Vertex-Transitive Digraphs of Order p^5 are Hamiltonian

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Abstract

We prove that connected vertex-transitive digraphs of order p^5 (where p is a prime) are Hamiltonian, and a connected digraph whose automorphism group contains a finite vertex-transitive subgroup G of prime power order such that G' is generated by two elements or elementary abelian is Hamiltonian.

Keywords: vertex-transitive digraphs; Hamilton cycles; coset digraphs

1 Introduction

One of the most famous problems in vertex-transitive graphs theory is the problem of existence of Hamilton paths/cycles (that is, simple paths/cycles going through all vertices) in finite connected vertex-transitive graphs (or digraphs). Graphs (or digraphs) which have Hamilton cycles are called Hamiltonian. The interest in this problem grew out of a question posed byLovász [13], who asked whether every finite connected vertex-transitive graph has a Hamilton path. In fact, there are only four known nontrivial connected vertex-transitive graph that do not possess Hamilton cycles. These four graphs are the Petersen graph, the Coxeter graph and the two graphs obtained from them by replacing each vertex by a triangle. The fact that none of these four graphs is a Cayley graph has led to a folklore conjecture that every connected Cayley graph with order greater than 2 has a Hamilton cycle. A large number of articles directly or indirectly related to this problem (for the list of relevant references and a detailed description of the status of this problem see [8]), have appeared in the literature, affirming the existence of such paths in some special vertex-transitive graphs and, in some cases, also the existence of Hamilton

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cycles. Since the publication of the survey paper [8], some recent improvements on this subject appeared, see [23, 4, 9, 6, 5, 22] and so forth.

Let p be a prime number. It is known that connected vertex-transitive graphs of order kp, where $k \leqslant 4$, and p^i, where $i \leqslant 4$, and $2p^2$ (except for the Petersen graph and the Coxeter graph) contain a Hamilton cycle; see [1, 17, 7, 14, 3, 15]. A Hamilton path is known to exist in connected vertex-transitive graphs of order $5p$, $6p$ and order $10p$ removing some special cases; see [16, 11, 10]. As for Cayley graphs, perhaps the biggest achievement on this subject is due to Witte (now Morris) who proved that a connected Cayley digraph of any p-group has a Hamilton cycle [21].

It seems to be quite a challenge to generalize Witte's theorem on Hamilton cycles in Cayley digraphs of p-groups to arbitrary vertex-transitive digraphs of prime power order. The first successful attempt using this approach is due to Chen [3], who proved that vertex-transitive digraphs of order p^4 are Hamiltonian. In this paper, we give some conditions under which one can obtain Hamilton cycles of vertex-transitive digraphs of prime power order by lifting Hamilton cycles from their quotient graphs. Using our methods, one can affirm the existence of Hamilton cycles of many connected vertex-transitive digraphs of prime power order. In particular, we obtain the following two theorems:

Theorem 1.1. *Connected vertex-transitive digraphs of order p^5 are Hamiltonian.*

Theorem 1.2. *Let Γ be a connected digraph of which the automorphism group contains a finite vertex-transitive subgroup G of prime power order. Let G' be the derived subgroup of G. Then Γ is Hamiltonian if one of the following two conditions hold:*

(i) *G' is generated by two elements;*

(ii) *G' is elementary abelian.*

The paper is organized as follows. In Section 2, notations and lemmas in group theory for later use are introduced. In section 3, we review the concept of coset digraphs and the representations of paths and cycles in coset digraphs. In section 4, the main theorems of this paper are proved.

2 Notations and preliminary lemmas in group theory

In this section, we fix some notations and introduce some lemmas for later use. The following standard group-theoretic notations will be used throughout the rest of the paper.

$H \leqslant G,\ H < G$	H is a subgroup, a proper subgroup of the group G
$H \lhd G$	H is a normal subgroup of G
$\langle X \rangle$	Subgroup generated by the subset X of a group
$[x, y]$	$x^{-1}y^{-1}xy$, commutator of two elements x and y of a group
$[X, Y]$	Subgroup of G generated by all the commutators $[x, y]$

	where $x \in X$, $y \in Y$ and $X, Y \subseteq G$
G'	$[G, G]$, derived subgroup of G
$C_G(g)$	Centralizer of g in G
G/H	$\{gH \mid g \in G\}$, set of left cosets of H in G, and particularly the quotient group of N in G if $N \lhd G$
$\lvert G : H \rvert$	Index of the subgroup H in the group G
$H\backslash G/H$	$\{HgH \mid g \in G\}$, set of double cosets of H in G
H^G	$\langle gHg^{-1} \mid g \in G\rangle$, normal closure of H in G
H_G	$\bigcap_{g\in G} gHg^{-1}$, core of H in G
$Z(G)$	Center of G
$\Phi(G)$	Frattini subgroup of G

Remark. The Frattini subgroup $\Phi(G)$ of G is defined to be the intersection of all maximum subgroups of G. An element g of G is said to be a non-generator if $G = \langle X\rangle$ whenever $G = \langle g, X\rangle$, where X is a subset of G. It is well known that $\Phi(G)$ is the set of non-generators of G.

Below we introduce four lemmas that will be needed in the proof of our main results.

Lemma 2.1 ([19])**.** *Let G be a finite p-group and N be a normal subgroup of G. Then $N \cap Z(G) = \{1\}$ if and only if $N = \{1\}$.*

Lemma 2.2. *Let G be a finite p-group and w be an element of G with centralizer $C_G(w) < G$. Let X be a generating set of G. Then $[G, \langle w\rangle] = [G, w]$. Furthermore, there exists a minimal generating set $\{[x, w], [g_1, w], \ldots, [g_{d-1}, w]\}$ of $[G, w]$ where $x \in X$ and $g_i \in G - [G, w]C_G(w)$ for all $i = 1, \ldots, d - 1$.*

Proof. For any $g \in G$ and any $i > 1$, since

$$[g, w^i] = [g, w][g, w^{i-1}]^w = [g, w][g^w, w^{i-1}] = [g, w][g, w^{i-1}][[g, w^{i-1}], w]$$

we have $[g, w^i] = [g, w][g^w, w] \cdots [g^{w^{i-1}}, w]$ and $[[g, w], w] = [g, w]^{-1}[g^w, w]$. It follows that

$$[G, \langle w\rangle] = [G, w] = \langle [g, w] \mid g \in G - [G, w]C_G(w)\rangle. \tag{1}$$

Set $K = [G, w]$. Then $K \lhd G$. Since $\Phi(K)$ is a characteristic subgroup of K, we have $\Phi(K) \lhd G$. From $G = \langle X\rangle$, we have $[G, w] = \langle [x, w]^g \mid x \in X, g \in G\rangle$. Then, since $\Phi(K)$ is a proper subgroup of K, there exists $x \in X$ such that $[x, w] \notin \Phi(K)$. Let $|K : \Phi(K)| = d$. Then, by the Basis theorem of Burnside (see [19, Theorem 1.16 of Chapter 2] for example), there exists a minimal generating set $\{[x, w], y_1, \ldots, y_{d-1}\}$ of K. Furthermore, by (1), we can choose $y_i = [g_i, w]$ with $g_i \in G - [G, w]C_G(w)$ for all $i = 1, \ldots, d - 1$. ☐

Lemma 2.3. *Let G be a finite p-group and H be a proper subgroup of G. If $H \cap Z(G) = \{1\}$, then $Z(H)$ contains an element w of order p such that $[G, w] \cap H = \{1\}$.*

Proof. Set $H_0 = H$ and $H_i = [G, H_{i-1}]$ for all $i \geqslant 1$. Then there is a positive integer j such that $\{1\} < H_j \leqslant Z(G)$. Since $H \cap Z(G) = \{1\}$, we have $H \cap H_j = \{1\}$. Let k be the minimum positive integer such that $H \cap H_k = \{1\}$. Then $H \cap H_{k-1} > \{1\}$. Take an element w of $H \cap H_{k-1}$ with order p. Then $[G, w] \cap H = \{1\}$. Sine $[H, w] \leqslant [G, w] \cap H$, we have $[H, w] = \{1\}$, namely $w \in Z(H)$. $\square$

The following lemma is a direct corollary of [2, Theorem 1].

Lemma 2.4. *Let G be a finite p-group for which G' is generated by two elements. Then any subgroup of G' can be generated by at most two elements.*

3 Coset digraphs and cycles

The digraphs considered in this paper are finite, connected, with no loops or multiple edges. For a vertex-transitive digraphs Γ, the following proposition gives a nice way to represent it by using subgroups of its automorphism group. For proof and comments of this proposition, see [12] and [3] respectively.

Proposition 3.1. *Let G be a finite group, H a subgroup of G, and $\Omega \subseteq H\backslash G/H - \{H\}$. Define a digraph $\Gamma = \mathrm{Cos}(G, H, \Omega)$ as follows: the vertices set of Γ is G/H; the arcs set of Γ is $\{(g_1H, g_2H) \mid Hg_1^{-1}g_2H \in \Omega\}$. Then we have*

(i) *Γ is a well defined vertex-transitive digraph on which G/H_G acts vertex-transitively by left multiplication: $gH_GxH = gxH$ for any $gH_G \in G/H_G$ and $xH \in G/H$,*

(ii) *every vertex-transitive digraph can be represented as $\mathrm{Cos}(G, H, \Omega)$ for some G, H, Ω,*

(iii) *$\mathrm{Cos}(G, H, \Omega)$ is connected if and only if $G = \left\langle \bigcup_{HgH\in\Omega} HgH \right\rangle$.*

As for the vertex-transitive digraph of prime power order, we have the following proposition.

Proposition 3.2 ([3])**.** *If Γ is a vertex-transitive digraph of order p^n where p is a prime and n is a positive integer, then Γ admits a representation $\mathrm{Cos}(G, H, \Omega)$ such that G is a p-group, $H \bigcap Z(G) = \{1\}$ and $H < \Phi(G)$.*

Remark. By Lemma 2.1, the subgroup H of G in Proposition 3.2 is core-free, that is, $H_G = \{1\}$. Therefore, G acts vertex-transitively on Γ by left multiplication. In fact, G can be chosen as a minimum vertex-transitive p-subgroup of $\mathrm{Aut}(\Gamma)$.

The digraph $\Gamma = \mathrm{Cos}(G, H, \Omega)$ defined in Proposition 3.1 is usually called a *coset digraph* on G/H, which is actually a *generalized orbital graph* of G acting on G/H (the definition of generalized orbital graph can be found in many publications, see [18] for example). Particularly, if $H = \{1\}$, then $\Gamma = \mathrm{Cos}(G, H, \Omega)$ is a Cayley digraph and denoted by $\mathrm{Cay}(G, \Omega)$. Consider the action of G/H_G on G/H by left multiplication. If K

is a subgroup of G which contains H, then K/H is coincident with a block for G/H_G. It follows that K induces a quotient digraph Γ_K of Γ: the vertices set of Γ_K is the system of blocks containing K/H, and for any two such blocks Δ_1 and Δ_2, (Δ_1, Δ_2) is an arc of Γ_K if and only if there exist $g_1H \in \Delta_1$ and $g_2H \in \Delta_2$ such that (g_1H, g_2H) is an arc of Γ. The following proposition gives a representation of Γ_K.

Proposition 3.3. *Let G be a finite group, H a subgroup of G, and $\Gamma = \mathrm{Cos}(G, H, \Omega)$ a coset digraph on G/H. Let K be a subgroup of G which contains H, and Γ_K the quotient digraph of Γ induced by K. Then $\Gamma_K \cong \mathrm{Cos}(G, K, \Lambda)$ where $\Lambda = \{KxK \mid HxH \in \Omega\}$.*

Proof. Set $gK/H = \{gxH \mid x \in K\}$ for all $g \in G$. Then the vertices set of Γ_K is $\{gK/H \mid g \in G\}$. Since

$$g_1K = g_2K \Leftrightarrow g_1^{-1}g_2 \in K \Leftrightarrow g_1K/H = g_2K/H$$

for all $g_1, g_2 \in G$, we obtain a one to one mapping

$$\sigma: \ G/K \to \{gK/H \mid g \in G\}, \ gK \mapsto gK/H \text{ for all } gK \in G/K.$$

If (g_1K, g_2K) is an arc of $\mathrm{Cos}(G, K, \Lambda)$, then $Kg_1^{-1}g_2K \in \Lambda$. Therefore there exist $x_1, x_2 \in K$ such that $Hx_1g_1^{-1}g_2x_2H \in \Omega$. It follows that $(g_1x_1^{-1}H, g_2x_2H)$ is an arc of Γ, and then $(g_1K/H, g_2K/H)$ is an arc of Γ_K.

On the other hand, if $(g_1K/H, g_2K/H)$ is an arc of Γ_K, then there exist $x_1, x_2 \in K$ such that (g_1x_1H, g_2x_2H) is an arc of Γ. It follows that $Hx_1^{-1}g_1^{-1}g_2x_2H \in \Omega$ and then $Kg_1^{-1}g_2K \in \Lambda$. Therefore (g_1K, g_2K) is an arc of $\mathrm{Cos}(G, K, \Lambda)$.

By the above discussions, we get $\Gamma_K \cong \mathrm{Cos}(G, K, \Lambda)$. ☐

For a finite group G and a subgroup H of G, we use $(x_1, x_2, \ldots, x_n) \cdot H$ to denote the sequence of cosets

$$H, \, x_1H, \, x_1x_2H, \ldots, x_1x_2 \cdots x_nH$$

in G/H, and we call $(x_i, \ldots, x_j) \cdot H$ a *section* of $(x_1, x_2, \ldots, x_n) \cdot H$ for any $1 \leqslant i \leqslant j \leqslant n$. For two sequences $(x_1, x_2, \ldots, x_n) \cdot H$ and $(y_1, y_2, \ldots, y_m) \cdot H$, define

$$(x_1, x_2, \ldots, x_n)(x_{i+1}, \ldots, x_n)^{-1} \cdot H = (x_1, x_2, \ldots, x_i) \cdot H \quad \text{for any } 1 \leqslant i < n$$

and

$$(x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_m) \cdot H = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) \cdot H.$$

In particular, if $\Gamma = \mathrm{Cos}(G, H, \Omega)$ is a coset digraph on G/H and $Hx_iH \in \Omega$ for all $i = 1, 2, \ldots, n$, then $(x_1, x_2, \ldots, x_n) \cdot H$ is a walk visiting the vertices of Γ in the order

$$H, \, x_1H, \, x_1x_2H, \ldots, x_1x_2 \cdots x_nH.$$

The following proposition is apparent and we omit the proof.

Proposition 3.4. *Let $(x_1, x_2, \ldots, x_n) \cdot H$ be a cycle in the coset digraph $\mathrm{Cos}(G, H, \Omega)$, and let h, h' be two elements of H. Then,*

(i) $(hx_1, x_2, \ldots, x_n h') \cdot H$ *is a cycle in* $\mathrm{Cos}(G, H, \Omega)$.

(ii) $(x_1, \ldots, x_{i-1}h^{-1}, hx_i, x_{i+1}, \ldots, x_n) \cdot H$ *is a cycle in* $\mathrm{Cos}(G, H, \Omega)$ *for any* $1 \leqslant i \leqslant n$.

(iii) $(x_1, \ldots, x_{i-1}, x_i h, x_{i+1}, \ldots, x_n) \cdot H$ *is a cycle in* $\mathrm{Cos}(G, H, \Omega)$ *for any* $1 \leqslant i < n$ *if* $h \in H_G$.

(iv) $(x_{i+1}, \ldots, x_n, x_1, \ldots, x_i) \cdot H$ *is a cycle in* $\mathrm{Cos}(G, H, \Omega)$ *if* $x_1 x_2 \cdots x_n = 1$.

Now we give a lemma for later use.

Lemma 3.5 ([3]). *Let $\mathrm{Cos}(G, H, \Omega)$ be a vertex-transitive digraph and N be a normal subgroup of G with $N \bigcap H = \{1\}$. Set $K = NH$ and $\Lambda = \{KxK \mid HxH \in \Omega\}$. Suppose there are m directed Hamilton cycles $(x_1, \ldots, x_{n-1}, y_1) \cdot H, \ldots, (x_1, \ldots, x_{n-1}, y_m) \cdot H$ in $\mathrm{Cos}(G, K, \Lambda)$ such that $Hx_iH, Hy_jH \in \Omega$ for $i = 1, \ldots, n-1$ and $j = 1, \ldots, m$, and $K = \langle S \rangle$ where $S = \{x_1 \ldots x_{n-1} y_j \mid j = 1, \ldots, m\}$. Then $\mathrm{Cos}(G, H, \Omega)$ is Hamiltonian if the Cayley digraph $\mathrm{Cay}(K, S)$ is.*

4 Main results

Throughout this section, we assume p is a prime. Let Γ be a connected vertex-transitive digraph of order a power of p. Then, by Proposition 3.2, Γ is isomorphic to a coset digraph $\mathrm{Cos}(G, H, \Omega)$ on G/H where G is a p-group, $H \cap Z(G) = \{1\}$ and $H < \Phi(G)$. By Lemma 2.3, $Z(H)$ contains an element w of order p such that $[G, w] \cap H = \{1\}$. Since $[G, w] \lhd G$, $K := [G, w]H$ is a subgroup of G. By Proposition 3.3, K induces a quotient digraph of $\mathrm{Cos}(G, H, \Omega)$ which is isomorphic to the coset digraph $\mathrm{Cos}(G, K, \Lambda)$ with $\Lambda = \{KxK \mid HxH \in \Omega\}$. First, we prove a lemma which gives some sufficient conditions such that a Hamilton cycle of $\mathrm{Cos}(G, K, \Lambda)$ can be lifted to a Hamilton cycle of $\mathrm{Cos}(G, H, \Omega)$. Then, using this lemma, we give the proofs of **Theorem 1.1** and **1.2**.

Lemma 4.1. *Let G be a finite p-group, H be a proper subgroup of G such that $H \bigcap Z(G) = \{1\}$ and $H < \Phi(G)$, and $\Gamma = \mathrm{Cos}(G, H, \Omega)$ be a coset digraph on G/H. Let w be an element of order p in the center of H such that $[G, w] \cap H = \{1\}$. Set $K = [G, w]H$ and $\Lambda = \{KxK \mid HxH \in \Omega\}$. Suppose that $\Sigma = \mathrm{Cos}(G, K, \Lambda)$ is Hamiltonian. Then $\Gamma = \mathrm{Cos}(G, H, \Omega)$ is Hamiltonian if one of the following three holds:*

(i) $[G, w]$ *is generated by one or two elements;*

(ii) $[G, w]$ *is an elementary abelian group;*

(iii) $[G, w]$ *is of order* p^3.

Proof. Let $(x_1, x_2, \ldots, x_u) \cdot K$ be a Hamilton cycle of Σ. Note that $[G, w] < K_G$. By Proposition 3.4, we can assume that $Hx_iH \in \Omega$ for all $i = 1, \ldots, u$, and furthermore $x_1x_2 \cdots x_u \in [G, w]$. By the hamiltonicity of $(x_1, x_2, \ldots, x_u) \cdot K$, for any $g \in G - K$, there exists a positive integer $i < u$ such that $x_1 \cdots x_iH = g^{-1}H$. Moreover, we can just set $x_1 \cdots x_i = g^{-1}$ by Proposition 3.4 (ii). Then, by Proposition 3.4 (i) and (iii), we get a Hamilton cycle

$$(x_1, \ldots, x_{i-1}, x_iw^{-1}, x_{i+1}, \ldots, x_uw) \cdot K$$

of Σ with

$$x_1 \cdots x_{i-1}x_iw^{-1}x_{i+1} \cdots x_uw = x_1 \cdots x_u[g, w].$$

By the proof of Lemma 2.2, $[G, w] = \langle [g, w] \mid g \in G - [\langle w\rangle, G]C_G(w)\rangle$. Then, since $K \leqslant [G, w]C_G(w)$, we can eliminate all the factors of $x_1 \cdots x_u$ by finite steps of replacements. It follows that we can assume $x_1x_2 \cdots x_u = 1$.

In the following discussions, we always assume that $(x_1, x_2, \ldots, x_u) \cdot K$ is a Hamilton cycle of Σ such that $x_1x_2 \cdots x_u = 1$ and $Hx_iH \in \Omega$ for all $i = 1, \ldots, u$.

Proof of (i).

For the case that $[G, w]$ is generated by only one element. Set $[G, w] = \langle y\rangle$ where $y = [g, w]$ for some $g \in G-K$. As in the above paragraph, set $g^{-1} = x_1, \ldots, x_i$ for some positive integer $i < u$. Then we get a Hamilton cycle $(x_1, \ldots, x_{i-1}, x_iw^{-1}, x_{i+1}, \ldots, x_uw) \cdot K$ of Σ with $x_1 \cdots x_{i-1}x_iw^{-1}x_{i+1} \cdots x_uw = y$. Set $|\langle y\rangle| = v$. It is straightforward to check that $(x_1, \ldots, x_{i-1}, x_iw^{-1}, x_{i+1}, \ldots, x_uw)^v \cdot H$ is a Hamilton cycle of $\Gamma = \mathrm{Cos}(G, H, \Omega)$.

Now we deal with the case when $[G, w]$ is generated by two elements. By Lemma 2.2, let $[G, w] = \langle y, z\rangle$ where $y = [g, w]$ and $z = [x_j, w]$ for some $g \in G - K$ and $1 \leqslant j \leqslant u$. Since $x_1 \cdots x_u = 1$, by Proposition 3.4 (iv), we have that $(x_{j+1}, \ldots, x_u, x_1, \ldots, x_j) \cdot K$ is also a Hamilton cycle of Σ. Therefore we can assume $j = u$ without loss of generality. Since $g \in G \setminus K$, again we set $g^{-1} = x_1 \cdots x_i$ for some $1 \leqslant i < u$. Now we have $[G, w] = \langle y, z\rangle = \langle y, yz^{-1}\rangle$ and get two Hamilton cycles of Σ:

$$(x_1, \ldots, x_{i-1}, x_iw^{-1}, x_{i+1}, \ldots, x_{u-1}, x_uw) \cdot K$$

with

$$x_1 \cdots x_{i-1}x_iw^{-1}x_{i+1} \cdots x_{u-1}x_uw = y,$$

and

$$(x_1, \ldots, x_{i-1}, x_iw^{-1}, x_{i+1}, \ldots, x_{u-1}, wx_u) \cdot K$$

with

$$x_1 \cdots x_{i-1}x_iw^{-1}x_{i+1} \cdots x_{u-1}wx_u = yz^{-1}.$$

By the main result of Witte [21], the Cayley digraph $\mathrm{Cay}([G, w], \{y, yz^{-1}\})$ is Hamiltonian. Then, by Lemma 3.5, Γ is Hamiltonian.

Proof of (ii).

Assume $[G,w]$ is an elementary abelian group of order p^d for some integer d. Recall that $(x_1, x_2, \ldots, x_u) \cdot K$ is a Hamilton cycle of Σ such that $x_1 x_2 \cdots x_u = 1$ and $Hx_iH \in \Omega$ for all $i = 1, \ldots, u$. Set $g_i^{-1} = x_1 \cdots x_i$ for all $i = 1, \ldots, u$. Let u_1 be the smallest integer such that $[g_{u_1}, w] \neq 1$, and let u_i be the smallest integer such that $[g_{u_i}, w] \notin \langle [g_{u_1}, w], \ldots, [g_{u_{i-1}}, w] \rangle$ for any $i \geqslant 2$. Then $1 \leqslant u_1 < u_2 < \cdots < u_d < u$ and we get a minimal generating set

$$\{[g_{u_1}, w],\ [g_{u_2}, w],\ \ldots,\ [g_{u_d}, w]\}$$

of $[G,w]$. Set $y_i = [g_{u_i}, w]$ for all $i = 1, \ldots, d$. Then $[G,w] = \langle y_1 \rangle \times \cdots \times \langle y_d \rangle$. Set

$$(a_{i,1}, \ldots, a_{i,u}) = (x_1, \ldots, x_{u_i} w^{-1}, x_{u_i+1}, \ldots, x_u w)$$

for all $i = 1, \ldots, d$. Then we get d Hamilton cycles $(a_{i,1}, \ldots, a_{i,u}) \cdot K$ of the digraph Σ with $a_{i,1} \cdots a_{i,u} = y_i$ for all $1 \leqslant i \leqslant d$. Set

$$(b_{0,1}, \ldots, b_{0,u}) \cdot H = (a_{1,1}, \ldots, a_{1,u}) \cdot H$$

and

$$(b_{i,1}, \ldots, b_{i,p^i u}) \cdot H = (b_{i-1,1}, \ldots, b_{i-1,p^{i-1}u})^p (a_{i,1}, \ldots, a_{i,u})^{-1} (a_{i+1,1}, \ldots, a_{i+1,u}) \cdot H$$

for all $i = 1, \ldots, d-1$. It is straightforward to check that the following conditions hold:

(a) $b_{0,1} \cdots b_{0,u} = y_1$ and $b_{i,1} \cdots b_{i,p^i u} = y_i^{-1} y_{i+1}$ for all $i = 1, \ldots, d-1$;

(b) $(b_{0,1}, \ldots, b_{0,u})^p \cdot H$ is a cycle of Γ;

(c) for any $0 \leqslant i \leqslant d-1$, $b_{i,1} \cdots b_{i,j} \in \langle y_1, \ldots, y_i \rangle$ if $u \mid j$ and $j < p^i u$;

(d) for any $0 \leqslant i \leqslant d-1$, $b_{i,1} \cdots b_{i,j} K = b_{i,1} \cdots b_{i,l} K$ if and only if $j \equiv l \pmod{u}$;

(e) for any $0 \leqslant i \leqslant d-1$ and $1 \leqslant \lambda \leqslant p^i$, $b_{i,1} \cdots b_{i,\lambda u} \neq 1$.

(f) for any $0 \leqslant i \leqslant d-1$ and $1 \leqslant \lambda, \mu \leqslant p^i$, $b_{i,1} \cdots b_{i,\lambda u} = b_{i,1} \cdots b_{i,\mu u}$ if and only if $\lambda = \mu$.

To complete the proof, we need to prove that $(b_{i,1}, \ldots, b_{i,p^i u})^p \cdot H$ are cycles of Γ for all $i = 0, 1, \ldots, d-1$.

Suppose to the contrary that there is an integer $j \leqslant d-1$ such that $(b_{i,1}, \ldots, b_{i,p^i u})^p \cdot H$ are cycles of Γ for all $i \in \{0, 1, \ldots, j-1\}$, but $(b_{j,1}, \ldots, b_{j,p^j u})^p \cdot H$ is not a cycle of Γ. Then there exist distinct integer pairs (r,s) and (e,f) with $0 \leqslant r, e \leqslant p-1$ and $1 \leqslant s, f \leqslant p^j u$ such that

$$(y_j^{-1} y_{j+1})^r b_{j,1} \cdots b_{j,s} H = (y_j^{-1} y_{j+1})^e b_{j,1} \cdots b_{j,f} H. \qquad (2)$$

Therefore $b_{j,1} \cdots b_{j,s} K = b_{j,1} \cdots b_{j,f} K$. By condition (d), we have $s \equiv f \pmod{u}$. Set $s = \lambda u + l$ and $f = \mu u + l$ where $0 \leqslant \lambda, \mu \leqslant p^j - 1$ and $1 \leqslant l \leqslant u$. To obtain the contradiction, we divide the discussions into the following three cases.

Case 1. $b_{j,\lambda u+1}\cdots b_{j,\lambda u+l}=b_{j,\mu u+1}\cdots b_{j,\mu u+l}$.

In this case, set $c=b_{j,\lambda u+1}\cdots b_{j,\lambda u+l}$. Then Eq. (2) implies that

$$(y_j^{-1}y_{j+1})^r b_{j,1}\cdots b_{j,\lambda u}(cHc^{-1})=(y_j^{-1}y_{j+1})^e b_{j,1}\cdots b_{j,\mu u}(cHc^{-1}). \tag{3}$$

Since $[G,w]\cap H=\{1\}$ and $[G,w]\lhd G$, we have $[G,w]\cap cHc^{-1}=\{1\}$. Then, by the condition (c) and Eq. (3), we have

$$(y_j^{-1}y_{j+1})^r b_{j,1}\cdots b_{j,\lambda u}=(y_j^{-1}y_{j+1})^e b_{j,1}\cdots b_{j,\mu u}. \tag{4}$$

Recall that $0\leqslant\lambda,\mu\leqslant p^j-1$. By condition (c), both $b_{j,1}\cdots b_{j,\lambda u}$ and $b_{j,1}\cdots b_{j,\mu u}$ belong to $\langle y_1,\dots,y_j\rangle$. Then, by Eq. (4) and condition (f), we have $r=e$ and $\lambda=\mu$, a contradiction.

Case 2. $b_{j,\lambda u+l+1}\cdots b_{j,(\lambda+1)u}=b_{j,\mu u+l+1}\cdots b_{j,(\mu+1)u}$.

In this case, set $c=b_{j,\lambda u+l+1}\cdots b_{j,(\lambda+1)u}$. Then Eq. (2) implies that

$$(y_j^{-1}y_{j+1})^r b_{j,1}\cdots b_{j,(\lambda+1)u}(c^{-1}Hc)=(y_j^{-1}y_{j+1})^e b_{j,1}\cdots b_{j,(\mu+1)u}(c^{-1}Hc).$$

As the proof of *Case 1.*, we have

$$(y_j^{-1}y_{j+1})^r b_{j,1}\cdots b_{j,(\lambda+1)u}=(y_j^{-1}y_{j+1})^e b_{j,1}\cdots b_{j,(\mu+1)u}.$$

Suppose that $\lambda\leqslant\mu$. If $\lambda=\mu$, then $r=e$ and we get the contradiction. If $\mu<p^j-1$, then we can obtain the contradiction by the same reason as *Case 1.*. Now we assume $\lambda<\mu=p^j-1$, then

$$(y_j^{-1}y_{j+1})^r b_{j,1}\cdots b_{j,(\lambda+1)u}=(y_j^{-1}y_{j+1})^{e+1}.$$

By condition (c), $b_{j,1}\cdots b_{j,(\lambda+1)u}\in\langle y_1,\dots,y_j\rangle$. Therefore, $r=e+1$ and $b_{j,1}\cdots b_{j,(\lambda+1)u}=1$, which is in contradiction with the condition (e).

Case 3. $\begin{cases} b_{j,\lambda u+1}\cdots b_{j,\lambda u+l}\neq b_{j,\mu u+1}\cdots b_{j,\mu u+l} \\ b_{j,\lambda u+l+1}\cdots b_{j,(\lambda+1)u}\neq b_{j,\mu u+l+1}\cdots b_{j,(\mu+1)u} \end{cases}$.

In this case,

$$b_{j,\lambda u+1}\cdots b_{j,\lambda u+l}=x_1\cdots x_l \quad\text{or}\quad b_{j,\mu u+1}\cdots b_{j,\mu u+l}=x_1\cdots x_l.$$

Without loss of generality, let $b_{j,\lambda u+1}\cdots b_{j,\lambda u+l}=x_1\cdots x_l$. Set $c=x_{l+1}\cdots x_u$. Then, $b_{j,\mu u+l+1}\cdots b_{j,(\mu+1)u}=cw$. By Eq. (2), we have

$$(y_j^{-1}y_{j+1})^r(b_{j,1}\cdots b_{j,\lambda u})c^{-1}Hc=(y_j^{-1}y_{j+1})^e(b_{j,1}\cdots b_{j,(\mu+1)u})w^{-1}c^{-1}Hc.$$

Noting that $[G,w]\cap c^{-1}Hc=\{1\}$, we have

$$(y_j^{-1}y_{j+1})^r b_{j,1}\cdots b_{j,\lambda u}=(y_j^{-1}y_{j+1})^e(b_{j,1}\cdots b_{j,(\mu+1)u})[w,c]. \tag{5}$$

Assume that

$$b_{j,\lambda u+1}\cdots b_{j,(\lambda+1)u} = y_\alpha \quad\text{and}\quad b_{j,\mu u+1}\cdots b_{j,(\mu+1)u} = y_\beta$$

for some $\alpha, \beta \in \{1, 2, \ldots, j+1\}$. Then $1 \leqslant u_\beta \leqslant l < u_\alpha \leqslant u_{j+1}$. It follow that both $b_{j,1}\cdots b_{j,\lambda u}$ and $(b_{j,1}\cdots b_{j,(\mu+1)u})[w,c]$ belong to $\langle y_1, \ldots, y_j\rangle$. Then, by Eq. (5), we have $r = e$. Therefore, by Eq. (2), we have $b_{j,1}\cdots b_{j,s}H = b_{j,1}\cdots b_{j,f}H$. Note that both $(b_{j,1}, \ldots, b_{j,s}) \cdot H$ and $(b_{j,1}, \ldots, b_{j,f}) \cdot H$ are sections of $(b_{j-1,1}, \ldots, b_{j-1,p^{j-1}u})^p \cdot H$. Since $(b_{j-1,1}, \ldots, b_{j-1,p^{j-1}u})^p \cdot H$ is a cycle of Γ, we have $s = f$, a contradiction.

So far we have proved that $(b_{i,1}, \ldots, b_{i,p^i u})^p \cdot H$ are cycles of Γ for all $i = 0, 1, \ldots, d-1$. In particular, $(b_{d-1,1}, \ldots, b_{d-1,p^{d-1}u})^p \cdot H$ is a cycle of Γ. Noting that the digraph Γ is of order $|G : H| = |G : K||K : H| = up^d$, which is coincident with the length of the sequence $(b_{d-1,1}, \ldots, b_{d-1,p^{d-1}u})^p$, we have that $(b_{d-1,1}, \ldots, b_{d-1,p^{d-1}u})^p \cdot H$ is a Hamilton cycle of Γ.

Proof of (iii).

Assume $[G, w]$ is of order p^3. Then $\big|[G, w] : \Phi([G, w])\big| \leqslant p^2$ or $[G, w]$ is an elementary abelian group and therefore the assertion follows from (i) or (ii) respectively. □

Proof of Theorem 1.1. Since Cayley digraphs of order a prime power are always Hamiltonian, it suffices to consider the non-Cayley case. By Proposition 3.2, a non-Cayley vertex-transitive digraph of order p^5 admits a representation as $\Gamma = \mathrm{Cos}(G, H, \Omega)$ where G is a p-group, $|G : H| = p^5$, $H \cap Z(G) = \{1\}$ and $\{1\} < H < \Phi(G)$. Let w be an element of order p in the center of H such that $[G, w] \cap H = \{1\}$. Set $K = [G, w]H$ and $\Lambda = \{KxK \mid HxH \in \Omega\}$. Then $\Sigma = \mathrm{Cos}(G, K, \Lambda)$ is of order not bigger than p^4 and hence Hamiltonian. Since $[G, w] \leqslant G'$ and $[G, w] < \Phi(G)$, we have $K \leqslant \Phi(G)$. Noting that G is not a cyclic group, we get $|G : K| \geqslant p^2$. Therefore, from $[G, w] \cap H = \{1\}$ and $|G : H| = p^5$, we have $\big|[G, w]\big| \leqslant p^3$. Then the assertion follows from Lemma 4.1 (iii). □

Proof of Theorem 1.2. Let Γ be a connected digraph of which the automorphism group contains a finite vertex-transitive subgroup G of prime power order.

(i) Suppose that G' is generated by two elements. By Lemma 2.4, we can assume that G is a minimum vertex-transitive p-subgroup of $\mathrm{Aut}(\Gamma)$ without loss of any generality. Let H be a vertex stabilizer in G. Assume that the order of Γ is p^n. Then $|G : H| = p^n$, $H \cap Z(G) = \{1\}$, $H < \Phi(G)$ and Γ admits a representation $\Gamma = \mathrm{Cos}(G, H, \Omega)$ for some $\Omega \subseteq H\backslash G/H - \{H\}$.

We proceed the remainder proof by induction on the order of Γ. Assume that the assertion holds for any such vertex-transitive digraph of order a power of p smaller than p^n. Let w be an element of order p in the center of H such that $[G, w] \cap H = \{1\}$. Set $K = [G, w]H$ and $\Lambda = \{KxK \mid HxH \in \Omega\}$. By Proposition 3.1, the automorphism group of $\Sigma = \mathrm{Cos}(G, K, \Lambda)$ contains a vertex-transitive subgroup isomorphic to G/K_G. Since G' is generated by two elements, we have $(G/K_G)'$ can be generated by two elements. By induction hypothesis, Σ is Hamiltonian. By Lemma 2.4, $[G, w]$ can be generated by at most two elements. It follows from Lemma 4.1 (i) that Γ is Hamiltonian.

(ii) Note that any subgroup of an elementary abelian group is also elementary abelian. Then, by using the same method as in the proof of (i) together with Lemma 4.1 (ii), one can prove the assertion. The details are omitted. □

References

[1] B. Alspach. Hamilton cycles in vertex-transitive graphs of order $2p$. In: Proc. Tenth Southeastern Conference on Combinatorics, Graph Theory and Computing, 1979, pp. 131–139. Congress. Numer., XXIII–XX, *Utilitas Math.*, Winnipeg, Man.

[2] N. Blackburn. On prime-power groups in which the derived group has two generators. *Proc. Camb. Phil. Soc.*, 53:19–27, 1957.

[3] Y. Q. Chen. On Hamiltonicity of vertex-transitive graphs and digraphs of order p^4. *J. Combin. Theory Ser. B* , 72:110–121, 1998.

[4] E. Ghaderpour and D. W. Morris. Cayley graphs of order 30p are hamiltonian. *Discrete Math.*, 312: 3614–3625, 2012.

[5] E. Ghaderpour and D. W. Morris. Cayley graphs on nilpotent groups with cyclic commutator subgroup are hamiltonian. *Ars Contemp. Math.*, 7:55–72, 2014.

[6] H. Glover, K. Kutnar, A. Malnič and D. Marušič. Hamilton cycles in (2, odd, 3)-Cayley graphs. *Proc. London Math. Soc.*, 104:1171–1197, 2012.

[7] K. Kutnar, D. Marušič. Hamiltonicity of vertex-transitive graphs of order $4p$. *European J. Combin.*, 29:423–438, 2008.

[8] K. Kutnar, D. Marušič. Hamilton cycles and paths in vertex-transitive graphs–Current directions. *Discrete Math.*, 309:5491–5500, 2009.

[9] K. Kutnar, D. Marušič, J. Morris, D. W. Morris, and P. Šparl. Hamiltonian cycles in Cayley graphs whose order has few prime factors. *Ars Math. Contemp.* 5:27–71, 2012.

[10] K. Kutnar, D. Marušič, C. Zhang. Hamilton paths in vertex-transitive graphs of order $10p$. *European J. Combin.*, 33:1043–1077, 2012.

[11] K. Kutnar, P. Šarl. Hamilton paths and cycles in vertex-transitive graphs of order $6p$. *Discrete Math.*, 309:5444–5460, 2009.

[12] P. Lorimer. Vertex-transitive graphs: Symmetric graphs of prime valency. *J. Graph Theory*, 8:55–68, 1984.

[13] L. Lovász. Combinatorial Structures and their Applications. In *Problem*, volume 11 of *Proc. Calgary Internat. Conf., Calgary, Alberta, 1969,* pages 243–246. Gordon and Breach, 1970.

[14] D. Marušič. Vertex-transitive graphs and digraphs of order p^k. *Ann. Discrete Math.*, 27:115–128, 1985.

[15] D. Marušič. Hamilton cycles in vertex symmetric graphs of order $2p^2$. *Discrete Math.*, 66:169–174, 1987.

[16] D. Marušič, T. D. Parsons. Hamiltonian paths in vertex-symmetric graphs of order $5p$. *Discrete Math.*, 42:227–242, 1982.

[17] D. Marušič, T. D. Parsons. Hamiltonian paths in vertex-symmetric graphs of order $4p$. *Discrete Math.*, 43:91–96, 1983.

[18] C. E. Praeger, R. J. Wang, M. Y. Xu. Symmetric graphs of order aproduct of two distinct primes. *J. Combin. Theory Ser. B.*, 58:299–318, 1993.

[19] M. Suzuki. Group Theory, Vol. I. Springer-Verlag, 1982.

[20] M. Suzuki. Group Theory, Vol. II. Springer-Verlag, 1986.

[21] D. Witte. Cayley digraphs of prime-power order are Hamiltonian. *J. Combin. Theory Ser. B.*, 40:107–112, 1986.

[22] D. W. Morris. Odd-order Cayley graphs with commutator subgroup of order pq are hamiltonian. *Ars Math. Contemp.*, 8:1–28, 2015.

[23] C. Zhang, J. Shi, W. Shi. Two sufficient conditions for vertex-transitive Hamilton graphs of prime-power order. *Algebra Colloq.* 16:525–534, 2009.