Generalized cages *

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Abstract

Let $2 \leqslant k_1 < k_2 < \cdots < k_t$, $3 \leqslant g_1 < g_2 < \cdots < g_s < N$ be integer parameters. A $(k_1, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$-graph is a graph that contains vertices of degrees $k_1, k_2, \ldots, k_t$ but no other degrees and cycles of lengths $g_1, g_2, \ldots, g_s$ but no other cycles of length $< N$. For any given set of parameters satisfying the above conditions, we present an explicit construction of $(k_1, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$-graphs and extend the concept of a cage (a smallest graph of given degree and girth) to that of a generalized cage – a smallest $(k_1, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$-graph. We introduce several infinite families of generalized cages and study their basic properties in the context of connected, bipartite, and vertex-transitive graphs, as well as combinatorial configurations (in the context of multilaterals).

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1 Introduction

The main motivation for our paper comes from [4] that addressed among other topics the question of the existence of trivalent graphs with certain prescribed and prohibited cycle lengths. This called, for example, for the construction of a trivalent graph of girth 6 that contains an 8-cycle, no 10-cycle and a 12-cycle, and all the possible combinations of prescribed or prohibited 6-, 8-, and 10-cycles. The authors have been able to construct a trivalent graph for any combination of these, and became interested in characterizing all possible combinations of cycle lengths for regular graphs. In the original version of this paper, we were able to show that essentially any sequence of cycle lengths can be the beginning of the cycle spectrum of a finite graph of degree k. Only after solving our original problem, we found out that a more general version of the above problem has been considered by Sachs in 1963 [22]. He was able to prove the following:

Theorem 1 ([22])**.** *Corresponding to any two integers $k \geqslant 3$ and $N \geqslant 4$ and any $N-2$ integers $\alpha_3, \dots, \alpha_N$ it is possible to construct an unlimited number of mutually non-isomorphic connected regular graphs of degree k without bridges or cut-vertices, which contain exactly α_i cycles of length i for $i = 3, \dots, N$ and in which all the cycles of length $\leqslant N$ are mutually disjoint.*

In our approach, we think of the problem of finding finite graphs with prescribed degrees and girths as a generalization of the well-known (k, g)-*Cage Problem* [8]:

Problem 2. Given integers $k \geqslant 2$ and $g \geqslant 3$, construct a k-regular graph of girth g of the smallest possible order.

The smallest k-regular graphs of girth g are called (k, g)*-cages*. The aim of our paper is to generalize the well-known problem of finding (k, g)-cages to that of finding *generalized cages* – smallest graphs for which we prescribe and limit both the vertex degrees and cycle lengths. Special cases of generalized cages have already been considered under several different settings. One of these directions allowed for extending the degrees considered and involved the study of graphs of prescribed bi-degree and girth [1, 10] or prescribed set of degrees and girth [6, 3]. In another direction, Harary and Kovács introduced the problem of the construction of regular graphs with prescribed even and odd girth [13, 14], a topic of considerable interest [5, 2, 1, 16, 24]. Obviously, each change of the original requirements required a separate proof of the existence of finite graphs with the characteristics under consideration. From this point of view, the graphs constructed in our proofs of Corollary 8 and Theorem 10 serve as universal examples for all of the above generalizations of (the original) concept of cages.

The precise definition of generalized cages we use in this paper proceeds as follows. A $(k_1, k_2, \dots, k_t; g_1, g_2, \dots, g_s; N)$*-graph*, $2 \leqslant k_1 < k_2 < \cdots < k_t$, $3 \leqslant g_1 < g_2 < \cdots < g_s < N$, is a graph that contains vertices of degrees $k_1, k_2, \dots, k_t$ but no other degrees and cycles of lengths $g_1, g_2, \dots, g_s$ but no other cycles of length $< N$. A $(k_1, k_2, \dots, k_t; g_1, g_2, \dots, g_s; N)$*-cage* is defined accordingly. We need to stress that even though we specify the degrees required of our graphs, we do not specify how many of the vertices that have the required degrees (beyond the fact that we always need at least one vertex of the particular order) must exist and we only prescribe the beginning of the cycle spectrum for our graphs (hence, specifically, we do not specify whether a cycle of length N must or must not exist). Occasionally, in the case when we only want to limit the girth of the graphs up to g_s, we will leave out the upper bound N, and will talk about the $(k_1, \ldots, k_t; g_1, \ldots, g_s;)$-graphs that are simply graphs with the required degrees $k_1, \ldots, k_t$ (and no others) and cycles of the required lengths $g_1, \ldots, g_s$ (and no others of length $\leqslant g_s$). Similarly, the notation $(k_1, \ldots, k_t; ; N)$ indicates graphs of girth at least N.

The problem of finding (k, g)-cages has been quite popular during the 1960's and 70's. After this initial period, the remaining problems were deemed hard, and not much has been happening with regard to cages until the introduction of algebraic and topological methods in the 1990's. These marked a revival of the topic with the focus on the construction of infinite families of small graphs of given degree and girth (see, for example, the survey paper [8]).

Following this general trend, we focus specifically on topological constructions and as a part of our treatise we investigate a construction that is a a generalization of the truncation construction particularly popular in topological graph theory. It is also a generalization of a method introduced by Sachs in [22] which helped him to prove Theorem 1. Using this construction in combination with the voltage graph construction, we prove a positive answer to the following question for all but the 1-regular graphs:

Problem 3. Given integers $1 \leqslant k_1 < k_2 < \cdots < k_t$, $3 \leqslant g_1 < g_2 < \cdots < g_s < N$, does there exist a $(k_1, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$-graph?

Unlike the case of the original cages (which are all connected), several generalized cages found in this paper turn out to be disconnected (as seen, for example, in the first figure of our paper). In view of this, throughout the paper we are careful to stress the difference between the results concerning connected and disconnected graphs. In the second half of our paper, in addition to proving the general existence of $(k_1, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$-graphs, we obtain bounds on the order of the $(k_1, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$-cages, refine the problem to the classes of connected, bipartite, and vertex-transitive graphs, and investigate the connection of such graphs to incidence structures.

2 The cases $k_1 = 1$ and $k_1 = 2$

Before introducing our main construction, note that graphs satisfying $k_1 = 1$ or $k_1 = 2$ (graphs possessing vertices of degree 1 or 2) are very easy to understand (see, for example, the next figure). Due to their simplicity, we deal with these cases first.

Theorem 4. *(i) There are no $(1; g_1, g_2, \ldots, g_s; N)$-graphs.*

(ii) A $(1, 2; g_1, g_2, \ldots, g_s; N)$-graph exists for every set of parameters $3 \leqslant g_1 < \cdots < g_s$, but there are no connected $(1, 2; g_1, g_2, \ldots, g_s; N)$-graphs.

(iii) If $k_1 = 1$ and $t > 1$, a $(1, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$-graph exists if a $(k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$-graph exists.

(iv) *If $k_1 = 1$ and $t > 1$, a connected $(k_1, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$-graph exists if there exists a connected $(k'_2, \ldots, k'_t; g_1, g_2, \ldots, g_s; N)$-graph satisfying the property $k'_i \leqslant k_i$ for all $2 \leqslant i \leqslant t$ and $k'_i < k_i$ for at least one i.*

Proof. There are no cycles in 1-regular graphs and no graph containing cycles and only vertices of degrees 1 and 2 is connected. To complete the proof of item (ii), note that a disjoint union of g_i-cycles $\mathcal{C}_{g_i}$, $1 \leqslant i \leqslant s$, and a single edge is a disconnected $(1, 2; g_1, g_2, \ldots, g_s; N)$-graph. Item (iii) can be proved by adding two vertices joined by a single edge which makes a $(k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$-graph into a disconnected $(1, k_2, \ldots, k_t;\ g_1, g_2, \ldots, g_s; N)$-graph. Similarly for (iv), connecting extra edges to vertices of degree $k'_i < k_i$ makes a connected $(k'_2, \ldots, k'_t; g_1, g_2, \ldots, g_s; N)$-graph satisfying the property from the theorem into a connected $(1, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$-graph. $\square$

While the order $n(1, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$ of the smallest $(1, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$-graph can be deduced from the above constructions, it is ultimately not completely easy to determine. For example, a straightforward analysis of all possible cases shows that the smallest $(1, 2; 3; 4)$-graph is simply the 3-cycle united with a single disconnected edge, $n(1, 2; 3; 4) = 5$, and the smallest $(1, 2, 3; 3; 4)$-graph is the 3-cycle with a pendant edge attached at one of the vertices, $n(1, 2, 3; 3; 4) = 4$ (see Figure 1). Somewhat surprisingly, the generalized cage with a larger set of degrees and the same set of required girths happens to be of a smaller order.

Figure 1: The unique $(1, 2; 3; 4)$- and $(1, 2, 3; 3; 4)$-cages.

The case when we allow for degree 2 is of a slightly different nature.

Theorem 5. *(i) If $k_1 = 2$ and $t = 1$, a $(2; g_1, g_2, \ldots, g_s; N)$-graph exists for every set of parameters $3 \leqslant g_1 < g_2 < \cdots < g_s$ and every $(2; g_1, g_2, \ldots, g_s; N)$ graph is a disjoint union of (at least one for each g_i) cycles of lengths $g_1, g_2, \ldots, g_s$, combined (possibly) with cycles of length $\geqslant N$.*

(ii) *There are no connected $(2; g_1, g_2, \ldots, g_s; N)$-graphs for s bigger than 1 and the $(2; g_1, g_2, \ldots, g_s; N)$-cages are of order $g_1 + g_2 + \cdots + g_s$, consist of disjoint unions of single copies of g_i-cycles, $1 \leqslant i \leqslant s$, and are the unique $(2; g_1, g_2, \ldots, g_s; N)$-graphs of that order.*

(iii) *If $k_1 = 2$ and $t > 1$, there exist connected $(2, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$-graphs for any choice of the parameters $2 < k_2 < \cdots k_t$ and $3 \leqslant g_1 < g_2 < \cdots < g_s < N$.*

Proof. The case $t = 1$ follows from the fact that all 2-regular graphs consist of disjoint cycles.

If $k_i = 2m$ for $m > 1$, joining m cycles through a single vertex results in a vertex of degree k_i and all other vertices of degree 2. If, on the other hand, $k_i = 2m+1$, $m \geqslant 1$, joining two bouquets of m cycles through a single edge results in two vertices of degree k_i and all other vertices of degree 2. It is not hard to see that further manipulation together with choosing the right cycle lengths can result in connected $(2, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$-graphs. $\square$

Again, the order of a smallest $(2, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$-graph can be determined from the above constructions but may be tricky to find. For example, $n(2, 3; 3, 5; 6) = 6$ as the $(2, 3; 3, 5; 6)$-cage is 3-cycle sharing an edge with a 5-cycle, while $n(2, 3; 3, 4; 6) = 4$ as the $(2, 3; 3, 4; 6)$-cage is the 4-cycle with a chord (see Figure 2).

Figure 2: The unique $(2, 3; 3, 4; 6)$- and $(2, 3; 3, 5; 6)$-cages.

In view of these results, we shall assume from now on that $k_1 \geqslant 3$.

3 Generalized truncation construction

The truncation of a map is well-known construction from topological graph theory in which one saws off the vertices of a map (an embedding of a graph into a surface) together with small immediate neighborhoods and then attaches cycles in their place [11]. Our main construction is a generalization of this construction.

Let G be a finite graph with the *degree set* $\{d_1, d_2, \ldots, d_t\}$ (i.e., the degree d_v of each vertex $v \in V(G)$ is equal to exactly one of the d_i's). Each edge of G can be associated with two opposing *directed edges*, each of which starts at other end-vertex of the edge. Let $D(G)$ denote the set of these oriented edges; note that $|D(G)| = 2|E(G)|$. While orientable topological maps naturally come with an inherent ordering of the oriented edges emanating from a vertex, in the case of graphs, we are free to choose this ordering ourselves: A *vertex-neighborhood labeling* of G is a function ρ from the set $D(G)$ into the set of positive integers that orders the oriented edges adjacent to the vertices of G, i.e., a function that maps the oriented edges starting from a vertex v bijectively onto the set $\{1, 2, \ldots, d_v\}$, for all $v \in V(G)$.

We note that every vertex-neighborhood labeling of a connected graph G uniquely determines an orientable embedding of G [11]. In that sense, the truncation of a graph G with a vertex neighborhood labeling ρ is simply a truncation of a specific embedding of G. The main difference between our truncation and the topological truncation, in which vertices are always replaced by cycles, lies in the fact that we truncate by all kinds of graphs: Let $H_1, H_2, \ldots, H_t$ be a family of graphs of orders $d_1, d_2, \ldots, d_t$, respectively, with the vertices of the H_i's labeled by the numbers $1, 2, \ldots, |V(H_i)| = d_i$, for $1 \leqslant i \leqslant t$. The *generalized truncation of the graph G with a vertex-neighborhood labeling ρ by the family $H_1, H_2, \ldots, H_t$* is the graph $T(G, \rho; H_1, H_2, \ldots, H_t)$ obtained from G by replacing *each* of the vertices of G by one the graphs H_i according to the following rule: Let v be a vertex of G of degree d_i, and let e_i, $1 \leqslant i \leqslant d_i$, be the oriented edges emanating from v labeled according to the vertex-neighborhood labeling ρ, i.e., labeled so that $\rho(e_i) = i$. Then, v is replaced by the graph H_i of order d_i by first removing the vertex v together with a part of each of the edges e_i, $1 \leqslant i \leqslant d_i$, and then attaching H_i to the oriented edges e_i in such a way so that a vertex of H labeled by i is attached to the edge e_i. An example of one such local truncation is included in Figure 3 in which a vertex of degree 4 is replaced by a labeled 4-cycle with a chord.

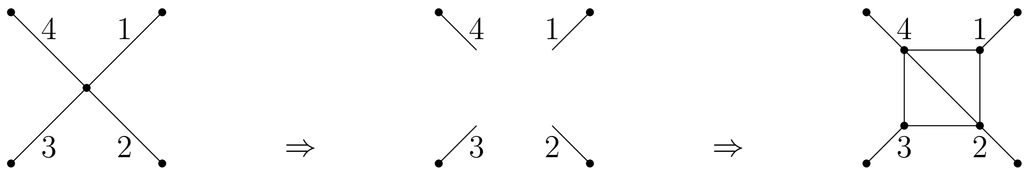


Figure 3: Truncation of a vertex of degree 4 by a 4-cycle with a chord

The significance of the generalized truncation construction in constructing graphs with prescribed and prohibited degrees and cycles is summarized in the following theorem.

Theorem 6. *Let G be a finite $(d_1, d_2, \ldots, d_t; g)$-graph with vertex-neighborhood labeling ρ. Let $H_1, H_2, \ldots, H_t$ be a truncation family for (G, ρ) of labeled $(k_1^i, \ldots, k_{t_i}^i; g_i)$-graphs, $1 \leqslant i \leqslant t$.*

The generalized truncation graph $\tilde{G} = T(G, \rho; H_1, H_2, \ldots, H_t)$ is a $(K; g)$ graph of girth not smaller than $g_{min} = \min\{2g, g_1, g_2, \ldots, g_t\}$, and degree set $K = \{k_i^j + 1 \mid 1 \leqslant i \leqslant t_j, 1 \leqslant j \leqslant t\}$.

Moreover, if both G and the H_i's are connected, so is $\tilde{G}$, and if $g_{min} < 2g$, then g_{min} is the exact girth of $\tilde{G}$.

Proof. In $\tilde{G}$, color the new edges (the edges of the graphs H_i) red, and the old edges (the original edges of G) blue. The blue edges form a 1-factor of $\tilde{G}$, and each vertex v of $\tilde{G}$ is incident with one blue edge and k_i^j red edges of $\tilde{G}$ (for some $1 \leqslant i \leqslant t_j, 1 \leqslant j \leqslant t$). Thus, the degree set of $\tilde{G}$ equals $K = \{k_i^j + 1 \mid 1 \leqslant i \leqslant t_j, 1 \leqslant j \leqslant t\}$, as claimed. Furthermore, no cycle $\mathcal{C}$ of $\tilde{G}$ has two consecutive blue edges, and we have two possibilities to consider. If $\mathcal{C}$ contains no blue edges at all, it is completely red, and its vertex set must be a subset of a copy of H_i for some $1 \leqslant i \leqslant t$. Thus, in this case, $\mathcal{C}$ is a cycle of H_i and its length must be $\geqslant g_i$. The other possibility is that $\mathcal{C}$ contains both red and blue edges. For any 2-colored cycle, if we remove the red edges, the sequence of blue edges is non-repeating and any two adjacent blue edges (with a red path between them) must have been adjacent in G. Thus, the blue edges constitute a walk in the original graph G that contains at least one cycle of G, and the red/blue cycle is of length at least $2g$. The last statement of our theorem follows from the fact that $\tilde{G}$ contains a red cycle of length g_i for each $1 \leqslant i \leqslant s$. The connectivity of $\tilde{G}$ in the case when both G and the H_i's are connected is easy to see. □

Note that the above proof shows an even stronger result. Namely, in the case when all the graphs H_i are connected, each cycle $\mathcal{C}$ of the graph G is 'lifted' into a cycle of the graph $\tilde{G}$ with the length of the lifted cycle being *at least* twice the length of $\mathcal{C}$.

4 Disconnected $(k_1, \ldots, k_t; g_1, \ldots, g_s; N)$-graphs

In this section, we solve the general question of the existence of $(k_1, \ldots, k_t; g_1, \ldots, g_s; N)$-graphs. As stated in the introduction, we will construct such graph for every set of parameters $2 \leqslant k_1 < k_2 < \cdots < k_t$ and $3 \leqslant g_1 < g_2 < \cdots < g_s < N$. It is important to note that if we do not require the graph to be connected, the problem is quite a bit easier to handle. As in the case of the 2-regular graphs covered in Theorem 5, we can compose the $(k_1, \ldots, k_t; g_1, \ldots, g_s; N)$-graph of disconnected components, each of them being a $(k; g; N)$-graph, $k \in \{k_1, \ldots, k_t\}$, $g \in \{g_1, \ldots, g_s\}$. Our main claim thus follows from the following theorem that is a special case of the more general Theorem 1 of Sachs. Its proof provides us with a nice example of the use of the generalized truncation.

Theorem 7. *Let $k \geqslant 2$ and $3 \leqslant g < N$. Then there exists a $(k; g; N)$-graph.*

Proof. We proceed by induction on k. The case of $k = 2$ has been settled in Theorem 5. The graph can be taken to be a single g-cycle.

For $k = 3$, let G be any $(g, \lceil \frac{N}{2} \rceil)$-graph (the existence of which is guaranteed for example by the result of [22]). Note that g is the degree of the graph while $\lceil \frac{N}{2} \rceil$ is its girth. Let H be the g-cycle, and ρ be any vertex-neighborhood labeling of G. It follows from Theorem 6 that the generalized truncation graph $T(G, \rho; H)$ is a $(3; g; N)$-graph.

For the induction step, assume the existence of a $(k; g; N)$-graph H for some $k \geqslant 2$ and $3 \leqslant g \leqslant N$ of order n. Let G be any n-regular graph of girth at least $\lceil \frac{N}{2} \rceil$ (guaranteed again by [22]), and let ρ be any vertex-neighborhood labeling of G. Theorem 6 asserts that the truncated graph $T(G, \rho; H)$ is a $(k + 1; g; N)$-graph. □

Corollary 8. *Let $1 \leqslant k_1 < k_2 < \cdots < k_t$ and $3 \leqslant g_1 < g_2 < \cdots < g_s < N$. If $k_1 \geqslant 2$ or $t > 1$, then there exists a $(k_1, \ldots, k_t; g_1, \ldots, g_s; N)$-graph.*

Proof. If $k_1 \geqslant 2$, the graph can be taken to be the disjoint union of graphs $G_{i,j}$, where $G_{i,j}$ is a $(k_i; g_j; N)$-graph whose existence is guaranteed by Theorem 6, $1 \leqslant i \leqslant t, 1 \leqslant j \leqslant s$. If $t > 1$, the result follows from Theorem 4 (*iii*). □

The reader may find it a bit disappointing that we broke the intriguing problem of mutually coexisting cycles into separate problems by constructing a disconnected graph. As seen in Theorem 5, the disconnectedness of our solution may appear inherent to the problem. This is however not the case, and the only sets of parameters that do not allow for connected graphs are those from Theorems 4 and 5. In the next two sections we will construct *connected* $(k_1, \ldots, k_t; g_1, \ldots, g_s; N)$-graphs for all other parameter sets.

5 Voltage graph lifts

Our construction of *connected* $(k; g_1, g_2, \ldots, g_s; N)$-graphs includes the use of a lift of a graph with a voltage assignment (see, e.g., [11]). We briefly describe this method and prove the existence of a connected graph with properties needed for our main construction.

The *base graph* B is allowed to have multiple edges and multiple loops and we replace each edge and loop of B by a pair of opposing (oriented) *darts* (with $D(B)$ denoting the set of darts of B). A *voltage assignment* on B is any mapping α from $D(B)$ into a group Γ that satisfies the condition $\alpha(e^{-1}) = (\alpha(e))^{-1}$ for all $e \in D(B)$.

The *derived regular cover (lift)* of B with respect to the voltage assignment α is denoted by B^α, and has the vertex set $V(B) \times \Gamma$ (written in the form u_g, $u \in V(B), g \in \Gamma$). Two vertices u_g and v_f are adjacent in B^α if $e = (u, v) \in D(B)$ and $f = g \cdot \alpha(e)$. The set of vertices $\{u_g \mid g \in \Gamma\}$ is called the *fiber* of u. If $\mathcal{C} = e_1, e_2, \ldots, e_\ell$ is an oriented cycle of B, the product $f = \alpha(e_1) \cdot \alpha(e_2) \cdot \cdots \cdot \alpha(e_\ell)$ (in Γ) is called the *net voltage* of $\mathcal{C}$, and a lift of $\mathcal{C}$ is a cycle of length equal to the $lcm(\ell, |f|)$, where $|f|$ denotes the order of f in Γ.

A lift of B is said to be *cyclic* if the voltage group used is a cyclic group $\mathbb{Z}_n$ of order n (which we will write in the additive notation).

We are ready to prove the main theorem of our section.

Lemma 9. *Let $k \geqslant 2$, $g \geqslant 3$, and B be a (k, g)-graph. Let $z \geqslant 1$. Then there exists a cyclic lift B^α that contains a connected component G such that*

(i) G is k-regular,

(ii) the girth of G is at least g,

(iii) no 2 vertices of G that belong to the same fiber share mutual neighbors, and

(iv) at least one fiber of G contains at least z vertices.

Proof. Let B be a k-regular graph of girth g and order n. Take $m > z \cdot n$ and relatively prime to g and let u be a vertex of B that lies on a g-cycle of B. Choose a direction for the g-cycle containing u and define a voltage assignment α from $D(B)$ into $\mathbb{Z}_m$ by assigning 1 to one of each pair of opposing darts of B and -1 to the opposite dart in such a way that the darts of the g-cycle containing u oriented along the direction chosen for this cycle all receive the voltage 1. Let (u, g) be any lift of u in B^α, and let G be the connected component of (u, g). Then G satisfies the conditions of our lemma.

First, each vertex of B^α is of degree k, and so must be the degree of each vertex of G.

Consider a cycle $\mathcal{C}$ of B^α of length ℓ. If $\mathcal{C}$ contains no two vertices from the same fiber of B^α (other than the beginning and end vertex), the projection of $\mathcal{C}$ into B must be a cycle of B of length ℓ as well, and hence $\ell \geqslant g$. If $\mathcal{C}$ does contain at least two vertices of the same fiber, it contains a sub-path $\mathcal{P}$ whose end-points belong to the same fiber, but no other two vertices do. The projection of $\mathcal{P}$ into B cannot be a self-reversing path as the base graph B has no loops and no multiple edges and thus the projection is a cycle of B, and hence $\ell \geqslant g$ again. It follows that the girth of G is at least g.

Let $v_f, v_h \in V(G)$, and suppose that there exists a vertex w_r that is a mutual neighbor of v_f and v_h. Then there must exist an edge e of B that connects v and w. Suppose that the voltage of the dart from v to w is a (where a is either 1 or -1). Since w_r is a neighbor of v_f, $r = f + a$. Similarly, since w_r is a neighbor of v_h, $r = h + a$. But f and h were assumed different, and we obtain a contradiction.

Finally, it is not hard to see that the order of G is at least m (the order of the voltage group). The argument is quite simple. Let $\mathcal{C}$ be the g-cycle through u whose edges have been assigned 1's in accordance with its direction. Then the net voltage of $\mathcal{C}$ is equal to g, and the length of each lift of $\mathcal{C}$ in B^α is equal to $g \cdot |g|$ where $|g|$ denotes the order of g in $\mathbb{Z}_m$. Since m is assumed to be relatively prime to g, the order of g in $\mathbb{Z}_m$ is m and hence the lift of $\mathcal{C}$ passing through u_g is a cycle of length gm. As all the vertices of $\mathcal{C}$ belong to G, the order of G is greater than $m > z \cdot n$, where n is the number of fibers. By the Pigeonhole Principle, G contains at least z vertices of at least one fiber of B^α. $\square$

6 Connected $(k_1, \dots, k_t; g_1, \dots, g_s; N)$-graphs

We prove the existence of a connected $(k_1, \dots, k_t; g_1, \dots, g_s; N)$-graph for each set of parameters satisfying the condition $k_1 \geqslant 3$. It is once again the case that the first part of our proof, namely the part where we prove the existence of the $(k_1; g_1, g_2, \dots, g_s; N)$-graphs, can already be deduced from Theorem 1 (but we prove it differently). Graphs with varied degrees are then obtained by extending the initial $(k_1; g_1, g_2, \dots, g_s; N)$-graphs.

Theorem 10. *Let* $2 < k_1 < \cdots < k_t$ *and* $3 \leqslant g_1 < \cdots < g_s < N$*, and suppose that* $t > 1$ *or* $k_1 \geqslant 3$*. Then there exists a connected* $(k_1, \dots, k_t; g_1, \dots, g_s; N)$*-graph.*

Proof. For $t = 1$ and $k_1 = 3$, let B be any connected (k', g')-graph with $k', g' > max\{N, 3\}$, and let G be the connected component of a lift of B that we constructed in Lemma 9 with $z = s$. Then G is of degree $k' > 3$ and girth at least $g' > N$. Let $v_1, v_2, \dots, v_s$ be the s vertices that belong to the same fiber of B whose existence is guaranteed by Lemma 9 (iv). Then all their neighborhoods are mutually disjoint and each of the neighborhoods is of size k'. For each of the numbers g_j, $1 \leqslant j \leqslant t$, decrease the degree of the vertex v_j to g_j (by removing the appropriate number of adjacent edges). The removal of these edges will result in a graph H with one vertex of degree g_j for each $1 \leqslant j \leqslant s$, and the rest of the vertices of degrees $k'-1$ or k' (depending on whether they had been a neighbor of a vertex whose degree has been decreased and the connecting edge has been removed or not). If ρ is any vertex-neighborhood labeling of H, for each $1 \leqslant i \leqslant s$, $\mathcal{C}_{g_i}$ is a g_i-cycle labeled at random from $\{1, \dots, g_i\}$, and $\mathcal{C}_{k'-1}, \mathcal{C}_{k'}$ are randomly labeled $k'-1$ and k'-cycles, then the generalized truncated graph $T(H, \rho; \mathcal{C}_{g_1}, \mathcal{C}_{g_2}, \dots, \mathcal{C}_{g_s}, \mathcal{C}_{k'-1}, \mathcal{C}_{k'})$ can be easily seen to be a $(3; g_1, g_2, \dots, g_s; N)$-graph. (Since k' is taken to be greater than N, the cycles $\mathcal{C}_{k'-1}$ and $\mathcal{C}_{k'}$ do not have to be listed among the cycles shorter than N.)

The proof for the cases $t = 1$ and $k_1 > 3$ can now be completed using the 'usual' recursion via inserting a $(k-1; g_1, g_2, \dots, g_s; N)$-graph of order n into any (n, g)-graph with $g > N$ to obtain a $(k; g_1, g_2, \dots, g_s; N)$-graph.

Consider finally the case $t > 1$ and $k_1 \geqslant 3$. In this case, we construct the $(k_1, \dots, k_t; g_1, \dots, g_s; N)$-graphs from the above $(k_1; g_1, g_2, \dots, g_s; N)$-graph G by increasing the degrees of some of the vertices of the graph G to the desired values $k_2, k_3, \dots, k_t$. Let us assume without loss of generality that G contains at least $t-1$ non-incident edges $e_2, \ldots, e_t$ with end-points x_i, y_i, $2 \leqslant i \leqslant t$, respectively. Let H be any connected k_1-regular graph of girth $g > N+1$, let e be an edge of H with end-points x and y, and let H' be the graph H with the edge e (but not the vertices x, y) removed. For every i, $2 \leqslant i \leqslant t$, attach $k_i - k_1$ disjoint copies of H' to the vertices x_i, y_i with each of the copies attached via a pair of edges connecting x to x_i and y to y_i. We claim that the resulting graph is our desired $(k_1, \ldots, k_t; g_1, \ldots, g_s; N)$-graph: The degrees of all but the vertices x, y, x_i, y_i, $2 \leqslant i \leqslant t$, remain unchanged (equal to k_1), the degrees the vertices of x, y in each copy of H' equal $k_1 - 1 + 1 = k_1$, and the degrees of the vertices x_i, y_i clearly equal k_i, for all $2 \leqslant i \leqslant t$. Thus, the resulting graph has the correct degree sequence. As G already contains cycles of all of the lengths g_i, $1 \leqslant i \leqslant s$, and no edges have been removed from G, the resulting graph contains all the required cycles. It remains to show that the resulting graph does not contain cycles of length smaller than N that are not among the required ones. As no edges have been added between any two vertices of G, and no edges have been added between any two vertices of H', if such a cycle existed, it would have to contain edges from both G and some copy of H', and thus would have to contain both vertices x and y of some copy of H'. However, since the girth of H was taken to be greater than $N+1$, any path between x and y that consists entirely of edges in H' must be of length at least N as otherwise this path together with the (removed) edge between x and y would have given rise to a cycle of length smaller than $N+1$. Thus, any cycle of the resulting graph that contains both x and y must be longer than N, and therefore the resulting graph does not contain cycles that contain edges from G and some copy of H' and are shorter than N. This completes the proof of the theorem for this last case. $\square$

We feel obliged to note that the above proof is a simplification of our original proof. This has been achieved using ideas submitted by one of our referees.

7 Bipartite $(k_1, \ldots, k_t; g_1, \ldots, g_s; N)$-graphs and configurations

As mentioned in the introduction, the motivation for considering bipartite graphs with these properties comes from the study of combinatorial configurations [4].

A (v_k) *configuration* is an incidence structure of v points and v lines satisfying the properties that each line is incident with k points, each point is incident with k lines, and any two distinct points are incident with at most one common line. The *incidence graph* (also called a *Levi graph*) of a (v_k) configuration is a bipartite graph with the vertices in one bipartition set representing the points, the vertices in the other bipartition set representing the lines, and the adjacency relation defined by the incidence relation in the configuration. From the definition it immediately follows that a Levi graph of a (v_k) configuration is a k-regular graph of girth at least 6. Conversely, each bipartite k-regular graph with girth at least 6 determines a pair of mutually dual (v_k) configurations.

A cycle of length g (which is an even number) in the Levi graph of a configuration $\mathcal{C}$ corresponds to a $\frac{g}{2}$-lateral (or a $\frac{g}{2}$-gon) in $\mathcal{C}$. Formally, an *n-lateral* (or an *n-gon*) in a configuration is a cyclically ordered set $\{p_0, \ell_0, p_1, \ell_1, \ldots, p_{n-2}, \ell_{n-2}, p_{n-1}, \ell_{n-1}\}$ of pairwise distinct points p_i and pairwise distinct lines ℓ_i such that p_i is incident to ℓ_{i-1} and ℓ_i.

The question posed in [12] (though in the context of geometric configurations) and partially answered in [4] asks about all possible combinations of existence and non-existence of n-laterals in configurations. The same question (via Levi graphs) can be expressed as a problem of existence of *bipartite* $(k; g_1, g_2, \dots, g_s; N)$-graphs, $6 \leqslant g_1 < g_2 < \dots < g_s < N$, for each set of parameters with all the g_i's even.

Theorem 11. *Let $3 \leqslant k_1 < \dots < k_s$ and $4 \leqslant g_1 < g_2 < \dots < g_s < N$, each g_i even. Then there exists a (connected) bipartite $(k_1, \dots, k_t; g_1, g_2, \dots, g_s; N)$-graph.*

Proof. If the $(k_1, \dots, k_t; g_1, g_2, \dots, g_s; N)$-graph constructed in the proof of Theorem 10, is already bipartite, then we are done. Otherwise we construct its double cover – a $\mathbb{Z}_2$ lift with each dart receiving the voltage 1. The lifted graph is twice the size of the original, has only cycles of even length, and most importantly, lifts even length cycles of the original graph into cycles of the same length, and lifts odd cycles to cycles of doubled size. Since all the edges of the lift go between the two copies of the original graph, the double cover of a $(k_1, \dots, k_t; g_1, g_2, \dots, g_s; N)$-graph with all g_i's even is a bipartite graph that contains cycles of length g_i for each $1 \leqslant i \leqslant s$. It remains to show that the lift does not contain cycles of 'wrong' length. Let $\mathcal{C}$ be any cycle of the lifted graph. If $\mathcal{C}$ contains no two distinct vertices from the same fiber, $\mathcal{C}$ projects onto a base graph cycle, and hence is of good length, as the base graph contains only cycles of good length. If $\mathcal{C}$ contains two distinct vertices u, v from the same fiber, since we have no vertical edges within fibers, $\mathcal{C}$ contains a non-trivial path connecting u and v. Let $\mathcal{P}$ be a shortest sub-path of $\mathcal{C}$ that connects two vertices from the same fiber. Then $\mathcal{P}$ projects onto a cycle of the base graph. As the end-points of the lift of this cycle are not the same vertex, this cycle must be of odd length and hence of length at least N. It follows that $\mathcal{C}$ is also of length at least N. Thus, the lifted graph is a $(k_1, \dots, k_t; g_1, g_2, \dots, g_s; N)$-graph. □

The following corollary of the above theorem answers the related question concerning configurations.

Corollary 12. *Let $k \geqslant 3$ and $3 \leqslant n_1 < n_2 < \dots < n_s < N$. Then there exists a (v_k) configuration that contains laterals of lengths $n_1, n_2, \dots, n_s$ but no other laterals of length $< N$.*

Proof. Using Theorem 11 construct a bipartite $(k; 2n_1, 2n_2, \dots, 2n_s; 2N)$-graph which is a Levi graph of the desired configuration. □

Combining the ideas from Section 5 with this Section we notice that if we want to extend the generalized girth problem to incidence graphs of non-balanced configurations, i.e. configurations with semi-regular bipartite Levi graphs, a special notation is needed. In [12], there are several examples involving configurations with the symbol (v_r, b_k), where $vr = bk$ and the corresponding Levi graph has girth at least 6, has v black vertices and b white vertices, the black vertices have degree r and the white vertices have degree k. In our notation it is a bipartite $(r, k; ; 6)$ graph. In order to have more precise information, we have to amend the notation. Let $[k_1, k_2, \dots, k_t; g_1, g_2, \dots, g_s; N]$ denote a

$(k_1, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N)$ graph with a t-vertex coloring and the property that each vertex of color i has degree k_i. Using this notation we no longer require that the k_i be distinct (and ordered). The generalized girth problem has now a special subproblem for $[k_1, k_2, \ldots, k_t; g_1, g_2, \ldots, g_s; N]$-graphs.

8 Vertex-transitive $(k; g_1, g_2, \ldots, g_s; N)$-graphs

A graph G is called *vertex-transitive* if for any ordered pair (u, v) of vertices of G there exists an automorphism ϕ of G such that $\phi(u) = v$. As every vertex-transitive graph must be k-regular for some k, none of the non-regular graphs constructed so far in our paper are vertex-transitive. Thus, in this section, in the context of vertex-transitive graphs, we will only consider $(k; g_1, \ldots, g_s; N)$-graphs. In view of the vertex-transitive constructions in [19, 15, 9], it is interesting to ask whether we may be able to construct a vertex-transitive $(k; g_1, g_2, \ldots, g_s; N)$-graph for any parameter set $(k; g_1, g_2, \ldots, g_s; N)$. To answer this question, we present a negative result that shows that for certain parameter sets there exist no vertex-transitive graphs.

Lemma 13. *If $g_1 > 3$ is odd, $g_2 = 2g_1$, and $g_1 + g_2 = 3g_1 < N$, then there is no vertex-transitive $(3; g_1, g_2; N)$-graph.*

Proof. Due to their high level of symmetry, vertex-transitive graphs have the same cycle structure through each vertex of the graph. Thus, if a trivalent vertex-transitive graph G contains a g_1- and a g_2-cycle, each vertex of G must lie on at least one g_1- and at least one g_2-cycle. Since G is trivalent, this means that there exist a g_1-cycle $\mathcal{C}_1$ and g_2-cycle $\mathcal{C}_2$ of G that share at least one edge. Let $P_1, P_2, \ldots, P_r$ be the maximal subpaths shared by $\mathcal{C}_1$ and $\mathcal{C}_2$ of lengths $\ell_1, \ell_2, \ldots, \ell_r$, respectively.

If $r = 1$, i.e., the two cycles share just one path, consider the cycle constructed from the union of $\mathcal{C}_1$ and $\mathcal{C}_2$ with the path P_1 removed. Its length is $g_1 + g_2 - 2\ell_1 = 3g_1 - 2\ell_1$, which is an odd number smaller than N. The only admissible odd cycle length smaller than N is g_1, and hence $g_1 = 3g_1 - 2\ell_1$ or $0 = 2g_1 - 2\ell_1$. But $\ell_1 < g_1$, and thus $0 < 2g_1 - 2\ell_1$. The assumption $r = 1$ leads to a contradiction.

If $r > 1$, consider the cycles obtained from the union of $\mathcal{C}_1$ and $\mathcal{C}_2$ after removing the paths $P_1, P_2, \ldots, P_r$. There is exactly r of them, say of lengths $g_1, g_2, \ldots, g_r$. Then $g_1 + g_2 + \cdots + g_r + 2\ell_1 + 2\ell_2 + \cdots + 2\ell_r = g_1 + g_2 = 3g_1$, and since each of the g_i's is either equal to g_1 or $2g_1$, and all the ℓ_i's are $\geqslant 1$, there cannot be more than 2 such cycles and they both have to be of length g_1. This is once again impossible, as $g_1 + g_1 + 2\ell_1 + 2\ell_2 = 2(g_1 + \ell_1 + \ell_2)$ is even while $3g_1$ is odd. $\square$

Note that our non-existence result bears further insight into the question about the distribution of face lengths in embeddings of Cayley graphs addressed in [23] as well as in the constructions of vertex-transitive cages [15, 9]. It appears extremely likely that there are many further restrictions of the above type, and the classification of all the parameters sets $(k; g_1, g_2, \ldots, g_s; N)$ that admit the existence of a vertex-transitive $(k; g_1, g_2, \ldots, g_s; N)$-graph may prove to be very interesting (and complicated).

9 Smallest order $(k_1, \dots, k_t; g_1, \dots, g_s; N)$-graphs

As mentioned in the introduction, a k-regular graph with girth g of the smallest possible order is called a (k, g)-cage; we denote the order of a (k, g)-cage by $n(k, g)$. Following this terminology, we refer to a $(k_1, \dots, k_t; g_1, \dots, g_s; N)$-graph of the smallest possible order as a $(k_1, \dots, k_t; g_1, \dots, g_s; N)$*-cage* and denote its order by $n(k_1, \dots, k_t; g_1, \dots, g_s, N)$. Using a simple computer program written in *C* (which checks for the presence of cycles of various lengths in a given graph) and the graph generator *geng* by B. D. McKay [18], we have been able to determine the values of $n(3; g_1, g_2, \dots, g_s; N)$ and $n(2, 3; g_1, g_2, \dots, g_s; N)$ for all possible combinations of cycle lengths and $N = 4, 5, \dots, 8$. The results of our calculations are included in Tables 1–8.

$(k_1, \dots, k_t; g_1, \dots, g_s; N)$	order
$(3; ; 4)$	6
$(3; 3; 4)$	4

$(k_1, \dots, k_t; g_1, \dots, g_s; N)$	order
$(3; ; 5)$	10
$(3; 3; 5)$	10
$(3; 4; 5)$	6
$(3; 3, 4; 5)$	4

Table 1: Orders of the smallest $(3; g_1, g_2, \dots, g_s; 4)$-graphs (left) and $(3; g_1, g_2, \dots, g_s; 5)$-graphs (right) for all possible combinations. All of them are connected.

$(k_1, \dots, k_t; g_1, \dots, g_s; N)$	order	$(k_1, \dots, k_t; g_1, \dots, g_s; N)$	order
$(3; ; 6)$	14	$(3; 3, 4; 6)$	4
$(3; 3; 6)$	12	$(3; 3, 5; 6)$	10
$(3; 4; 6)$	6	$(3; 4, 5; 6)$	8
$(3; 5; 6)$	10	$(3; 3, 4, 5; 6)$	6

Table 2: Orders of the smallest $(3; g_1, g_2, \dots, g_s; 6)$-graphs for all possible combinations. All these graphs are connected.

$(k_1, \dots, k_t; g_1, \dots, g_s; N)$	order	$(k_1, \dots, k_t; g_1, \dots, g_s; N)$	order
$(3; ; 7)$	24	$(3; 4, 5; 7)$	12
$(3; 3; 7)$	16	$(3; 4, 6; 7)$	6
$(3; 4; 7)$	16	$(3; 5, 6; 7)$	10
$(3; 5; 7)$	18	$(3; 3, 4, 5; 7)$	10
$(3; 6; 7)$	14	$(3; 3, 4, 6; 7)$	8
$(3; 3, 4; 7)$	4	$(3; 3, 5, 6; 7)$	10
$(3; 3, 5; 7)$	20	$(3; 4, 5, 6; 7)$	8
$(3; 3, 6; 7)$	12	$(3; 3, 4, 5, 6; 7)$	6

Table 3: Orders of the smallest $(3; g_1, g_2, \dots, g_s; 7)$-graphs for all possible combinations. All these graphs are connected.

$(k_1,\ldots,k_t;g_1,\ldots,g_s;N)$	order	$(k_1,\ldots,k_t;g_1,\ldots,g_s;N)$	order
$(3;;8)$	30	$(3;3,4,5;8)$	10
$(3;3;8)$	18	$(3;3,4,6;8)$	10
$(3;4;8)$	18	$(3;3,4,7;8)$	14
$(3;5;8)$	20	$(3;3,5,6;8)$	20
$(3;6;8)$	14	$(3;3,5,7;8)$	20
$(3;7;8)$	24	$(3;3,6,7;8)$	12
$(3;3,4;8)$	4	$(3;4,5,6;8)$	16
$(3;3,5;8)$	26^*	$(3;4,5,7;8)$	12
$(3;3,6;8)$	22	$(3;4,6,7;8)$	12
$(3;3,7;8)$	16	$(3;5,6,7;8)$	12
$(3;4,5;8)$	18	$(3;3,4,5,6;8)$	6
$(3;4,6;8)$	6	$(3;3,4,5,7;8)$	14
$(3;4,7;8)$	16	$(3;3,4,6,7;8)$	8
$(3;5,6;8)$	10	$(3;3,5,6,7;8)$	10
$(3;5,7;8)$	18	$(3;4,5,6,7;8)$	8
$(3;6,7;8)$	18	$(3;3,4,5,6,7;8)$	8

Table 4: Orders of the smallest $(3;g_1,g_2,\ldots,g_s;8)$-graphs for all possible combinations. All graphs are connected except of the smallest $(3;3,4,6;8)$-graph which is the union of K_4 and $K_{3,3}$. In case of the smallest $(3;4,5,6;8)$-graph there exist both a connected and a disconnected graph of order 16. The $(3;3,5;8)$-graph was found by G. Exoo.

Furthermore, it is not hard to see that for all $2 \leqslant k$ and $3 \leqslant g$

$$n(k;g;g+1) = n(k,g),$$

and for all $2 \leqslant k$ and $3 \leqslant g_1 < \cdots < g_s < N$,

$$n(k,g_1) \leqslant n(k;g_1,\ldots,g_s;N),$$

as the class of the $(k;g_1,\ldots,g_s;N)$-graphs is a subclass of the class of the (k,g)-graphs.

The following well-known *Moore bound* [8] serves therefore as a lower bound for

$(k_1,\ldots,k_t;g_1,\ldots,g_s;N)$	order
$(2,3;;4)$	5
$(2,3;3;4)$	4

$(k_1,\ldots,k_t;g_1,\ldots,g_s;N)$	order
$(2,3;;5)$	7
$(2,3;3;5)$	6
$(2,3;4;5)$	5
$(2,3;3,4;5)$	4

Table 5: Orders of the smallest $(2,3;g_1,\ldots,g_s;4)$-graphs (left) and $(2,3;g_1,\ldots,g_s;5)$-graphs (right) for all possible combinations. All of them are connected.

$(k_1,\ldots,k_t;g_1,\ldots,g_s;N)$	order	$(k_1,\ldots,k_t;g_1,\ldots,g_s;N)$	order
$(2,3;;6)$	8	$(2,3;3,4;6)$	4
$(2,3;3;6)$	6	$(2,3;3,5;6)$	6
$(2,3;4;6)$	5	$(2,3;4,5;6)$	6
$(2,3;5;6)$	7	$(2,3;3,4,5;6)$	5

Table 6: Orders of the smallest $(2,3;g_1,\ldots,g_s;6)$-graphs for all possible combinations. All these graphs are connected.

$(k_1,\ldots,k_t;g_1,\ldots,g_s;N)$	order	$(k_1,\ldots,k_t;g_1,\ldots,g_s;N)$	order
$(2,3;;7)$	10	$(2,3;4,5;7)$	6
$(2,3;3;7)$	6	$(2,3;4,6;7)$	6
$(2,3;4;7)$	5	$(2,3;5,6;7)$	7
$(2,3;5;7)$	8	$(2,3;3,4,5;7)$	5
$(2,3;6;7)$	8	$(2,3;3,4,6;7)$	7
$(2,3;3,4;7)$	4	$(2,3;3,5,6;7)$	6
$(2,3;3,5;7)$	8	$(2,3;4,5,6;7)$	7
$(2,3;3,6;7)$	7	$(2,3;3,4,5,6;7)$	6

Table 7: Orders of the smallest $(2,3;g_1,\ldots,g_s;7)$-graphs for all possible combinations. All these graphs are connected.

$n(k;g_1,g_2,\ldots,g_s;N)$ as well.

$$n(k,g) \geqslant \begin{cases} 1+k+k(k-1)+\cdots+k(k-1)^{(g-3)/2}, & g \text{ odd} \\ 2(1+(k-1)+\cdots+(k-1)^{(g-2)/2}), & g \text{ even} \end{cases} \tag{1}$$

A (k,g)-graph of order equal to the Moore bound is called a *Moore graph* and Moore graphs are known to exist (even though only for very special sets of parameters [8]). Thus, there are instances of our generalized problem where the Moore bound is actually achieved. Namely, the Moore bound is at least achieved for any set of parameters $(k;g;g+1)$ for which a (k,g)-Moore graph exists.

The general situation is however quite different, and it appears likely that the 'trivial' cases are the only cases where the order of a generalized cage matches the order of the corresponding cage. A specific situation supporting our argument is described in the following lemma.

Lemma 14. *Let $3 \leqslant k,g$ and suppose that $N > 2g$. Then*

$$n(k;g;N) \geqslant \begin{cases} g(1+(k-2)+(k-2)(k-1)+\cdots+(k-2)(k-1)^{(g-3)/2}), & g \textit{ odd}, \\ g(1+(k-2)+(k-2)(k-1)+\cdots+(k-2)(k-1)^{(g-2)/2}), & g \textit{ even}. \end{cases} \tag{2}$$

Proof. Let k,g and N be as above and suppose first that g is odd. By definition, any $(k;g;N)$-graph G contains at least one g-cycle. Observe that no two g-cycles in such

$(k_1,\ldots,k_t;g_1,\ldots,g_s;N)$	order	$(k_1,\ldots,k_t;g_1,\ldots,g_s;N)$	order
$(2,3;;8)$	11	$(2,3;3,4,5;8)$	5
$(2,3;3;8)$	6	$(2,3;3,4,6;8)$	9
$(2,3;4;8)$	5	$(2,3;3,4,7;8)$	8
$(2,3;5;8)$	8	$(2,3;3,5,6;8)$	6
$(2,3;6;8)$	8	$(2,3;3,5,7;8)$	10
$(2,3;7;8)$	10	$(2,3;3,6,7;8)$	7
$(2,3;3,4;8)$	4	$(2,3;4,5,6;8)$	9
$(2,3;3,5;8)$	8	$(2,3;4,5,7;8)$	7
$(2,3;3,6;8)$	9	$(2,3;4,6,7;8)$	9
$(2,3;3,7;8)$	8	$(2,3;5,6,7;8)$	8
$(2,3;4,5;8)$	6	$(2,3;3,4,5,6;8)$	6
$(2,3;4,6;8)$	6	$(2,3;3,4,5,7;8)$	8
$(2,3;4,7;8)$	8	$(2,3;3,4,6,7;8)$	7
$(2,3;5,6;8)$	7	$(2,3;3,5,6,7;8)$	7
$(2,3;5,7;8)$	9	$(2,3;4,5,6,7;8)$	7
$(2,3;6,7;8)$	9	$(2,3;3,4,5,6,7;8)$	7

Table 8: Orders of the smallest $(2,3;g_1,\ldots,g_s;8)$-graphs for all possible combinations. All graphs are connected. Note that in case of the smallest $(2,3;3,4,6;8)$-graph there exist both a connected and a disconnected graph of order 9.

a graph can share an edge as such an occurrence would lead to the existence of g'-cycle of length strictly between g and N. Let $\mathcal{C}$ be a g-cycle of G, assume that the vertices of $\mathcal{C}$ are the vertices $v_1,v_2,\ldots,v_g$, and consider the $g(k-2)$ edges incident to the vertices v_i not included in $\mathcal{C}$. We claim that each v_i is attached to a separate 'tree' of $(k-2)+(k-2)(k-1)+\cdots+(k-2)(k-1)^{(g-3)/2}$ vertices (with any two trees mutually disjoint). The argument is similar to the argument used to prove the Moore bound. Consider the end-points of the $(k-2)$ edges adjacent to v_i and not belonging to $\mathcal{C}$. Each of these must be adjacent to $k-1$ new vertices not shared with any other branch of the tree at v_i and neither with any other tree rooted at v_j, $j \neq i$. This is due to the fact that we cannot create shorter cycles than g and cannot even create cycles of length g that would share a part of $\mathcal{C}$. The first time an edge from a tree from v_i can go to another branch of that tree or to one of the other trees is at the distance $(g-1)/2$ from v_i. This completes the proof for odd g.

The case for even g differs from the odd case in that there might exist two cycles of length g that share at least an edge. More precisely, if $g=2m$, there might exist two g-cycles that share an m-path. If no such two g-cycles exist in G, a very similar proof to the odd case shows the bound stated. If G contains two g-cycles sharing an m-path, trees attached to vertices antipodal with respect to these cycles (i.e., lying on the same cycle and of distance m) may form another m-path and thus the attached trees may share vertices. It is not hard to see, however, that this is not possible. Namely, if there existed two distinct vertices anywhere on these two g-cycles that were joined by an additional path of length

m, one could always construct a cycle involving this path and parts of the original g-cycles of length smaller than $2g$ but not equal to g. Thus, all the trees attached to the $3m$ vertices of the two g-cycles must be disjoint in this case as well, and the number of vertices of the graph G must be at least $3m(1+(k-2)+(k-2)(k-1)+\cdots+(k-2)(k-1)^{(g-2)/2})$. □

Two upper bounds follow from the proof of Corollary 8:

$$n(k;g_1,\ldots,g_s;N) \leqslant \sum_{i=1}^{s} n(k;g_i;N),$$

$$n(k_1,\ldots,k_t;g_1,\ldots,g_s;N) \leqslant \sum_{i=1}^{t}\sum_{j=1}^{s} n(k_i;g_j;N),$$

where the second bound could clearly be improved. Both upper bounds are obtained by considering disconnected graphs which yields another interesting aspect of the problem of the minimal order of an $(k;g_1,g_2,\ldots,g_s;N)$-graph that does not occur with regard to (k,g)-cages: the order $n(k;g_1,g_2,\ldots,g_s;N)$ can be achieved by a disconnected union of smaller graphs (as is the case for $k=2$). The smallest such example for $k=3$ is the $(3;3,4,6;8)$-cage which turns out to be a union of the complete graph K_4, the smallest $(3;3,4;8)$-graph, and the complete bipartite graph $K_{3,3}$, the smallest $(3;4,6;8)$-graph. It has order 10 while the smallest connected $(3;3,4,6;8)$-graph has order 14. It is also possible that $n(k;g_1,g_2,\ldots,g_s;N)$ is achieved by both a connected and a disconnected graph. The smallest case for $k=3$ is $n(3;4,5,6;8)$: there exist connected and disconnected $(3;4,5,6;8)$-graphs of order 16 which is the smallest value for this set of parameters. On the other hand, there exists an infinite family of triples $(g_i,g_i';N_i)$, $g_i<g_i'<N_i$ and $k\geqslant 3$, such that

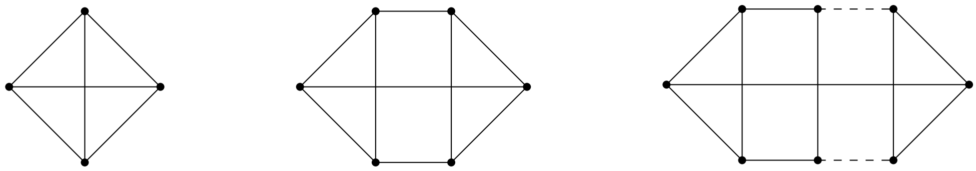
$$n(k;g_i,g_i';N_i) < n(k;g_i;N_i)+n(k;g_i';N_i),$$

and the corresponding generalized cage is connected. A somewhat disappointing example of this situation can be obtained as follows:

$$n(3;3,4;N)=4<n(3;3;N)+n(3;4;N),$$

where N is assumed to be >4. At this point, we do not have a more meaningful example of this situation.

One of the fundamental problems when trying to determine a precise value of $n(k,g)$ (the order of the smallest (k,g)-graph) is that beside having to construct a graph of order $n(k,g)$, one also has to establish the non-existence of a smaller (k,g)-graph. Although obviously one is to expect the same problem to occur with regard to our generalized cages, there are certain parameter sets where establishing the non-existence of smaller graphs of those parameters proves out to be a bit less demanding. One such example are the sets of parameters with $k=3$ and the required cycle lengths covering the whole range of cycle lengths from the girth 3 through $N-1$ – the $(3;3,4,\ldots,N-1;N)$-cages. These are depicted in Figure 4 for $N\geqslant 4$ and it is easy to see that the graphs G_N have cycles of all the lengths $3,4,\ldots,N-1$ while being of the smallest possible orders (N for N even and



$G_4 = G_5$ $\quad$ $G_6 = G_7$ $\quad$ $G_N = G_{N+1}$, N even

Figure 4: Smallest $(3; 3, 4, \ldots, N-1; N)$-graphs G_N, $N \geqslant 4$.

$N-1$ for N odd) – due to the fact that they must contain an $N-1$ cycle. These graphs can be thought of as in some sense opposite to the graphs considered in Lemma 14.

To conclude our paper, we point out that the above cages are not unique with respect their parameters. It is easy to see that the graph depicted in Figure 5 is also a $(3; 3, 4, 5, 6, 7, 8; 9)$-graph but it is not isomorphic to the graph $G_8 = G_9$.

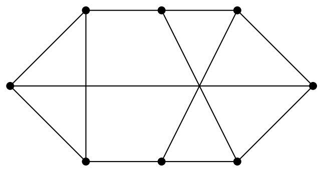


Figure 5: Smallest $(3; 3, 4, 5, 6, 7, 8; 9)$-graph not isomorphic to $G_8 = G_9$.

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