Simplicial complexes of whisker type

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Abstract

Let $I \subset K[x_1, \ldots, x_n]$ be a zero-dimensional monomial ideal, and $\Delta(I)$ be the simplicial complex whose Stanley–Reisner ideal is the polarization of I. It follows from a result of Soleyman Jahan that $\Delta(I)$ is shellable. We give a new short proof of this fact by providing an explicit shelling. Moreover, we show that $\Delta(I)$ is even vertex decomposable. The ideal $L(I)$, which is defined to be the Stanley–Reisner ideal of the Alexander dual of $\Delta(I)$, has a linear resolution which is cellular and supported on a regular CW-complex. All powers of $L(I)$ have a linear resolution. We compute depth $L(I)^k$ and show that depth $L(I)^k = n$ for all $k \geqslant n$.

Keywords: depth function; linear quotients; vertex decomposable; whisker complexes; zero-dimensional ideals

1 Introduction

Graphs with whiskers have first been considered by Villarreal in [19]. They all share the nice property that they are Cohen-Macaulay. Various extensions of this concept and generalizations of his result have been considered in the literature, see [2, 8, 13, 18]. The edge ideal of a whisker graph is obtained as the polarization of a monomial ideal $I \subset S$,

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where $S = K[x_1, \ldots, x_n]$ is the polynomial ring over a field K, I is generated in degree 2 and $\dim S/I = 0$. In particular, I contains the squares $x_1^2, \ldots, x_n^2$. More generally, given a simplicial complex Γ, the whisker complex $W(\Gamma)$ is studied in [15]. Its facet ideal is the polarization of a monomial ideal in S which contains all the x_i^2. In [15], Loiskekoski shows that the Stanley–Reisner ideal of the Alexander dual of the independence complex of $W(\Gamma)$ has a linear resolution, as well as its powers.

In the present paper we generalize the above mentioned results by considering the polarization of any monomial ideal $I \subset S$ with $\dim S/I = 0$. The simplicial complex $\Theta(I)$, whose facet ideal coincides with the polarization $I^{\wp}$ of I, is called of *whisker type* – the whiskers being the simplices corresponding to the polarization of the pure powers contained in I. The independence complex of $\Theta(I)$, denoted $\Delta(I)$, is characterized by the property that the Stanley–Reisner ideal $I_{\Delta(I)}$ coincides with $I^{\wp}$. Note that $F \in \Delta(I)$ if and only if F does not contain any facet of $\Theta(I)$.

Given an arbitrary monomial ideal $I \subset S$, a multicomplex is associated with I, as defined by Popescu and the second author in [11]. Soleyman Jahan defines in [17, Proposition 3.8] a bijection between the facets of the multicomplex given by I and the facets of the simplicial complex associated with $I^{\wp}$. In Theorem 1 we present a short proof of this bijection when $\dim S/I = 0$, by using multiplicity theory. This result allows us to describe in Corollary 2 the facets of $\Delta(I)$. By applying the Eagon–Reiner Theorem it is then shown in Corollary 3 that the ideal $L(I)$ has a linear resolution, where $L(I)$ is generated by the monomials $x_{1,a_1+1} \cdots x_{n,a_n+1}$ for which $x_1^{a_1} \cdots x_n^{a_n}$ is a monomial in S not belonging to I.

In the case that $\dim S/I = 0$, the case we consider here, the corresponding multicomplex is pretty clean, see [11]. Soleyman Jahan showed in [17, Theorem 4.3] that if I defines a pretty clean multicomplex, then the simplicial complex associated with $I^{\wp}$ is clean, which, by a theorem of Dress [5], implies that the simplicial complex attached to $I^{\wp}$ is shellable. Applied to our situation it follows that $\Delta(I)$ is shellable. We give a direct proof of this fact by showing that $L(I)$ has linear quotients. This provides an explicit shelling of $\Delta(I)$, and as a side result we obtain a formula for the Betti numbers of $L(I)$ in terms of the h-vector of S/I, see Corollary 6. We conclude Section 2 with Corollary 8, where it is shown that the minimal graded free resolution of $L(I)$ is cellular and supported on a regular CW-complex. The proof is based on a result of Dochtermann and Mohammadi [4, Theorem 3.10], who showed that the minimal graded free resolution of any ideal with regular decomposition function, as defined in [12], have such nice cellular structure.

In Section 3 we show that $\Delta(I)$ is not only shellable but even vertex decomposable. This was already known for whisker graphs (see [3, Theorem 4.4]). Finally in Section 4 we prove that all powers of $L(I)$ have linear quotients, see Theorem 10. Analyzing the linear quotients, the depth function $f(k) = \operatorname{depth} S/L(I)^k$ can be computed. In Corollary 11 a formula for the depth function is given and $\lim_{k\to\infty} \operatorname{depth} S/L(I)^k$ is determined.

2 Independence complex of a whisker type simplicial complex

Throughout this paper S denotes the polynomial ring $K[x_1, \dots, x_n]$ and $I \subset S$ a monomial ideal with $\dim S/I = 0$, unless otherwise stated. The (finite) set of monomials in S which belong to S but not to I will be denoted by $\mathrm{Mon}(S \setminus I)$. For an arbitrary monomial ideal I, we denote by $G(I)$ the unique minimal set of monomial generators of I. We will consider the polarization of I, denoted $I^{\wp}$. The polynomial ring in which $I^{\wp}$ is defined will be denoted by $S^{\wp}$.

In the following theorem (cf. [17, Proposition 3.8]) we determine the set $\mathrm{Min}(I^{\wp})$ of minimal prime ideals of $I^{\wp}$.

Theorem 1. *Let $I \subset S$ be a monomial ideal with $\dim S/I = 0$. The map ϕ which assigns to each monomial $u = x_1^{a_1} \cdots x_n^{a_n} \in S \setminus I$ the monomial prime ideal $\phi(u) = (x_{1,a_1+1}, \dots, x_{n,a_n+1}) \subset S^{\wp}$, establishes a bijection between $\mathrm{Mon}(S \setminus I)$ and $\mathrm{Min}(I^{\wp})$.*

Proof. We first observe that $\phi(\mathrm{Mon}(S \setminus I)) \subset S^{\wp}$. Indeed, since $\dim S/I = 0$, there exists for each $1 \leqslant i \leqslant n$ an integer $b_i > 0$ such that $x_i^{b_i} \in I$ and $x_i^{b_i - 1} \notin I$. It follows that $S^{\wp}$ is the polynomial ring in the variables $x_{i,1}, \dots, x_{i,b_i}$ with $1 \leqslant i \leqslant n$. Now let $u = x_1^{a_1} \cdots x_n^{a_n} \in \mathrm{Mon}(S \setminus I)$. Then $a_i < b_i$ for all i, and this implies that $\phi(u) \in S^{\wp}$.

Next we show that $\phi(\mathrm{Mon}(S \setminus I)) \subset \mathrm{Min}(I^{\wp})$. In fact, let $u = x_1^{a_1} \cdots x_n^{a_n}$ be an element in $\mathrm{Mon}(S \setminus I)$, and let $v \in G(I)$. We claim that there exists an integer i such that x_{i,a_i+1} divides $v^{\wp}$, where $v^{\wp}$ is the polarization of v. From this claim it follows that $I^{\wp} \subset \phi(u)$. Since height $I^{\wp}$ = height $I = n$ and since height $\phi(u) = n$, we then see that $\phi(u)$ is in fact a minimal prime ideal of $I^{\wp}$.

Let $v = x_1^{b_1} \cdots x_n^{b_n}$. In order to prove the claim, note that $v^{\wp} = \prod_{i=1}^{n} (\prod_{j=1}^{b_i} x_{i,j})$. Since v does not divide u, there exists an integer i such that $b_i > a_i$. Therefore, x_{i,a_i+1} divides $v^{\wp}$, as desired.

Clearly, ϕ is injective. We will show that $|\mathrm{Mon}(S \setminus I)| = |\mathrm{Min}(I^{\wp})|$. This will then imply that $\phi : \mathrm{Mon}(S \setminus I) \to \mathrm{Min}(I^{\wp})$ is bijective. In order to see that these two sets have the same cardinality we observe that the multiplicity $e(S/I)$ of S/I is equal to the length $\ell(S/I)$ of S/I, because $\dim S/I = 0$, see [1, Corollary 4.7.11(b)]. Since $\ell(S/I) = \dim_K S/I$ and since the elements of $\mathrm{Mon}(S \setminus I)$ form a K-basis of S/I, we see that $e(S/I) = \dim_K S/I = |\mathrm{Mon}(S \setminus I)|$. On the other hand, since S/I is isomorphic to $S^{\wp}/I^{\wp}$ modulo a regular sequence of linear forms [9, Proposition 1.6.2], and since $S^{\wp}/I^{\wp}$ is reduced and equidimensional, [1, Corollary 4.7.8] implies that $e(S/I) = e(S^{\wp}/I^{\wp}) = |\mathrm{Min}(I^{\wp})|$. □

We denote by $\Delta(I)$ the simplicial complex whose Stanley-Reisner ideal is $I^{\wp}$. We view the variables $x_{i,j} \in S^{\wp}$ as the vertices of $\Delta(I)$. As an immediate consequence of Theorem 1 we obtain

Corollary 2. *Let $\mathcal{S}$ be the set of variables of $S^{\wp}$. Then $F \subset \mathcal{S}$ is a facet of $\Delta(I)$ if and only if there exists $x_1^{a_1} \cdots x_n^{a_n} \in \mathrm{Mon}(S \setminus I)$ such that*

$$F = \mathcal{S} \setminus \{x_{1,a_1+1}, \dots, x_{n,a_n+1}\}.$$

Since $\Delta(I)$ is Cohen–Macaulay, the Eagon–Reiner Theorem [6] (see also [9, Theorem 8.1.9]) implies that $I_{\Delta(I)^\vee}$ has a linear resolution. Here $\Delta(I)^\vee$ denotes the Alexander dual of $\Delta(I)$. Recall that, if Δ is an arbitrary simplicial complex on the vertex set $[n] = \{1, \ldots, n\}$ and $I_\Delta = \bigcap_F P_F$ where $P_F = (x_i : i \in F)$, then $I_{\Delta^\vee}$ is generated by the monomials u_F where $u_F = \prod_{i \in F} x_i$. These facts applied to our case yield

Corollary 3. *The ideal $L(I)$ generated by the monomials $x_{1,a_1+1} \cdots x_{n,a_n+1}$, with $x_1^{a_1} \cdots x_n^{a_n} \in \mathrm{Mon}(S \setminus I)$, has a linear resolution.*

In the following we consider the special case that $x_i^2 \in I$ for all i. In that case all other generators of I are square-free. In simplified notation, the polarization $I^\wp$ of I is generated by the square-free monomials in I and by the monomials $x_i y_i$ for $i = 1, \ldots, n$.

Let Γ be the simplicial complex with $I(\Gamma) = J$ and $W(\Gamma)$ be the simplicial complex with $I(W(\Gamma)) = (J, x_1 y_1, \ldots, x_n y_n)$. The edges of $W(\Gamma)$ corresponding to the $x_i y_i$ are called the *whiskers* of $W(\Gamma)$ and $W(\Gamma)$ is called the *whisker complex* of Γ.

Given a simplicial complex Σ, the independence complex Λ of Σ is the simplicial complex such that $I_\Lambda = I(\Sigma)$. Notice that $F \in \Lambda$ if and only if no face of Σ is contained in F.

Corollary 4. *Let Γ be a simplicial complex on the vertex set $[n]$, $I' = I(\Gamma)$ the facet ideal of Γ and $W(\Gamma)$ its whisker complex. Let $I = (I', x_1^2, x_2^2, \ldots, x_n^2)$. Then $\Delta(I)$ is the independence complex of $W(\Gamma)$ and $L(I)$ is generated by the monomials $\prod_{i \in [n] \setminus F} x_i \prod_{i \in F} y_i$ with $F \in \Delta$, where Δ is the independence complex of Γ.*

3 Linear quotients

Let $I \subset S$ be a monomial ideal with $\dim S/I = 0$. The main purpose of this section is to show that $L(I)$ not only has a linear resolution, but even has linear quotients.

Theorem 5. *The ideal $L(I)$ has linear quotients.*

Proof. Let $u, v \in G(L(I))$, $u = x_{1,a_1+1} \cdots x_{n,a_n+1}$ and $v = x_{1,b_1+1} \cdots x_{n,b_n+1}$. We set $u \leqslant v$ if $a_i \leqslant b_i$ for all i, and extend this partial order to a total order on $G(L(I))$. We claim that, with respect to this total order of the monomial generators of $L(I)$, the ideal $L(I)$ has linear quotients. Indeed, let $x_{1,a_1+1} \cdots x_{n,a_n+1}$ be the largest element in $G(L(I))$. Then $u = x_1^{a_1} \cdots x_n^{a_n} \in \mathrm{Mon}(S \setminus I)$ and $x_i u \in I$ for all i. Set $I' = I + (u)$. Then the polarization $(I')^\wp$ of I' is equal to $I_{\Delta(I')}$. Notice that $L(I') \subset L(I)$ and $\ell(S/I') < \ell(S/I)$. In particular, $L(I) = (L(I'), x_{1,a_1+1} \cdots x_{n,a_n+1})$. Arguing by induction on the length, we may assume that $L(I')$ has linear quotients. Thus we just need to compute the colon ideal $Q = L(I') : x_{1,a_1+1} \cdots x_{n,a_n+1}$. We claim that

$$Q = (x_{1,1}, x_{1,2}, \ldots, x_{1,a_1}, x_{2,1}, \ldots, x_{2,a_2}, \ldots, x_{n,1}, \ldots, x_{n,a_n}). \tag{1}$$

Suppose that $j \in \{1, \ldots, a_i\}$ for some i. Then $x_1^{a_1} \cdots x_i^{j-1} \cdots x_n^{a_n} \in \mathrm{Mon}(S \setminus I)$ and

$$\phi(x_1^{a_1} \cdots x_i^{j-1} \cdots x_n^{a_n}) = x_{1,a_1+1} \cdots x_{i,j} \ldots x_{n,a_n+1} \in L(I').$$

It follows that $x_{i,j} \in Q$.

On the other hand, the elements $v/\gcd(v, x_{1,a_1+1} \cdots x_{n,a_n+1})$ with $v \in G(L(I'))$ generate Q, see for example [9, Proposition 1.2.2]. In fact, let $v \in G(L(I'))$. Then $v = x_{1,c_1+1} \cdots x_{n,c_n+1}$ and $x_1^{c_1} \cdots x_n^{c_n} \in \mathrm{Mon}(S \setminus I')$. There exists i such that $c_i < a_i$ because $x_i u \in I$ for all i. Hence x_{i,c_i+1} does not divide $x_{1,a_1+1} \cdots x_{n,a_n+1}$, and therefore x_{i,c_i+1} divides $v/\gcd(v, x_{1,a_1+1} \cdots x_{n,a_n+1})$. Since $c_i + 1 \leqslant a_i$, the desired conclusion follows. □

Corollary 6. *For every $i \geqslant 0$,*

$$\beta_i(S^{\wp}/L(I)) = \sum_{j \geqslant 0} h_j \binom{j}{i-1},$$

where $h_j = h_j(S/I)$ is the j-th component of the h-vector of S/I. In particular, $\operatorname{proj\,dim} S^{\wp}/L(I) = \max\{\deg u : u \in \mathrm{Mon}(S \setminus I)\} + 1$.

Proof. As in the previous proof, let $u = x_1^{a_1} \cdots x_n^{a_n} \in \mathrm{Mon}(S \setminus I)$ with $x_i u \in I$ for all i. Set $I' = I + (u)$, and consider the short exact sequence

$$0 \to L(I)/L(I') \to S^{\wp}/L(I') \to S^{\wp}/L(I) \to 0.$$

Notice that $L(I)/L(I') \cong S^{\wp}/Q(-n)$ with Q as in (1). Hence its minimal free resolution is the Koszul complex $\mathbb{K}$ on the variables $x_{i,j}$ with $x_{i,j} \in G(Q)$. Thus the minimal free resolution of $S^{\wp}/L(I)$ can be obtained as a mapping cone of $\mathbb{K}$ and the minimal free resolution of $S^{\wp}/L(I')$. Therefore $\beta_0(S^{\wp}/L(I)) = \beta_0(S^{\wp}/L(I'))$, and for $i \geqslant 1$ we obtain

$$\begin{aligned} \beta_i(S^{\wp}/L(I)) &= \beta_i(S^{\wp}/L(I')) + \text{rank } (K_{i-1}) = \beta_i(S^{\wp}/L(I')) + \binom{\deg u}{i-1} \\ &= \sum_{u \in \mathrm{Mon}(S \setminus I)} \binom{\deg u}{i-1} = \sum_{j \geqslant 0} h_j \binom{j}{i-1}. \end{aligned}$$

□

It is easily seen that the geometric realization of $\Delta(I)$ is a sphere if I is a complete intersection, and a ball otherwise. Both topological spaces admit shellable triangulations, though in general not all triangulations of these spaces are shellable, see [16] and [14]. However, due to Theorem 5 we have

Corollary 7. *The simplicial complex $\Delta(I)$ is shellable.*

As a further consequence of Theorem 5 we have

Corollary 8. *The graded minimal free resolution of $L(I)$ is cellular and supported on a regular CW-complex.*

Proof. Since $L(I)$ has linear quotients we may apply [4, Theorem 3.10] and only need to show that $L(I)$ admits a regular decomposition function. In order to explain this, let $J = (u_1, \ldots, u_m)$ be an ideal with linear quotients with respect to the given order of the generators. The *decomposition function* of J (with respect to the given order of the generators of J) is the map $b : \mathrm{Mon}(J) \to G(J)$ with $b(u) = u_j$, where j is the smallest number such that $u \in (u_1, \ldots, u_j)$. For each $u_j \in G(J)$, let $\mathrm{set}(u_j)$ be the set of all x_i such that $x_i u_j \in (u_1, \ldots, u_{j-1})$. According to [12], the decomposition function b is called *regular*, if $\mathrm{set}(b(x_i u_j)) \subset \mathrm{set}(u_j)$ for all $u_j \in G(J)$ and all $x_i \in \mathrm{set}(u_j)$.

Now let $u \in G(L(I))$, $u = x_{1,a_1+1} \cdots x_{n,a_n+1}$. By (1) we have

$$\mathrm{set}(u) = \{x_{1,1}, x_{1,2}, \ldots, x_{1,a_1}, x_{2,1}, \ldots, x_{2,a_2}, \ldots, x_{n,1}, \ldots, x_{n,a_n}\}.$$

Let $x_{i,j} \in \mathrm{set}(u)$. Then $b(x_{i,j}u) = x_{i,j}(u/x_{i,a_i+1})$, and so

$$\mathrm{set}(b(x_{i,j}u)) = \mathrm{set}(u) \setminus \{x_{i,j+1}, \ldots, x_{i,a_i}\} \subset \mathrm{set}(u),$$

as desired. □

4 Vertex decomposability

In [3, Theorem 4.4] it was shown that for any graph, the independence complex of its whisker graph is vertex decomposable. Here we extend this result by showing that $\Delta(I)$ is vertex decomposable for any monomial ideal I with $\dim S/I = 0$. Recall that a simplicial complex Δ is called *vertex decomposable* if Δ is a simplex, or Δ contains a vertex v such that

(i) any facet of $\mathrm{del}_\Delta(v)$ is a facet of Δ, and

(ii) both $\mathrm{del}_\Delta(v)$ and $\mathrm{link}_\Delta(v)$ are vertex decomposable.

Here $\mathrm{link}_\Delta(v) = \{G \in \Delta : v \notin G \text{ and } G \cup \{v\} \in \Delta\}$ is the *link* of v in Δ and $\mathrm{del}_\Delta(v) = \{G \in \Delta : v \notin G\}$ is the *deletion* of v from Δ.

A vertex v which satisfies condition (i) is called a *shedding vertex* of Δ.

For the proof of the next result we observe the following fact: let Δ be a simplicial complex, $\mathcal{F}(\Delta)$ the set of its facets and v a vertex not belonging to Δ. The *cone* of v over Δ, denoted by $v * \Delta$, is the simplicial complex whose set of facets is $\mathcal{F}(v * \Delta) = \{\{v\} \cup F : F \in \mathcal{F}(\Delta)\}$. If Δ is vertex decomposable, then $v * \Delta$ is again vertex decomposable (with respect to the same shedding vertex).

Theorem 9. *Let I be a monomial ideal in $S = K[x_1, \ldots, x_n]$ with $\dim S/I = 0$. Then $\Delta(I)$ is vertex decomposable.*

Proof. By assumption, for each $1 \leqslant i \leqslant n$ there exists $b_i \geqslant 1$ such that $x_i^{b_i} \in G(I)$. Then $\Delta(I)$ is a simplicial complex on $\mathcal{S} = \{x_{1,1}, \ldots, x_{1,b_1}, \ldots, x_{n,1}, \ldots, x_{n,b_n}\}$. We proceed by induction on $\sum_{i=1}^n b_i$. If $\sum_{i=1}^n b_i = n$, then $I = (x_1, \ldots, x_n)$, which is a trivial case. Suppose that $\sum_{i=1}^n b_i > n$. Hence we may assume $b_n > 1$.

We first show that the vertex $x_{n,1}$ is a shedding vertex of $\Delta(I)$. Clearly,

$$\mathrm{del}_{\Delta(I)}(x_{n,1}) = \{F : F \in \Delta(I), x_{n,1} \notin F\} \cup \{F \setminus \{x_{n,1}\} : F \in \Delta(I), x_{n,1} \in F\}.$$

Obviously, any facet of $\mathrm{del}_{\Delta(I)}(x_{n,1})$ with $x_{n,1} \notin F$ is a facet of $\Delta(I)$. On the other hand, if we consider $F \setminus \{x_{n,1}\}$ with $F \in \mathcal{F}(\Delta(I))$ and $x_{n,1} \in F$, then $F \setminus \{x_{n,1}\}$ is not a facet of $\mathrm{del}_{\Delta(I)}(x_{n,1})$. Indeed, since $F \in \mathcal{F}(\Delta(I))$, there exists $u \in \mathrm{Mon}(S \setminus I)$ such that $\phi(u) = P_{\mathcal{S}\setminus F}$. Let t be the largest integer such that x_n^t divides u. Then $x_{n,t+1} \in P_{\mathcal{S}\setminus F}$ and so $x_{n,j} \in F$ for all $j \neq t+1$. Since $x_{n,1} \in F$, we have $t+1 \neq 1$. Let $u' = u/x_n^t$. Then $u' \in \mathrm{Mon}(S \setminus I)$ and $\phi(u') = P_{((\mathcal{S}\setminus F)\setminus\{x_{n,t+1}\})\cup\{x_{n,1}\}}$. Thus $G = (F \setminus \{x_{n,1}\}) \cup \{x_{n,t+1}\} \in \mathcal{F}(\Delta(I))$. Since $G \in \mathrm{del}_{\Delta(I)}(x_{n,1})$, the claim follows. Consequently, $\mathcal{F}(\mathrm{del}_{\Delta(I)}(x_{n,1})) = \{F : F \in \mathcal{F}(\Delta(I)), x_{n,1} \notin F\}$ which implies that $x_{n,1}$ is a shedding vertex of $\Delta(I)$.

We now prove that $\mathrm{del}_{\Delta(I)}(x_{n,1})$ and $\mathrm{link}_{\Delta(I)}(x_{n,1})$ are vertex decomposable.

First we consider $\mathrm{del}_{\Delta(I)}(x_{n,1})$. Let J_1 be the ideal in S with $\mathrm{Mon}(S \setminus J_1) = \{u : u \in \mathrm{Mon}(S \setminus I), x_n \text{ does not divide } u\}$. Then $\Delta(J_1)$ is a simplicial complex on $\mathcal{S} \setminus \{x_{n,1}, \ldots, x_{n,b_n}\}$. By using Corollary 2 we see that

$$\mathrm{del}_{\Delta(I)}(x_{n,1}) = x_{n,b_n} * (x_{n,b_n-1} * (\cdots * (x_{n,2} * \Delta(J_1)))).$$

Our induction hypothesis implies that $\Delta(J_1)$ is vertex decomposable, hence $\mathrm{del}_{\Delta(I)}(x_{n,1})$ is vertex decomposable.

As for $\mathrm{link}_{\Delta(I)}(x_{n,1})$, let Γ be the simplicial complex whose faces are obtained from the faces of $\mathrm{link}_{\Delta(I)}(x_{n,1})$ as follows: for every $F \in \mathrm{link}_{\Delta(I)}(x_{n,1})$, we replace each $x_{n,j} \in F$ by $x_{n,j-1}$. Hence Γ is a simplicial complex on $\mathcal{S} \setminus \{x_{n,b_n}\}$ and $\Gamma \cong \mathrm{link}_{\Delta(I)}(x_{n,1})$. Let J_2 be the monomial ideal in S such that $\mathrm{Mon}(S \setminus J_2) = \{u/x_n : u \in \mathrm{Mon}(S \setminus I), x_n \text{ divides } u\}$. Then Corollary 2 implies that $\Gamma = \Delta(J_2)$, which is vertex decomposable by induction hypothesis. It follows that $\mathrm{link}_{\Delta(I)}(x_{n,1})$ is vertex decomposable, as desired. □

5 Powers

In this section we study the powers of $L(I)$. The main result is

Theorem 10. *Let $I \subset S$ be a monomial ideal with $\dim S/I = 0$. Then $L(I)^k$ has linear quotients for all k. In particular, all powers of $L(I)$ have a linear resolution.*

Proof. Any $u \in L(I)^k$ can be written in the form $u = u_1'u_2'\cdots u_n'$, where $u_i' = x_{i,j(i)_1}x_{i,j(i)_2}\cdots x_{i,j(i)_k}$ for $i = 1, \ldots, n$ with $j(i)_1 \leqslant j(i)_2 \leqslant \cdots \leqslant j(i)_k$. We define a partial order on $G(L(I)^k)$ by setting $v \leqslant u$, if, with respect to the lexicographical order, $u_i' \leqslant v_i'$ for all i, and we extend this partial order to a total order on the set of monomial generators of $L(I)^k$.

Now let $v, u \in L(I)^k$ with $v < u$. We need to show that there exists $w \in L(I)^k$ with $w < u$ such that $w/\gcd(w,u)$ is of degree 1 and such that $w/\gcd(w,u)$ divides

$v/\gcd(v,u)$. Indeed, since $v < u$, there exists i such that $u'_i < v'_i$ in the lexicographical order. Thus if $v'_i = x_{i,j'(i)_1} x_{i,j'(i)_2} \cdots x_{i,j'(i)_k}$ with $j'(i)_1 \leqslant j'(i)_2 \leqslant \cdots \leqslant j'(i)_k$, then there exists ℓ such that $j'(i)_s = j(i)_s$ for $s < \ell$ and $j'(i)_\ell < j(i)_\ell$. We let $w = w'_1 w'_2 \cdots w'_n$ with $w'_t = u'_t$ for $t \neq i$ and $w'_i = x_{i,j'(i)_\ell}(u'_i/x_{i,j(i)_\ell})$.

It is clear that $w < u$. Furthermore, $w \in G(L(I)^k)$. In fact, $u = u_1 \cdots u_k$ with $u_i \in L(I)$, and $x_{i,j(i)_\ell}$ divides one of these factors, say it divides u_r. Then $\bar{u}_r = x_{i,j'(i)_\ell}(u_r/x_{i,j(i)_\ell})$ belongs to $L(I)$ since $j'(i)_\ell < j(i)_\ell$, and hence $w = u_1 \cdots \bar{u}_r \cdots u_k$ belongs to $G(L(I)^k)$. Note further that $w/\gcd(w,u) = x_{i,j'(i)_\ell}$ and that $x_{i,j'(i)_\ell}$ divides $v/\gcd(v,u)$. This completes the proof. ☐

Corollary 11. *For $i = 1, \ldots, n$, let b_i be the smallest integer such that $x_i^{b_i} \in I$. Then*

$$\operatorname{depth} S^{\wp}/L(I)^k = \sum_{i=1}^{n} b_i - \max\{\deg(\operatorname{lcm}(u_1, \ldots, u_k)) : u_1, \ldots, u_k \in \operatorname{Mon}(S \setminus I)\} - 1.$$

In particular, $\operatorname{depth} S^{\wp}/L(I)^k = n - 1$ *for all $k \geqslant n$, and*

$$\operatorname{depth} S^{\wp}/L(I)^k > \operatorname{depth} S^{\wp}/L(I)^{k+1},$$

as long as $\operatorname{depth} S^{\wp}/L(I)^k > n - 1$.

Proof. In general, let $J \subset K[x_1, \ldots, x_n]$ be a graded ideal generated by a sequence $f_1, \ldots, f_s$ with linear quotients, and denote by $q_j(J)$ the minimal number of linear forms generating the ideal $(f_1, f_2, \ldots, f_{j-1}) : f_j$. Then $\operatorname{depth} K[x_1, \ldots, x_n]/J = n - q(J) - 1$, where $q(J) = \max\{q_j(J) : 2 \leqslant j \leqslant s\}$, see [10, Formula (1)].

We apply this formula to $S^{\wp}/L(I)^k$. Since the Krull dimension of $S^{\wp}$ is equal to $\sum_{i=1}^n b_i$, it remains to be shown that

$$q(L(I)^k) = \max\{\deg(\operatorname{lcm}(u_1, u_2, \ldots, u_k)) : u_1, u_2, \ldots, u_k \in \operatorname{Mon}(S \setminus I)\}. \quad (2)$$

To see this, let $u_1, u_2, \ldots, u_k \in \operatorname{Mon}(S \setminus I)$ where $u_j = x_1^{a_1(j)} \cdots x_n^{a_n(j)}$ for $j = 1, \ldots, k$. Then $u = u'_1 u'_2 \cdots u'_n$ with $u'_i = \prod_{j=1}^k x_{i,a_i(j)+1}$ is a generator of $L(I)^k$. We may assume that u is the j-th element in the given total order of the elements of $G(L(I)^k)$. As shown in the proof of Theorem 10, $q_j(L(I)^k)$ is the cardinality of the set

$$\{x_{1,1}, \ldots, x_{1,c_1}, x_{2,1}, \ldots, x_{2,c_2}, \ldots, x_{n,1}, \ldots, x_{n,c_n}\},$$

where $c_i = \max\{a_i(1), \ldots, a_i(k)\}$ for $i = 1, \ldots, n$. If follows that

$$q_j(L(I)^k) = \deg(\operatorname{lcm}(u_1, u_2, \ldots, u_k)),$$

and hence equation (2) follows.

Suppose now that $k \geqslant n$. Then we may choose $u_i = x_i^{b_i - 1}$ for $i = 1, \ldots, n$ and $u_i \in \operatorname{Mon}(S \setminus I)$ arbitrary for $i > n$, and obtain $\deg(\operatorname{lcm}(u_1, u_2, \ldots u_k)) = \sum_{i=1}^n (b_i - 1) = \sum_{i=1}^n b_i - n$. Since this is the largest possible least common multiple of sequences of elements of $\operatorname{Mon}(S \setminus I)$, it follows that $\operatorname{depth} S^{\wp}/L(I)^k = n - 1$ for all $k \geqslant n$.

Finally, suppose that depth $S^{\wp}/L(I)^k > n-1$. Then the formula for depth $S^{\wp}/L(I)^k$ implies that $\max\{\deg(\mathrm{lcm}(u_1,\ldots,u_k)) : u_1,\ldots,u_k \in \mathrm{Mon}(S \setminus I)\} < \sum_{i=1}^{n}(b_i - 1)$.

Let $x_1^{a_1}\cdots x_n^{a_n} = \mathrm{lcm}(u_1,\ldots,u_k)$ attain this maximal degree. Since $\sum_{i=1}^{n} a_i < \sum_{i=1}^{n}(b_i-1)$, there exists an index i such that $a_i < b_i - 1$. Let $u_{k+1} = x_i^{b_i-1}$. Then $\deg(\mathrm{lcm}(u_1,\ldots,u_k,u_{k+1})) > \deg(\mathrm{lcm}(u_1,\ldots,u_k))$. Consequently, depth $S^{\wp}/L(I)^k >$ depth $S^{\wp}/L(I)^{k+1}$, as desired. □

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