in the plane share their x- or y-coordinate, then these n points with the L_1 metric either induce at least n distinct lines or else they induce a line that consists of all of them.

(For sets of n points in the plane that are allowed to share their coordinates, [13] provides a weaker conclusion: these n points with the L_1 metric either induce at least $n/37$ distinct lines or else they induce a line that consists of all of them.)

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