The concavity and convexity of the Boros–Moll sequences

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Abstract

In their study of a quartic integral, Boros and Moll discovered a special class of sequences, which is called the Boros–Moll sequences. In this paper, we consider the concavity and convexity of the Boros–Moll sequences $\{d_i(m)\}_{i=0}^m$. We show that for any integer $m \geqslant 6$, there exist two positive integers $t_0(m)$ and $t_1(m)$ such that $d_i(m)+d_{i+2}(m) > 2d_{i+1}(m)$ for $i \in [0, t_0(m)] \bigcup [t_1(m), m-2]$ and $d_i(m)+d_{i+2}(m) < 2d_{i+1}(m)$ for $i \in [t_0(m)+1, t_1(m)-1]$. When m is a square, we find $t_0(m) = \frac{m-\sqrt{m}-4}{2}$ and $t_1(m) = \frac{m+\sqrt{m}-2}{2}$. As a corollary of our results, we show that

$$\lim_{m\to+\infty} \frac{\operatorname{card}\{i \mid d_i(m) + d_{i+2}(m) < 2d_{i+1}(m), 0 \leqslant i \leqslant m-2\}}{\sqrt{m}} = 1.$$

Keywords: Boros–Moll sequences; concavity; convexity; log-concavity; log-convexity

1 Introduction and Main Results

The object of this paper is to study the concavity and convexity of the Boros–Moll sequences. Boros and Moll [4, 5, 6, 7, 8] explored a special class of Jacobi polynomials in their study of a quartic integral. They have shown that for any $a > -1$ and any nonnegative integer m,

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a), \tag{1}$$

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where

$$P_m(a) = \sum_{j,\ k} \binom{2m+1}{2j} \binom{m-j}{k} \binom{2k+2j}{k+j} \frac{(a+1)^j (a-1)^k}{2^{3(k+j)}}. \tag{2}$$

Using Ramanujan's Master Theorem, Boros and Moll [7, 18] derived the following formula

$$P_m(a) = 2^{-2m} \sum_k 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} (a+1)^k, \tag{3}$$

which indicates that the coefficients of a^i in $P_m(a)$ are positive for $0 \leqslant i \leqslant m$. Chen, Pang and Qu [12] gave a combinatorial proof to show that (2) is equal to (3). Let $d_i(m)$ be defined by

$$P_m(a) = \sum_{i=0}^{m} d_i(m) a^i. \tag{4}$$

The polynomials $P_m(a)$ will be called the Boros–Moll polynomials, and the sequences $\{d_i(m)\}_{i=0}^m$ of the coefficients will be called the Boros–Moll sequences. It follows from (3) and (4) that

$$d_i(m) = 2^{-2m} \sum_{k=i}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{k} \binom{k}{i}. \tag{5}$$

The readers can find in [3] many proofs of this formula. Recall that $P_m(a)$ can be expressed as a hypergeometric function

$$P_m(a) = 2^{-2m} \binom{2m}{m} {}_2F_1 \left(-m, m+1; \frac{1}{2} - m; \frac{a+1}{2} \right), \tag{6}$$

from which one sees that $P_m(a)$ can be viewed as the Jacobi polynomial $P_m^{(\alpha,\beta)}(a)$ with $\alpha = m + \frac{1}{2}$ and $\beta = -(m + \frac{1}{2})$, where $P_m^{(\alpha,\beta)}(a)$ is given by

$$P_m^{(\alpha,\beta)}(a) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m+\beta}{m-k} \binom{m+k+\alpha+\beta}{k} \left(\frac{1+a}{2} \right)^k. \tag{7}$$

Some combinatorial properties of the Boros–Moll sequences have been established. Boros and Moll [5] proved that the sequence $\{d_i(m)\}_{i=0}^m$ is unimodal and the maximum element appears in the middle, namely,

$$d_0(m) < d_1(m) < \cdots < d_{[\frac{m}{2}]}(m) > d_{[\frac{m}{2}]+1}(m) > \cdots > d_m(m). \tag{8}$$

They also established the unimodality of the sequence $\{d_i(m)\}_{i=0}^m$ by taking a different approach [6]. Amdeberhan, Dixit, Guan, Jiu and Moll [1] presented another proof of (8). Amdeberhan, Manna and Moll [2] analyzed properties of the 2-adic valuation of an integer sequence and gave a combinatorial interpretation of the valuations of the integer sequence which is related to the Boros–Moll sequences. Moll [18] conjectured that the sequence $\{d_i(m)\}_{i=0}^m$ is log-concave. Kauers and Paule [16] proved this conjecture based on the following four recurrence relations found by using the WZ-method [20]:

$$d_i(m+1) = \frac{m+i}{m+1} d_{i-1}(m) + \frac{(4m+2i+3)}{2(m+1)} d_i(m), \qquad 0 \leqslant i \leqslant m+1, \tag{9}$$

$$d_i(m+1) = \frac{(4m-2i+3)(m+i+1)}{2(m+1)(m+1-i)} d_i(m) - \frac{i(i+1)}{(m+1)(m+1-i)} d_{i+1}(m), \qquad 0 \leqslant i \leqslant m, \tag{10}$$

$$d_i(m+2) = \frac{-4i^2+8m^2+24m+19}{2(m+2-i)(m+2)} d_i(m+1) - \frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)} d_i(m), \qquad 0 \leqslant i \leqslant m+1, \tag{11}$$

and for $0 \leqslant i \leqslant m+1$,

$$(m+2-i)(m+i-1)d_{i-2}(m) - (i-1)(2m+1)d_{i-1}(m) + i(i-1)d_i(m) = 0. \tag{12}$$

In fact, the recurrences (11) and (12) are also derived independently by Moll [19] by using the WZ-method [20]. Chen and Gu [11] showed that the Boros–Moll sequences satisfy the reverse ultra log-concavity. Chen and Xia [13] proved that the Boros–Moll sequences satisfy the ratio monotone property which implies the log-concavity and the spiral property. They [14] also confirmed a conjecture given by Moll in [19]. By constructing an intermediate function, Chen and Xia [15] proved the 2-log-concavity of the Boros–Moll sequences. Chen, Dou and Yang [10] proved two conjectures of Brändén [9] on the real-rootedness of the polynomials $Q_n(x)$ and $R_n(x)$ which are related to the Boros–Moll polynomials $P_n(x)$. The first conjecture implies the 2-log-concavity of the Boros–Moll sequences, and the second conjecture implies the 3-log-concavity of the Boros–Moll sequences.

In this paper, we consider the concavity and convexity of the Boros–Moll sequences. Let $\{a_i\}_{i=0}^n$ be a sequence of real numbers. Recall that the sequence $\{a_i\}_{i=0}^n$ is said to be convex (resp. concave) if

$$a_i + a_{i+2} \geqslant 2a_{i+1} \quad (\text{resp.} \quad a_i + a_{i+2} \leqslant 2a_{i+1}) \tag{13}$$

for $0 \leqslant i \leqslant n-2$. It is easy to see that for positive sequences, the log-convexity implies the convexity and the concavity implies the log-concavity.

The main results of this paper can be stated as follows.

Theorem 1. *Let m, i be integers and $m \geqslant 6$. We have $d_i(m) + d_{i+2}(m) > 2d_{i+1}(m)$ for $i \in [0, t_0(m)] \bigcup [t_1(m), m-2]$ and $d_i(m) + d_{i+2}(m) < 2d_{i+1}(m)$ for $i \in [t_0(m)+1, t_1(m)-1]$,* *where* $t_0(m) = \frac{m}{2} - \frac{\sqrt{m}}{2} - 2$ *and* $t_1(m) = \frac{m}{2} + \frac{\sqrt{m}}{2} - 1$ *when* m *is a square;* $t_0(m) = [\frac{m}{2} - \frac{\sqrt{m}}{2} - 2]$ *or* $[\frac{m}{2} - \frac{\sqrt{m}}{2} - 1]$ *and* $t_1(m) = [\frac{m}{2} + \frac{\sqrt{m}}{2} - 1]$ *or* $[\frac{m}{2} + \frac{\sqrt{m}}{2}]$ *when* m *is not a square.*

In order to prove Theorem 1, we establish the following two Theorems:

Theorem 2. *Let* m, i *be integers and* $m \geqslant 6$. *We have* $d_i(m) + d_{i+2}(m) > 2d_{i+1}(m)$ *for* $i \in [0, \frac{m}{2} - \frac{\sqrt{m}}{2} - 2] \bigcup [\frac{m}{2} + \frac{\sqrt{m}}{2} - 1, m-2]$.

Theorem 3. *Let* m, i *be integers and* $m \geqslant 6$. *We have* $d_i(m) + d_{i+2}(m) < 2d_{i+1}(m)$ *for* $i \in [\frac{m}{2} - \frac{\sqrt{m}}{2} - 1, \frac{m}{2} + \frac{\sqrt{m}}{2} - 2]$.

Note that $\frac{m}{2} - \frac{\sqrt{m}}{2}$ is an integer if and only if m is a square. Therefore, from Theorems 2 and 3, we immediately prove Theorem 1. By Theorems 2 and 3, we can obtain the following corollary:

Corollary 4. *We have*

$$\lim_{m \to +\infty} \frac{\text{card}\{i | d_i(m) + d_{i+2}(m) < 2d_{i+1}(m),\ 0 \leqslant i \leqslant m-2\}}{\sqrt{m}} = 1. \tag{14}$$

To conclude this section, we propose an open problem. Determine the signs of the differences $d_{[\frac{m}{2} - \frac{\sqrt{m}}{2} - 1]}(m) + d_{[\frac{m}{2} - \frac{\sqrt{m}}{2} + 1]}(m) - 2d_{[\frac{m}{2} - \frac{\sqrt{m}}{2}]}(m)$ and $d_{[\frac{m}{2} + \frac{\sqrt{m}}{2} - 1]}(m) + d_{[\frac{m}{2} + \frac{\sqrt{m}}{2} + 1]}(m) - 2d_{[\frac{m}{2} + \frac{\sqrt{m}}{2}]}(m)$ when m is not a square.

2 Proofs of the Main Results

In this section, we present proofs of the main results. We first represent $d_i(m) + d_{i+2}(m) - 2d_{i+1}(m)$ in terms of $d_i(m)$ and $d_i(m+1)$.

Lemma 5. *For* $1 \leqslant i \leqslant m-2$, *we have*

$$d_i(m) + d_{i+2}(m) - 2d_{i+1}(m) = A(m,i)d_i(m+1) + B(m,i)d_i(m) \tag{15}$$

where

$$A(m,i) = -\frac{(m+1)(m+1-i)(2m-2i-3)}{i(i+1)(i+2)}, \tag{16}$$

$$B(m,i) = \frac{(8m^3 - 15m - 5i - 8mi^2 - 6m^2 i - 20mi + 2m^2 + 12i^2 + 8i^3 - 9)}{2i(i+1)(i+2)}. \tag{17}$$

Proof. It follows from (10) and (12) that for $1 \leqslant i \leqslant m-1$,

$$d_{i+1}(m) = \frac{(4m-2i+3)(m+i+1)}{2i(i+1)} d_i(m) - \frac{(m+1-i)(m+1)}{i(i+1)} d_i(m+1) \tag{18}$$

and

$$d_{i+2}(m) = \frac{2m+1}{i+2} d_{i+1}(m) - \frac{(m-i)(m+i+1)}{(i+1)(i+2)} d_i(m). \tag{19}$$

Lemma 5 follows from (18) and (19). This completes the proof.

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. It is a routine to verify that Theorem 2 holds for $6 \leqslant m \leqslant 9$. So, we can assume that $m \geqslant 10$. It is easy to check that for $1 \leqslant i \leqslant m-2$,

$$\begin{aligned} -2(m+1)(m+1-i)\frac{B(m,i)}{A(m,i)} &- (4m^2+7m+3-2i^2) \\ &= \frac{i(4i^2-4im+2m^2+6i-6m+1)}{(2m-2i-3)} > 0. \end{aligned} \tag{20}$$

It is easy to verify that

$$\left(\frac{i(4i^2-4im+2m^2+6i-6m+1)}{(2m-2i-3)}\right)^2 - i^2(4m+4i^2+1) = \frac{4i^2 G(m,i)}{(2m-2i-3)^2}, \tag{21}$$

where

$$\begin{aligned} G(m,i) =&(4m^2-16m+1)i^2 - (4m^3-26m^2+30m)i \\ &+ m^4 - 10m^3 + 21m^2 - 9m - 2. \end{aligned} \tag{22}$$

In Section 3, we will prove that for $i \in [0, \frac{m}{2} - \frac{\sqrt{m}}{2} - 2] \bigcup [\frac{m}{2} + \frac{\sqrt{m}}{2} - 1, m-2]$ and $m \geqslant 10$,

$$G(m,i) \geqslant 0. \tag{23}$$

Combining (20), (21) and (23), we deduce that

$$\frac{i(4i^2-4im+2m^2+6i-6m+1)}{(2m-2i-3)} > i\sqrt{4m+4i^2+1}. \tag{24}$$

It follows from (20) and (24) that for $i \in [1, \frac{m}{2} - \frac{\sqrt{m}}{2} - 2] \bigcup [\frac{m}{2} + \frac{\sqrt{m}}{2} - 1, m-2]$ and $m \geqslant 10$,

$$-\frac{B(m,i)}{A(m,i)} > \frac{4m^2+7m+3-2i^2+i\sqrt{4m+4i^2+1}}{2(m+1)(m+1-i)}. \tag{25}$$

In order to establish the reverse ultra log-concavity of $\{d_i(m)\}_{i=0}^m$, Chen and Gu [11] gave an upper bound of the ratio $d_i(m+1)/d_i(m)$. They proved that for $m \geqslant 2$ and $0 \leqslant i \leqslant m$,

$$\frac{d_i(m+1)}{d_i(m)} \leqslant \frac{4m^2+7m+3-2i^2+i\sqrt{4m+4i^2+1}}{2(m+1)(m+1-i)}. \tag{26}$$

By (25) and (26), we see that for $i \in [1, \frac{m}{2} - \frac{\sqrt{m}}{2} - 2] \bigcup [\frac{m}{2} + \frac{\sqrt{m}}{2} - 1, m-2]$ and $m \geqslant 10$,

$$-\frac{B(m,i)}{A(m,i)} > \frac{d_i(m+1)}{d_i(m)}. \tag{27}$$

It should be noted that $A(m,i) < 0$ for $1 \leqslant i \leqslant m-1$. Thus, the above inequality can be rewritten as

$$A(m,i)d_i(m+1) + B(m,i)d_i(m) > 0. \tag{28}$$

By Lemma 5 and (28), we see that $d_i(m) + d_{i+2}(m) - 2d_{i+1}(m) > 0$ for $m \geqslant 10$ and $i \in [1, \frac{m}{2} - \frac{\sqrt{m}}{2} - 2] \bigcup [\frac{m}{2} + \frac{\sqrt{m}}{2} - 1, m-2]$.

It remains to verify that $d_0(m) + d_2(m) > 2d_1(m)$ for $m \geqslant 10$. In (12), let $i = 2$, we have

$$d_0(m) = \frac{2m+1}{m(m+1)}d_1(m) - \frac{2}{m(m+1)}d_2(m). \tag{29}$$

Therefore,

$$d_0(m) + d_2(m) - 2d_1(m) = \frac{m^2+m-2}{m(m+1)}d_2(m) - \frac{2m^2-1}{m(m+1)}d_1(m). \tag{30}$$

Employing (5), we deduce that

$$d_{m-2}(m) = \frac{(m-1)(4m^2+2m+1)}{2^{m+2}(2m-1)}\binom{2m}{m} \tag{31}$$

and

$$d_{m-3}(m) = 2^{-m}\frac{(m-2)(2m+1)(4m^2-2m+3)}{24(2m-1)}\binom{2m}{m}. \tag{32}$$

By (31), (32) and the ratio monotone property of the Boros–Moll sequences established by Chen and Xia in [13], we have

$$\frac{d_2(m)}{d_1(m)} > \frac{d_{m-3}(m)}{d_{m-2}(m)} = \frac{(m-2)(2m+1)(4m^2-2m+3)}{6(m-1)(4m^2+2m+1)} > \frac{2m^2-1}{m^2+m-2}, \tag{33}$$

which implies that the left hand side of (30) is positive for $m \geqslant 10$. This completes the proof.

Now we turn to prove Theorem 3.

Proof of Theorem 3. It is easy to check that Theorem 3 is true for $6 \leqslant m \leqslant 931$ by Maple. In the following, we assume that $m \geqslant 932$. It is easy to verify that

$$\begin{aligned}&\frac{4m^2+7m+\frac{m}{i+2}i+4}{2(m+1)(m+1-i)} + \frac{B(m,i)}{A(m,i)}\\ =&\frac{-H(m,i)}{2(m+1)(m+1-i)(i+2)(2m-2i-3)},\end{aligned} \tag{34}$$

where

$$H(m,i) = 8i^4 - 8i^3m + 2i^2m^2 + 28i^3 - 20i^2m + 2im^2 + 27i^2 - 11im + 9i - 4m + 6. \tag{35}$$

We can prove that for $m \geqslant 932$ and $i \in [\frac{m}{2} - \frac{\sqrt{m}}{2} - 1, \frac{m}{2} + \frac{\sqrt{m}}{2} - 2]$,

$$H(m,i) < 0. \tag{36}$$

The proof of (36) is analogous to the proof of (23), and hence is omitted. In Section 4, we will prove that for $m \geqslant 55$ and $[\frac{2m}{5}] + 1 \leqslant i \leqslant m-1$,

$$\frac{d_i(m+1)}{d_i(m)} \geqslant \frac{4m^2 + 7m + \frac{m}{i+2}i + 4}{2(m+1)(m+1-i)}. \tag{37}$$

In view of (34), (36) and (37), we find that for $i \in [\frac{m}{2} - \frac{\sqrt{m}}{2} - 1, \frac{m}{2} + \frac{\sqrt{m}}{2} - 2]$ and $m \geqslant 932$,

$$\frac{d_i(m+1)}{d_i(m)} \geqslant \frac{4m^2 + 7m + \frac{m}{i+2}i + 4}{2(m+1)(m+1-i)} > -\frac{B(m,i)}{A(m,i)}, \tag{38}$$

which implies

$$A(m,i)d_i(m+1) + B(m,i)d_i(m) < 0. \tag{39}$$

This is because $A(m,i) < 0$ for $1 \leqslant i \leqslant m-1$. In view of Lemma 5 and (39), we deduce that $d_i(m) + d_{i+2}(m) - 2d_{i+1}(m) < 0$ for $i \in [\frac{m}{2} - \frac{\sqrt{m}}{2} - 1, \frac{m}{2} + \frac{\sqrt{m}}{2} - 2]$ and $m \geqslant 932$. This completes the proof.

Proof of Corollary 4. It follows from Theorem 2 that

$$\lim_{m\to+\infty} \frac{\mathrm{card}\{i | d_i(m) + d_{i+2}(m) < 2d_{i+1}(m), 0 \leqslant i \leqslant m-2\}}{\sqrt{m}} \leqslant 1. \tag{40}$$

Theorem 3 implies that

$$\lim_{m\to+\infty} \frac{\mathrm{card}\{i | d_i(m) + d_{i+2}(m) < 2d_{i+1}(m), 0 \leqslant i \leqslant m-2\}}{\sqrt{m}} \geqslant 1. \tag{41}$$

Corollary 4 follows from (40) and (41). The proof is complete.

3 Proof of (23)

In this section, we present a proof of (23).

It is a routine to verify that for $m \geqslant 10$,

$$2(4m^2 - 16m + 1)\left(\frac{m}{2} - \frac{\sqrt{m}}{2} - 2\right) - (4m^3 - 26m^2 + 30m) = -4m^{5/2} - 6m^2 + 16m^{3/2} + 35m - m^{1/2} - 4 < 0. \tag{42}$$

Also, it is easy to check that for $m \geqslant 6$,

$$2(4m^2-16m+1)\left(\frac{m}{2}+\frac{\sqrt{m}}{2}-1\right)-(4m^3-26m^2+30m)$$
$$=4m^{5/2}+2m^2-16m^{3/2}+3m+m^{1/2}-2>0. \tag{43}$$

It follows from (42) and (43) that for $m \geqslant 10$,

$$\frac{m}{2}-\frac{\sqrt{m}}{2}-2<\frac{4m^3-26m^2+30m}{2(4m^2-16m+1)}<\frac{m}{2}+\frac{\sqrt{m}}{2}-1. \tag{44}$$

It should be noted that $4m^2-16m+1>0$ for $m \geqslant 6$. Therefore, for $m \geqslant 10$ and $i \in [0, \frac{m}{2}-\frac{\sqrt{m}}{2}-2]\bigcup[\frac{m}{2}+\frac{\sqrt{m}}{2}-1, m-2]$, we obtain

$$G(m,i) \geqslant \min\left\{G(m,\frac{m}{2}-\frac{\sqrt{m}}{2}-2), G(m,\frac{m}{2}+\frac{\sqrt{m}}{2}-1)\right\}. \tag{45}$$

It is easy to verify that for $m \geqslant 10$,

$$G(m,\frac{m}{2}-\frac{\sqrt{m}}{2}-2)=3m^{5/2}-\frac{7}{4}m^2-\frac{35}{2}m^{3/2}-\frac{59}{4}m+2m^{1/2}+2>0 \tag{46}$$

and

$$G(m,\frac{m}{2}+\frac{\sqrt{m}}{2}-1)=m^{5/2}-\frac{15}{4}m^2+\frac{3}{2}m^{3/2}+\frac{17}{4}m-m^{1/2}-1>0. \tag{47}$$

Inequality (23) follows from (45), (46) and (47). This completes the proof.

4 Proof of (37)

In this section, we provide a proof of (37).

We are ready to prove (37) by induction on m. It is easy to check that (37) holds for $m = 55$. We assume that (37) is true for $n \geqslant 55$, i.e.,

$$\frac{d_i(n+1)}{d_i(n)} \geqslant \frac{4n^2+7n+\frac{n}{i+2}i+4}{2(n+1-i)(n+1)}, \qquad \left\lceil\frac{2n}{5}\right\rceil+1 \leqslant i \leqslant n-1. \tag{48}$$

We aim to prove that (37) holds for $n+1$, that is,

$$\frac{d_i(n+2)}{d_i(n+1)} \geqslant \frac{4(n+1)^2+7(n+1)+\frac{n+1}{i+2}i+4}{2(n+2-i)(n+2)}, \quad \left\lceil\frac{2(n+1)}{5}\right\rceil+1 \leqslant i \leqslant n. \tag{49}$$

It is easy to check that

$$\frac{4n^2+7n+\frac{n}{i+2}i+4}{2(n+1-i)(n+1)}-\frac{(2+i)(4n+5)(4n+3)(n+i+1)}{-2(8i^2+4i^3-8n^2-4n^2i-18n-8ni-8-3i)(n+1)}$$
$$=\frac{F(n,i)}{2(8n^2+4n^2i+18n+8ni+8+3i-8i^2-4i^3)(2+i)(n+1)(n+1-i)}, \tag{50}$$

where $F(n,i)$ is given by

$$F(n,i) = 4 - 4i + 8n - 7i^2 - 4n^2 i - 3n^2 i^2 - 6ni - 4i^3 - i^4 + 10ni^2 + 8ni^3. \tag{51}$$

Let $f(i)$ and $g(i)$ be defined by

$$f(i) = 8ni - 3n^2 - i^2, \qquad g(i) = 10ni - 4n^2 - 4i^2. \tag{52}$$

For $\left[\frac{2n+2}{5}\right] + 1 \leqslant i \leqslant n-1$, we find that

$$f(i) \geqslant f\left(\frac{2n}{5}\right) = \frac{n^2}{25}, \qquad g(i) \geqslant g\left(\frac{2n}{5}\right) = -\frac{16n^2}{25}. \tag{53}$$

In view of (51), (52) and (53), we deduce that for $\left[\frac{2n+2}{5}\right] + 1 \leqslant i \leqslant n-1$ and $n \geqslant 55$,

$$\begin{aligned} F(n,i) =& f(i)i^2 + g(i)i + (8n + 4 - 4i) - (7i^2 + 6ni) \\ >& \frac{1}{25}n^2 i^2 - \frac{16}{25}n^2 i - 13ni \geqslant \frac{ni}{25}\left(\frac{2n^2}{5} - 16n - 325\right) > 0. \end{aligned} \tag{54}$$

It follows from (50) and (54) that for $\left[\frac{2n+2}{5}\right] + 1 \leqslant i \leqslant n-1$ and $n \geqslant 55$,

$$\frac{4n^2 + 7n + \frac{n}{i+2}i + 4}{2(n+1-i)(n+1)} \geqslant \frac{(2+i)(4n+5)(4n+3)(n+i+1)}{-2(8i^2 + 4i^3 - 8n^2 - 4n^2 i - 18n - 8ni - 8 - 3i)(n+1)}. \tag{55}$$

By (48) and (55), we deduce that for $\left[\frac{2n+2}{5}\right] + 1 \leqslant i \leqslant n-1$,

$$\frac{d_i(n+1)}{d_i(n)} \geqslant \frac{(2+i)(4n+5)(4n+3)(n+i+1)}{-2(8i^2 + 4i^3 - 8n^2 - 4n^2 i - 18n - 8ni - 8 - 3i)(n+1)}. \tag{56}$$

It is a routine to verify that

$$\begin{aligned} &\frac{\frac{(n+i+1)(4n+3)(4n+5)}{4(n+2-i)(n+1)(n+2)}}{\frac{-4i^2 + 8n^2 + 24n + 19}{2(n+2-i)(n+2)} - \frac{4(n+1)^2 + 7(n+1) + \frac{n+1}{i+2}i + 4}{2(n+2)(n+2-i)}} \\ &\qquad = \frac{(2+i)(4n+5)(4n+3)(n+i+1)}{2(8n^2 + 4n^2 i + 18n + 8ni + 8 + 3i - 8i^2 - 4i^3)(n+1)}, \end{aligned} \tag{57}$$

which implies that

$$\frac{-4i^2 + 8n^2 + 24n + 19}{2(n+2-i)(n+2)} - \frac{4(n+1)^2 + 7(n+1) + \frac{n+1}{i+2}i + 4}{2(n+2)(n+2-i)} > 0. \tag{58}$$

By (57), we can rewrite (56) as follows

$$d_i(n+1) \geqslant \frac{\dfrac{(n+i+1)(4n+3)(4n+5)}{4(n+2-i)(n+1)(n+2)}}{\dfrac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} - \dfrac{4(n+1)^2+7(n+1)+\frac{n+1}{i+2}i+4}{2(n+2)(n+2-i)}} d_i(n). \tag{59}$$

It follows from (58) and (59) that for $\left[\frac{2n+2}{5}\right]+1 \leqslant i \leqslant n-1$,

$$\begin{aligned} &\frac{-4i^2+8n^2+24n+19}{2(n+2-i)(n+2)} d_i(n+1) - \frac{(n+i+1)(4n+3)(4n+5)}{4(n+2-i)(n+1)(n+2)} d_i(n) \\ &\qquad \geqslant \frac{4(n+1)^2+7(n+1)+\frac{n+1}{i+2}i+4}{2(n+2-i)(n+2)} d_i(n+1). \end{aligned} \tag{60}$$

By (11), we find that the left hand side of (60) equals $d_i(n+2)$. Thus we have verified the inequality (49) for $\left[\frac{2n+2}{5}\right]+1 \leqslant i \leqslant n-1$. It is still necessary to show that (49) is true for $i=n$, that is,

$$\frac{d_n(n+2)}{d_n(n+1)} \geqslant \frac{4(n+1)^2+7(n+1)+\frac{n+1}{n+2}n+4}{4(n+2)}. \tag{61}$$

By (5), we get

$$d_n(n+1) = 2^{-n-2}(2n+3)\binom{2n+2}{n+1}. \tag{62}$$

It follows from (31) and (62) that

$$\frac{d_n(n+2)}{d_n(n+1)} = \frac{(n+1)(4n^2+18n+21)}{2(n+2)(2n+3)} \geqslant \frac{4(n+1)^2+7(n+1)+\frac{n+1}{n+2}n+4}{4(n+2)}, \tag{63}$$

which yields (61). Hence the proof is complete by induction.

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References

[1] T. Amdeberhan, A. Dixit, X. Guan, L. Jiu, and V.H. Moll. The unimodality of a polynomial coming from a rational integral. Back to the original proof. *J. Math. Anal. Appl.*, 420:1154–1166, 2014.

[2] T. Amdeberhan, D. Manna, and V.H. Moll. The 2-adic valuation of a sequence arising from a rational integral. *J. Combin. Theory Ser. A*, 115:1474–1486, 2008.

[3] T. Amdeberhan and V.H. Moll. A formula for a quartic integral: a survey of old proofs and some new ones. *Ramanujan J.*, 18:91–102, 2009.

[4] G. Boros and V.H. Moll. An integral hidden in Gradshteyn and Ryzhik. *J. Comput. Appl. Math.*, 106:361–368, 1999.

[5] G. Boros and V.H. Moll. A sequence of unimodal polynomials. *J. Math. Anal. Appl.* 237:272–285, 1999.

[6] G. Boros and V.H. Moll. A criterion for unimodality. *Electron. J. Combin.* 6: R3, 1999.

[7] G. Boros and V.H. Moll. The double square root, Jacobi polynomials and Ramanujan's Master Theorem. *J. Comput. Appl. Math.*, 130:337–344, 2001.

[8] G. Boros and V.H. Moll. *Irresistible Integrals.* Cambridge University Press, Cambridge, 2004.

[9] P. Brändén. Iterated sequences and the geometry of zeros. *J. Reine Angew. Math.* 658:115–131, 2011

[10] W.Y.C. Chen, D.Q.J. Dou, and A.L.B. Yang. Brändén's conjectures on the Boros–Moll polynomials. *Inter. Math. Res. Notices*, 2013:4819–4828, 2013.

[11] W.Y.C. Chen and C.C.Y. Gu. The reverse ultra log-concavity of the Boros–Moll polynomials. *Proc. Amer. Math. Soc.*, 137:3991–3998, 2009.

[12] W.Y.C. Chen, S.X.M. Pang, and E.X.Y. Qu. On the combinatorics of the Boros–Moll polynomials. *Ramanujan J.*, 21:41–51, 2010.

[13] W.Y.C. Chen and E.X.W. Xia. The ratio monotonicity of Boros–Moll polynomials. *Math. Comput.*, 78:2269–2282, 2009.

[14] W.Y.C. Chen and E.X.W. Xia. Proof of Moll's minimum conjecture. *European J. Combin.*, 34:787–791, 2013.

[15] W.Y.C. Chen and E.X.W. Xia. 2-log-concavity of the Boros–Moll polynomials. *Proc. Edinburgh Math. Soc.*, 56:701–722, 2013.

[16] M. Kauers and P. Paule. A computer proof of Moll's log-concavity conjecture. *Proc. Amer. Math. Soc.*, 135:3847–3856, 2007.

[17] D.V. Manna and V.H. Moll. A remarkable sequence of integers. *Expo. Math.*, 27:289–312, 2009.

[18] V.H. Moll. The evaluation of integrals: A personal story. *Notices Amer. Math. Soc.*, 49:311–317, 2002.

[19] V.H. Moll. Combinatorial sequences arising from a rational integral. *Online J. Anal. Combin.*, 2:Article 4, 2007.

[20] H.S. Wilf and D. Zeilberger. An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. *Invent. Math.*, 108:575–633, 1992.