Solution to a Conjecture on the Maximum Skew-Spectral Radius of Odd-Cycle Graphs*

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Abstract

Let G be a simple graph with no even cycle, called an odd-cycle graph. Cavers et al. [Linear Algebra Appl. 436(12):4512-1829, 2012] showed that the spectral radius of G^σ is the same for every orientation σ of G, and equals the maximum matching root of G. They proposed a conjecture that the graphs which attain the maximum skew spectral radius among the odd-cycle graphs G of order n are isomorphic to the odd-cycle graph with one vertex degree $n-1$ and size $m = \lfloor 3(n-1)/2 \rfloor$. By using the Kelmans transformation, we give a proof to the conjecture. Moreover, sharp upper bounds of the maximum matching roots of the odd-cycle graphs with given order n and size m are given and extremal graphs are characterized.

Keywords: skew spectral radius, odd-cycle graphs, maximum matching root, Kelmans transformation

1 Introduction

Let G be a finite simple graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. For more notation and terminology that will be used in the sequel, we refer to [2, 3]. Given an orientation σ of G which makes any edge uv to be an arc $u \to v$ or $v \to u$, we get an oriented graph G^σ. The skew-adjacency matrix is the square matrix $S(G^\sigma) = [s_{ij}]$

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of order n, where $s_{ij} = 1$ and $s_{ji} = -1$ if $u \to v$, $s_{ij} = s_{ji} = 0$ otherwise. By the definition, $S(G^\sigma)$ is a skew symmetric matrix. Let the skew-characteristic polynomial be $\phi_s(G^\sigma, x) = det(xI_n - S(G^\sigma))$, where I_n is the unit matrix with order n. Thus the eigenvalues of $S(G^\sigma)$ are all pure imaginary numbers, which form the skew-spectrum of G^σ. The spectral radius $\rho(G^\sigma)$ of an oriented graph G^σ is the maximum norm of its all eigenvalues. In recent years, there are many results about the skew-spectrum and the skew-spectral radius, see [1, 5, 6, 17].

For a given graph G, different orientations may lead to different skew-spectra, and thus different skew-spectral radius. In [5], Cavers et al. introduced the maximum skew-spectral radius $\rho_s(G)$ which is defined as $\rho_s(G) = \max\{\rho(G^\sigma) : \sigma$ is an orientation of $G\}$. They studied a special class of graphs, called the odd-cycle graphs. A graph is called an odd-cycle graph if it have no even cycles. In [5], Cavers et al. showed that when the graph G is an odd-cycle graph, the skew-spectra are independent of the orientation σ. In fact, the skew-characteristic polynomial $\phi_s(G^\sigma, x)$ of G^σ can be determined by the matching polynomial $m(G, x)$ of the graph G. Anuradha and Balakrishnan also got some skew-spectral properties of the odd-cycle graphs in [1].

Now, we recall the definition of the matching polynomial [11].

Definition 1.1. *Let $m_r(G)$ denote the number of r independent edges in a graph G. Define the matching polynomial of G as*

$$m(G, x) = \sum_{k=0}^{n} (-1)^k m_k(G) x^{n-2k}.$$

Note that many results about the roots of the matching polynomial have been obtained; see [9, 10, 11, 14], such as, the matching roots are real-valued, and symmetric. Denote by $t(G)$ the maximum matching root of G. For the applications of the matching polynomial in chemistry and statistical physics, we refer the reader to [14, 13].

Given an odd-cycle graph G, the following lemma in [5] implies the relationship between the skew-characteristic polynomial of an oriented graph G^σ and the matching polynomial of the graph G.

Lemma 1.2. *([5]) Let G be a graph of order n. Then G is an odd-cycle graph if and only if $\phi_s(G^\sigma, x) = (-i)^n m(G, ix)$ for all orientations σ of G.*

From the above lemma, we note that when G is an odd-cycle graph, the skew-spectral radius $\rho(G^\sigma)$ under any orientation σ equals the maximum skew-spectral radius $\rho_s(G)$. Moreover, the maximum skew-spectral radius of the graph G equals the maximum matching root of the graph G, i.e. $\rho_s(G) = t(G)$.

In [5], Cavers et al. studied the upper bound of the maximum skew-spectral radius among the odd-cycle graphs with order n, and proposed a conjecture. Let H_n be the odd-cycle graph of order n with one vertex degree $n-1$ and size $m = \lfloor 3(n-1)/2 \rfloor$. It is easy to check that H_n is unique (up to isomorphism). Moreover, we can find that H_n is the maximal odd-cycle graph of order n, since the size m of an odd-cycle graph of order n is no more than $\lfloor 3(n-1)/2 \rfloor$ ([1], Theorem 3.2).

Conjecture 1.3. *([5]) If G is an odd-cycle graph of order n, then $\rho_s(G) \leqslant \rho_s(H_n)$, and equality holds if and only if $G \cong H_n$.*

Based on Lemma 1.2, we can find that Conjecture 1.3 can be equivalently rewritten as follows.

Conjecture 1.4. *If G is an odd-cycle graph of order n, then $t(G) \leqslant t(H_n)$, and equality holds if and only if $G \cong H_n$.*

For convenience, in the remainder of this paper, we just study the maximum matching root $t(G)$ of an odd-cycle graph. We will prove Conjecture 1.4. In fact, we get an even stronger result. We characterize the extremal graphs with maximum $t(G)$ among the odd-cycle graphs with order n and size m.

Theorem 1.5. *Let G be an odd-cycle graph with order n and size m, where $1 \leqslant m \leqslant \lfloor \frac{3(n-1)}{2} \rfloor$. Let $F(n,m)$ be an odd-cycle graph with one vertex degree $n-1$ and size m, which is unique up to isomorphism.*

(1) If $m = 1$, then $t(G) = 1$.

(2) If $m = 2$, then $t(G) \leqslant \sqrt{2}$, and equality holds if and only if G is the disjoint union of a star $K_{1,2}$ and $n-2$ isolated vertices.

(3) If $m = 3$, then $t(G) \leqslant \sqrt{3}$, and equality holds if and only if G is the disjoint union of a triangle and $n-3$ isolated vertices, or the disjoint union of a star graph $K_{1,3}$ and $n-4$ isolated vertices.

(4) If $4 \leqslant m \leqslant n-2$, then $t(G) \leqslant t(F(m+1,m))$, and equality holds if and only if G is the disjoint union of the graph $F(m+1,m)$ and $n-m-1$ isolated vertices.

(5) If $n-1 \leqslant m \leqslant \lfloor \frac{3(n-1)}{2} \rfloor$ and $m \geqslant 4$, then $t(G) \leqslant t(F(n,m))$, and equality holds if and only if $G \cong F(n,m)$.

We organize the rest of this paper as follows. In Section 2, we study the Kelmans transformation acting on odd-cycle graphs. In Section 3, we find the monotone property of the maximum matching roots after an acting of the Kelmans transformation. In Section 4, we give our proofs of Conjecture 1.4 and Theorem 1.5.

2   The Kelmans transformation on odd-cycle graphs

In [5], Cavers et al. suggested that the standard techniques called edge-switching will be useful to prove Conjecture 1.3. In this paper, we mainly use a graph transformation called the Kelmans transformation.

We first recall some results about the Kelmans transformation. In[15], Kelmans introduced a graph transformation for the extremal problems related with the synthesis of reliable networks. In [16], Satyanarayana et al. also introduced a reliability improving

graph transformation, named the "swing surgery", which can be thought as the inverse of the Kelmans transformation. Brown et al. [4] called the Kelmans transformation as shift operation, and studied many applications in network reliability. Csikvári in [7] made a breakthrough in solving a conjecture of Nikiforov by the Kelmans transformation. Moreover, Csikvári in [8] found several applications of the Kelmans transformation in the extremal problems on graph polynomials, such as, matching polynomial, independent polynomial, chromatic polynomial, Laplacian polynomial. For more interesting details, see [8]. Next, we give the definition of the Kelmans transformation.

Definition 2.1. *Let u and v be two vertices of a graph G. Let $N(v)$ be the neighbor set of v in G. For any $w \in N(v)$, if w is not u and u and w are nonadjacent, then we delete the edge vw and add the edge uw. After the above operations, we get a new graph G'. Then G' is a graph obtained from G by the Kelmans transformation between u and v; see Fig. 1.*
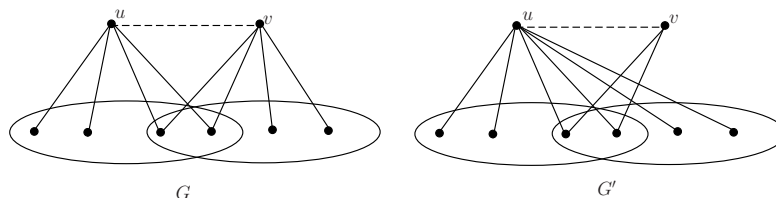

Figure 1: The Kelmans transformation.

For convenience, Csikvári [8] called u and v the beneficiary and the co-beneficiary of the transformation, respectively. For brevity, we denote by $KT(G, u, v)$ the Kelmans transformation acting on a graph G with the beneficiary u and the co-beneficiary v. It is easy to check that the Kelmans transformation does not change the number of edges.

Let G be a connected odd-cycle graph with order n and size m. Then $n - 1 \leqslant m \leqslant n + \lfloor (n-1)/2 \rfloor - 1$ ([1], Theorem 3.2). Let $F(n, m)$ be an odd-cycle graph with one vertex degree $n - 1$ and size m. It is easy to check that $F(n, m)$ is unique (up to isomorphism). If $m = n + \lfloor (n-1)/2 \rfloor - 1$, then $F(n, m)$ is the same as H_n.

Theorem 2.2. *Let G be a connected odd-cycle graph with order n and size m. Then $F(n, m)$ can be obtained from G by a number of Kelmans transformations.*

Proof. Let G be a connected odd-cycle graph with order n and size m, which implies that $n - 1 \leqslant m \leqslant n + \lfloor (n-1)/2 \rfloor - 1$. Then G must be a cactus, that is, every block of G is either a cycle or an edge ([2], EX. 3.2.3). It follows that any two cycles of G are edge-disjoint. Let $C(G)$ be the set of odd cycles with order larger than 3. Next, we will prove the theorem by three steps.

Step 1. If $C(G)$ is nonempty, assume that $C(G) = \{C_1, \ldots, C_t\}$. The number of the edges of $C(G)$ is $|E(C(G))| = \sum_{i=1}^{t} |C_t| \leqslant m$. Let $C \in C(G)$ and u, v be two vertices

of C whose distance in C is 2. Then there exists a path $uwvw'$ in C. By the Kelmans transformation $KT(G, u, v)$, we get a graph G'. It is easy to find that the degree of the vertex v in G' is 1; see Fig. 2.
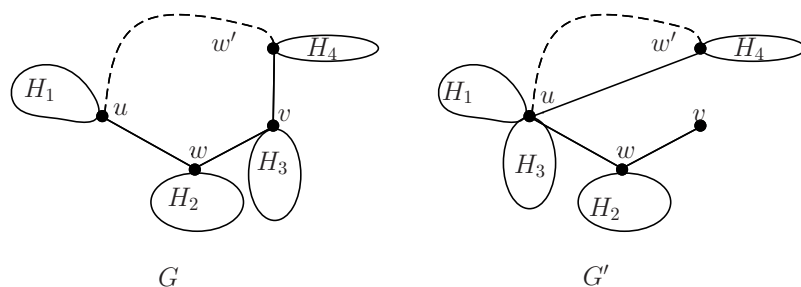

Figure 2: $KT(G, u, v)$ in step 1.

We first show that G' is connected. For every vertex u' in G, there is a path $u'Pu$ in G connected u' and u. If the vertex v is in the path P, then the path P can be decomposed as P_1vP_2. By the definition of the Kelmans transformation, we can check that $u'P_1u$ is in the graph G' and u' is connected to u in G'. If v is not in the path P, then u' and u are connected by the path P in G'.

Secondly, we claim that G' is still an odd-cycle graph. Suppose that there exists an even cycle C' in G'. Then u must be in C'. Let $C' = uh_1 \ldots h_su$. If uh_1 and uh_s both belong to G, then C' is also an even cycle of G, a contradiction. If neither uh_1 nor uh_s belongs to G, then $vh_1 \ldots h_sv$ is an even cycle of G, again a contradiction. Therefore, without loss of generality, suppose that uh_1 is in G and uh_s is not in G, which implies that $h_1 \neq h_s$ and vh_s is an edge of G. We then obtain a path $P = uh_1 \cdots h_sv$ with even length in G. Since G is a cactus and u, v are both in the cycle C, we can deduce that P is in the cycle C. It follows that $P = uwv$, contradicting with $h_1 \neq h_s$.

Finally, we consider the cycle set $C(G')$. By analyzing the corresponding relationship between $C(G)$ and $C(G')$, it can be verified that $|C(G')| = |C(G)|$ or $|C(G)| - 1$ and $|E(C(G'))| \leqslant |E(C(G))| - 2$. Thus, by less than the number $\frac{m}{2}$ of Kelmans transformations as above, we can get a connected odd-cycle graph G' whose cycles are all triangles. Then, we turn to Step 3.

Step 2. If $C(G)$ is empty, then G contains no cycles or each cycle of G is a triangle. Let $G' := G$.

Step 3. Assume that a vertex u of G' has the maximum degree in G' and its degree is denoted by $d_{G'}(u)$. If $d_{G'}(u)$ is less than $n - 1$, then there exists a vertex v such that the distance of u, v in G' is 2. Suppose that uwv is a 2-path in G'. We apply the Kelmans transformation $KT(G', u, w)$ and get a graph G''. By the similar analysis in the first step, we find that G'' is a connected odd-cycle graph whose cycles are all triangles. Moreover, we get $d_{G'}(u) \geqslant d_G(u) + 1$. Then by at most $n - 1$ Kelmans transformations as above, we get a graph G'' with the maximum degree $n - 1$, order n and size m. Moreover, G'' satisfies that all cycles are triangles. It is easy to see that the graph G'' is isomorphic to
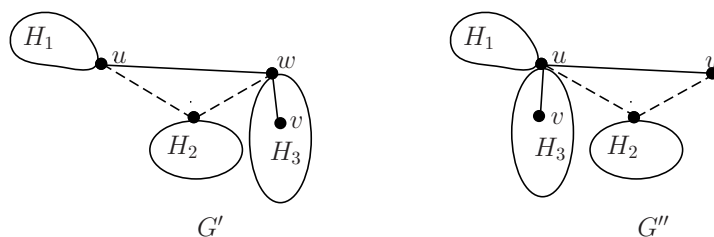
the graph $F(n,m)$; see Fig. 3.

Figure 3: $KT(G', u, w)$ in step 3.

The proof is thus complete.   □

3   Maximum matching root and Kelmans transformation

In this section, we will show that the maximum matching roots of the graph strictly increase after the Kelmans transformation. Before proceeding, we first review some useful lemmas of the matching polynomial.

Lemma 3.1. *([11]) Let edge $e = uv$. Then*

$$m(G, x) = m(G - e, x) - m(G - u - v, x) \tag{1}$$

Let G and H be two disjoint graphs. Then the graph $G \bigcup H$ denotes the union of G and H.

Lemma 3.2. *([11]) Let $G_1, \ldots, G_k$ be k disjoint graphs. Then*

$$m\left(\bigcup_{i=1}^{k} G_i, x\right) = \prod_{i=1}^{k} m(G_i, x).$$

In [12], Gutman showed some parallel results for the roots of the matching polynomial and the spectra of the characteristic polynomial.

Lemma 3.3. *([12]) If H is a subgraph of G, then $t(G) \geqslant t(H)$. If G is connected and H is a proper subgraph of G, then $t(G) > t(H)$.*

Now, we give some propositions of the maximum matching root under the Kelmans transformation.

Definition 3.4. *Define $G_1 \succeq G_2$ if for all $x \geqslant t(G_1)$ we have $m(G_2, x) \geqslant m(G_1, x)$. Define $G_1 \succ G_2$ if for all $x \geqslant t(G_1)$ we have $m(G_2, x) > m(G_1, x)$.*

It is easy to check that $G_1 \succ G_2$ can deduce $G_1 \succeq G_2$. Note that Csikvári [8] introduced the definition $\succ$ which has the same meaning as our definition $\succeq$, and they obtained many good results which inspire us. In the following, we obtain some results about $\succ$ and $\succeq$.

Before proceeding, we list some useful results about $\succeq$ which were proved in [8].

Proposition 3.5. *([8]) The relation $\succeq$ is transitive. If $G_1 \succeq G_2$, then $t(G_1) \geqslant t(G_2)$.*

Proposition 3.6. *([8]) If H is a spanning subgraph of G, then $G \succeq H$.*

Theorem 3.7. *([8]) Assume that G' is a graph obtained from G by a number Kelmans transformations, then $G' \succeq G$; in particular, $t(G') \geqslant t(G)$.*

Next, we give our results.

Proposition 3.8. *The relation $\succ$ is transitive. If $G_1 \succ G_2$, then $t(G_1) > t(G_2)$.*

Proof. Suppose $G_1 \succ G_2$. Since $t(G_1)$ is the maximum root of the matching polynomial, it implies that for $x \geqslant t(G_1)$, we have $m(G_2, x) > m(G_1, x) \geqslant 0$. Due to the fact that the leading coefficient of the matching polynomial is 1, we get $t(G_1) > t(G_2)$. If $G_1 \succ G_2 \succ G_3$, then $m(G_3, x) > m(G_2, x) > m(G_1, x)$ for $x \geqslant max\{t(G_1), t(G_2))\} = t(G_1)$. It follows that $G_1 \succ G_3$. $\square$

Proposition 3.9. *If $G_1 \succ G_2$ and $G_2 \succeq G_3$, then $G_1 \succ G_3$. If $G_1 \succeq G_2$ and $G_2 \succ G_3$, then $G_1 \succ G_3$.*

Proof. We only prove the first part, the second part can be proved similarly. For $x \geqslant t(G_1)$, we have $m(G_2, x) > m(G_1, x)$. For $x \geqslant t(G_2)$, we have $m(G_3, x) \geqslant m(G_2, x)$. By Propositions 3.5 and 3.8, it follows that $t(G_1) > t(G_2) \geqslant t(G_3)$. For $x \geqslant t(G_1)$, we get $m(G_3, x) > m(G_1, x)$, i.e. $G_1 \succ G_3$. $\square$

Proposition 3.10. *If G is connected and H is a proper spanning subgraph of G, then $G \succ H$.*

Proof. Since H is a proper spanning subgraph of G, suppose H is obtained by deleting l edges $\{e_1, \ldots, e_l\}$ from G. Then, $m(H, x) = m(G - e_1 - \cdots - e_l, x)$. Suppose e_1 is uv. By Lemma 3.1, we have

$$m(G, x) = m(G - e_1, x) - m(G - u - v, x), \tag{2}$$

Since G is connected, by Lemma 3.3, we have $t(G - u - v) < t(G)$ and $t(G - e_1) < t(G)$. Since the leading coefficient of the matching polynomial is 1, it follows that for $x \geqslant t(G)$, we get

$$m(G - e_1, x) - m(G, x) = m(G - u - v, x) > 0. \tag{3}$$

Thus we get $G \succ G - e_1$. Since H is a spanning subgraph of $G - e_1$, by Proposition 3.6, we have $G - e_1 \succeq H$. By Proposition 3.9, we deduce that $G \succ H$. $\square$

Proposition 3.11. *If H is a proper spanning subgraph of G, then for $x > t(G)$, we have $m(H, x) > m(G, x)$.*

Proof. If G is a connected graph, then by Proposition 3.10, we get $G \succ H$ and $m(G, x) > m(H, x)$ for $x > t(G)$. Next, we suppose that G has k connected components $G_1, \ldots, G_k$ where $k \geqslant 2$. Since H is an edge-deleted subgraph of G, let H be a union of k disjoint graphs $\{H_1, \ldots, H_k\}$, where H_i is the spanning subgraph of G_i. By Lemma 3.2, we know

$$m(G, x) = m\left(\bigcup_{i=1}^{k} G_i, x\right) = \prod_{i=1}^{k} m(G_i, x), \tag{4}$$

$$m(H, x) = m\left(\bigcup_{i=1}^{k} H_i, x\right) = \prod_{i=1}^{k} m(H_i, x). \tag{5}$$

Since H is the proper spanning subgraph of G, we can find that for some i, H_i is the proper spanning subgraph of G_i. Without loss of generality, assume that H_1 is the proper spanning subgraph of G_1. By Proposition 3.10, we have $G_1 \succ H_1$. Then, for $x \geqslant t(G_1)$, $m(H_1, x) > m(G_1, x) \geqslant 0$. For other subgraphs H_i where $i \geqslant 2$, by Proposition 3.6, we have $G_i \succeq H_i$. Then, for $x \geqslant t(G_i)$, $m(H_1, x) \geqslant m(G_1, x) \geqslant 0$. From Lemma 3.3, we know that $t(G) = \max\{t(G_1), \ldots, t(G_k)\}$. Thus, for $x > t(G)$, we get $m(H_1, x) > m(G_1, x) > 0$ and $m(H_i, x) \geqslant m(G_i, x) > 0$ for $2 \leqslant i \leqslant k$. Using Equations 4 and 5, we deduce that for $x > t(G)$,

$$m(H, x) = \prod_{i=1}^{k} m(H_i, x) > \prod_{i=1}^{k} m(G_i, x) = m(G, x).$$

The proof is now complete. $\qquad\square$

Theorem 3.12. *Let G be a connected graph. Assume that G' is obtained from G by a number of Kelmans transformations, and G' is not isomorphic to G. Then $G' \succ G$.*

Proof. Let G be a connected graph. Let G_1 be a graph obtained from G by the Kelmans transformation between u and v, where u is the beneficiary, and G_1 is not isomorphic to G. We just need to prove $G_1 \succ G$.

Firstly, we find that u has an adjacent vertex w which is not adjacent to v, and v has an adjacent vertex w' which is not in the set of neighbors of the vertex u. Otherwise, G_1 is isomorphic to G. This is a contradiction.

Secondly, we use Lemma 3.1 and get

$$m(G, x) = m(G - uw, x) - m(G - u - w, x),$$
$$m(G_1, x) = m(G_1 - uw, x) - m(G_1 - u - w, x).$$

Then we have the following equation

$$m(G, x) - m(G_1, x) = m(G - uw, x) - m(G_1 - uw, x) + m(G_1 - u - w, x) - m(G - u - w, x). \tag{6}$$

Next, we consider the right part of the above equation. If we apply the Kelmans transformation between u and v in $G - uw$, then we can get the graph $G_1 - uw$. Thus, by Theorem 3.7, we get $G_1 - uw \succeq G - uw$. That is,

$$m(G - uw, x) - m(G_1 - uw, x) \geqslant 0,$$

for $x \geqslant t(G_1 - uw)$. Since v has an adjacent vertex w' and uw' is not in $E(G)$, $G_1 - u - w$ is a proper spanning subgraph of $G - u - w$. By Proposition 3.11, we have

$$m(G_1 - u - w, x) - m(G - u - w, x) > 0,$$

for all $x > t(G - u - w)$. By Lemma 3.3, we find that $t(G_1) \geqslant t(G_1 - uw)$ and $G - u - w$ is isomorphic to a proper subgraph of G_1 by mapping v of $G - u - w$ to u of G_1. If G_1 is connected, then we have $t(G_1) > t(G - u - w)$ by Lemma 3.3. Otherwise, suppose that G is connected and G_1 is not connected. Then it follows that the set of neighbors of u and the set of neighbors of v are disjoint. Moreover, G_1 is the disjoint union of a isolated vertex v and a connected component H. Then, it implies that $G - u - w$ is isomorphic to a proper subgraph of H. Thus we have $t(G_1) = t(H) > t(G - u - w)$ from Lemma 3.3. We can conclude that $m(G - uw) - m(G_1 - uw) \geqslant 0$ and $m(G_1 - u - w, x) - m(G - u - w, x) > 0$ for $x \geqslant t(G_1)$. Finally, by Equation 6, we have $m(G, x) - m(G_1, x) > 0$ that is $G_1 \succ G$.

   Now we are ready to finish the proof of the theorem. Suppose that G' is obtained from G by k Kelmans transformations $\{KT(G, u, v), KT(G_1, u_1, v_1), \ldots, KT(G_{k-1}, u_{k-1}, v_{k-1})\}$ in order. From the above result, $G_1 \succ G$. Moreover, from Theorem 3.7, $G_1 \preceq G_2 \preceq G_3 \preceq \cdots \preceq G_{k-1} \preceq G'$. By Proposition 3.9, we get $G' \succ G$. The proof is thus complete. $\quad\square$

4   The maximum matching roots of odd-cycle graphs

In this section, we first prove a theorem about the maximum matching root of $F(n, m)$.

Theorem 4.1. *Given a positive integer n, for $n - 1 \leqslant m \leqslant \lfloor \frac{3(n-1)}{2} \rfloor - 1$, we have*

$$t(F(n, m)) < t(F(n, m + 1)).$$

Given a positive integer m with $m \geqslant 4$, for $n \leqslant m \leqslant \lfloor \frac{3(n-1)}{2} \rfloor$, we have

$$t(F(n, m)) < t(F(n + 1, m)).$$

Proof. For a positive integer n, when $n - 1 \leqslant m \leqslant \lfloor \frac{3(n-1)}{2} \rfloor - 1$, $F(n, m)$ is a proper spanning subgraph of $F(n, m + 1)$. Then, by Proposition 3.10, we have $t(F(n, m)) < t(F(n, m + 1))$.

   Given a positive integer m with $m > 4$, when $n \leqslant m \leqslant \lfloor \frac{3(n-1)}{2} \rfloor$, suppose that G' is the disjoint union of $F(n, m)$ and an isolate vertex named w. Since $n \leqslant m$, it follows that there exists a cycle in $F(n, m)$. Due to the definition of $F(n, m)$, the cycle in $F(n, m)$ is a triangle. Then there exists a triangle $\{v_1, v_2, v_3\}$ where v_1 has the maximum degree in $F(n, m)$. Up to isomorphism, the graph $F(n + 1, m)$ can be obtained from G' by deleting the edge $v_2 v_3$ and adding the edge $v_1 w$. By Lemma 3.1, we have

$$m(G', x) = m(G' - v_2 v_3, x) - m(G' - v_2 - v_3, x), \tag{7}$$
$$m(F(n + 1, m), x) = m(F(n + 1, m) - v_1 w, x) - m(F(n + 1, m) - v_1 - w, x). \tag{8}$$

Note that the graph $G' - v_2 v_3$ is isomorphic to $F(n+1, m) - v_1 w$. Then we get

$$m(G', x) - m(F(n+1, m), x) = m(F(n+1, m) - v_1 - w, x) - m(G' - v_2 - v_3, x). \quad (9)$$

Thanks to the structure of $F(n+1, m)$, the graph $F(n+1, m) - v_1 - w$ is the disjoint union of $m - n$ edges and $3n - 2m - 1$ isolated vertices. The graph $G' - v_2 - v_3$ is the disjoint union of the graph $F(n-2, m-3)$ and an isolated vertex w. By the definition of $F(n-2, m-3)$, we know that the graph $F(n+1, m) - v_1 - w$ is a subgraph of $G' - v_2 - v_3$. When $m > n$, $F(n+1, m) - v_1 - w$ is isomorphic to a proper spanning subgraph of $G' - v_2 - v_3$. When $m = n$, since $m \geqslant 4$, we deduce that $F(n+1, m) - v_1 - w$ is the disjoint union of $m - 1$ isolated vertices, and the graph $G' - v_2 - v_3$ is the disjoint union of the graph $F(m-2, m-3)$ and an isolated vertex w. Thus $F(n+1, m) - v_1 - w$ is isomorphic to a proper spanning subgraph of $G' - v_2 - v_3$.

Above all, by Proposition 3.11, for $x > t(G' - v_2 - v_3)$, we have $m(F(n+1, m) - v_1 - w, x) > m(G' - v_2 - v_3, x)$. Since $G' - v_2 - v_3$ is a proper subgraph of $F(n+1, m)$, by Lemma 3.3, we get $t(G' - v_2 - v_3) < t(F(n+1, m))$. Then for $x \geqslant t(F(n+1, m))$, we obtain $m(G', x) - m(F(n+1, m), x) > 0$, that is, $F(n+1, m) \succ G'$. Thus, we conclude that $t(F(n, m)) = t(G') < t(F(n+1, m))$. The proof is now complete. $\qquad \square$

Next, we give a proof of Conjecture 1.4 by proving the following theorem.

Theorem 4.2. *If G is an odd-cycle graph of order n, then $t(G) \leqslant t(H_n)$, and equality holds if and only if $G \cong H_n$.*

Proof. If G is disconnected, then there is a connected odd-cycle graph G' with order n which contains G as a proper subgraph. By Lemma 3.3, $t(G)$ is less than $t(G')$. Then, suppose that G is a connected odd-cycle graph with order n and size m.

By Kelmans transformations, we transfer the graph G into $F(n, m)$. By Theorem 3.12, we deduce that $t(G) \leqslant F(n, m)$ and equality holds if and only if $G \cong F(n, m)$. When $t < \lfloor \frac{3(n-1)}{2} \rfloor$, by Theorem 4.1 it follows that $t(F(n, t)) < t(F(n, \lfloor \frac{3(n-1)}{2} \rfloor)) = t(H_n)$. Then, we conclude that $t(G) \leqslant t(H_n)$ and equality holds if and only if $G \cong H_n$. The proof is thus complete. $\qquad \square$

Now we are ready to finish the proof of Theorem 1.5.

Proof of Theorem 1.5. For $m \leqslant 2$, it is easy to check that the first two parts of the theorem is true.

For $m = 3$, G is one of the following graphs: the disjoint union of three edges and $n - 6$ isolated vertices, the disjoint union of the star $K_{1,2}$, an edge and $n - 5$ isolated vertices, the disjoint union of the star $K_{1,3}$ and $n - 4$ isolated vertices, the disjoint union of a triangle and $n - 4$ isolated vertices. By some computations, it is easy to check that the third part of the theorem is true.

For $4 \leqslant m \leqslant n - 2$, suppose that G has s connected components $\{G_1, \cdots, G_s\}$ with order $\{n_1, \cdots, n_s\}$ and size $\{m_1, \cdots, m_s\}$, where $s \geqslant n - m$. Since every connected component is an odd-cycle graph, by the Kelmans transformations, we can obtained a

graph G' with s connected components $\{F(n_1, m_1), \cdots, F(n_s, m_s)\}$. Then by Lemma 3.2 and Lemma 3.12, we have

$$t(G) = \max\{t(G_1), \ldots, t(G_s)\} \leqslant \max\{t(F(n_1, m_1)), \cdots, t(F(n_s, m_s))\} = t(G').$$

In every nontrivial connected component $F(n_i, m_i)$, one of the vertices which have the maximum degree in $F(n_i, m_i)$ is named the root vertex of $F(n_i, m_i)$. By identifying all root vertices, we obtain a graph $F(n - s + 1, m)$. Let G'' be the graph obtained from the disjoint union of the graph $F(n - s + 1, m)$ and $s - 1$ isolated vertices.

If there are at least two nontrivial connected components, then every nontrivial connected component $F(n_i, m_i)$ is a proper subgraph of $F(n - s + 1, m)$. Then by Lemma 3.2 and Lemma 3.3, it implies that $t(G') < t(F(n - s + 1, m))$. It follows that

$$t(G) \leqslant t(G') < t(G'').$$

Next, by Theorem 4.1, for $s > n - m$ we have

$$t(F(n - s + 1, m)) < t(F(m + 1, m)).$$

Above all, together with Theorem 3.12, we conclude that $t(G) \leqslant t(F(m + 1, m))$ and the equality holds if and only if G has only one nontrivial connected component which is isomorphic to $F(m + 1, m)$. Thus, the fourth part of the theorem is true.

When $n - 1 \leqslant m \leqslant \lfloor \frac{3(n-1)}{2} \rfloor$ and $m \geqslant 4$, suppose that G is disconnected and has s connected components. Then by the similar method to the fourth part, we can deduce that the maximum matching root of G is smaller than the maximum matching root of $F(n - s + 1, m)$. By Theorem 4.1, for $s > 1$ we have

$$t(F(n - s + 1, m)) < t(F(n, m)).$$

Then, we conclude that $t(G) \leqslant t(F(n, m))$, and the equality holds if and only if G is connected and isomorphic to $F(n, m)$ from Theorem 3.12. Thus, the fifth part of the theorem is true.

The proof is thus complete. $\qquad\square$

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