A Deza–Frankl type theorem for set partitions

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Abstract

A set partition of $[n]$ is a collection of pairwise disjoint nonempty subsets (called blocks) of $[n]$ whose union is $[n]$. Let $\mathcal{B}(n)$ denote the family of all set partitions of $[n]$. A family $\mathcal{A} \subseteq \mathcal{B}(n)$ is said to be m-intersecting if any two of its members have at least m blocks in common. For any set partition $P \in \mathcal{B}(n)$, let $\tau(P) = \{x : \{x\} \in P\}$ denote the union of its singletons. Also, let $\mu(P) = [n] - \tau(P)$ denote the set of elements that do not appear as a singleton in P. Let

$$\mathcal{F}_{2t} = \{P \in \mathcal{B}(n) \ : \ |\mu(P)| \leqslant t\}\,;$$
$$\mathcal{F}_{2t+1}(i_0) = \{P \in \mathcal{B}(n) \ : \ |\mu(P) \cap ([n] \setminus \{i_0\})| \leqslant t\}\,.$$

In this paper, we show that for $r \geqslant 3$, there exists a constant $n_0 = n_0(r)$ depending on r such that for all $n \geqslant n_0$, if $\mathcal{A} \subseteq \mathcal{B}(n)$ is $(n-r)$-intersecting, then

$$|\mathcal{A}| \leqslant \begin{cases} |\mathcal{F}_{2t}|, & \text{if } r = 2t; \\ |\mathcal{F}_{2t+1}(1)|, & \text{if } r = 2t+1. \end{cases}$$

Moreover, equality holds if and only if

$$\mathcal{A} = \begin{cases} \mathcal{F}_{2t}, & \text{if } r = 2t; \\ \mathcal{F}_{2t+1}(i_0), & \text{if } r = 2t+1, \end{cases}$$

for some $i_0 \in [n]$.

Keywords: t-intersecting family, Erdős-Ko-Rado, set partitions

1 Introduction

Let $[n]=\{1,\ldots,n\}$, and let $\binom{[n]}{k}$ denote the family of all k-subsets of $[n]$. A family $\mathcal{A}$ of subsets of $[n]$ is *t-intersecting* if $|A\cap B|\geqslant t$ for all $A,B\in\mathcal{A}$. One of the most beautiful results in extremal combinatorics is the Erdős-Ko-Rado theorem.

Theorem 1 (Erdős, Ko, and Rado [13], Frankl [15], Wilson [46])**.** *Suppose $\mathcal{A}\subseteq\binom{[n]}{k}$ is t-intersecting and $n>2k-t$. Then for $n\geqslant(k-t+1)(t+1)$, we have*

$$|\mathcal{A}|\leqslant\binom{n-t}{k-t}.$$

Moreover, if $n>(k-t+1)(t+1)$ then equality holds if and only if $\mathcal{A}=\{A\in\binom{[n]}{k}\ :\ T\subseteq A\}$ for some t-set T.

In the celebrated paper [1], Ahlswede and Khachatrian extended the Erdős-Ko-Rado theorem by determining the structure of all t-intersecting set systems of maximum size for all possible n (see also [3, 14, 16, 17, 20, 25, 31, 36, 40, 42, 43, 45] for some related results). There have been many recent results showing that a version of the Erdős-Ko-Rado theorem holds for combinatorial objects other than set systems. For example, an analogue of the Erdős-Ko-Rado theorem for the Hamming scheme is proved in [41]. A complete solution for the t-intersection problem in the Hamming space is given in [2]. Intersecting families of permutations were initiated by Deza and Frankl in [10]. Some recent work done on this problem and its variants can be found in [5, 7, 8, 11, 12, 19, 26, 28, 35, 37, 38, 39, 44]. The investigation of the Erdős-Ko-Rado property for graphs started in [23], and gave rise to [4, 6, 21, 22, 24, 47]. The Erdős-Ko-Rado type results also appear in vector spaces [9, 18], set partitions [27, 29, 30] and weak compositions [32, 33, 34].

Let S_n denote the set of permutations of $[n]$. A family $\mathcal{A}\subseteq S_n$ is said to be m-intersecting if for any $\sigma,\delta\in\mathcal{A}$, there is an m-set $T\subseteq[n]$ such that $\sigma(j)=\delta(j)$ for all $j\in T$. Given any $\sigma\in S_n$, set $\overline{\mu}(\sigma)=\{j\in[n]\ :\ \mu(j)\neq j\}$, i.e., $\overline{\mu}(\sigma)$ is the set of all elements in $[n]$ that are not fixed by σ. Let

$$\overline{\mathcal{F}}_r=\begin{cases}\{\sigma\in S_n\ :\ |\overline{\mu}(\sigma)|\leqslant t\}\,, & \text{if } r=2t;\\ \{\sigma\in S_n\ :\ |\overline{\mu}(\sigma)\cap([n]\setminus\{1\})|\leqslant t\}\,, & \text{if } r=2t+1.\end{cases}$$

It can be verified easily that $\overline{\mathcal{F}}_r$ is $(n-r)$-intersecting. Furthermore, Deza and Frankl [10] proved the following theorem.

Theorem 2 (Deza-Frankl)**.** *For $r\geqslant 3$, there exists an $n_0=n_0(r)$ such that for all $n\geqslant n_0$, if $\mathcal{A}\subseteq S_n$ is $(n-r)$-intersecting, then*

$$|\mathcal{A}|\leqslant|\overline{\mathcal{F}}_r|.$$

A set partition of $[n]$ is a collection of pairwise disjoint nonempty subsets (called *blocks*) of $[n]$ whose union is $[n]$. Let $\mathcal{B}(n)$ denote the family of all set partitions of $[n]$.

It is well-known that the size of $\mathcal{B}(n)$ is the n-th Bell number, denoted by B_n. A block of size one is also known as a *singleton*. We denote the number of all set partitions of $[n]$ which are singleton-free (i.e. without any singleton) by $\tilde{B}_n$.

A family $\mathcal{A} \subseteq \mathcal{B}(n)$ is said to be *m-intersecting* if $|P \cap Q| \geqslant m$ for all $P, Q \in \mathcal{A}$, i.e., any two of its members have at least m blocks in common. Let $I(n, m)$ denote the set of all m-intersecting families of set partitions of $[n]$.

For any set partition $P \in \mathcal{B}(n)$, let $\tau(P) = \{x : \{x\} \in P\}$ denote the union of its singletons. Also, let $\mu(P) = [n] - \tau(P)$ denote the set of elements that do not appear as a singleton in P. For any two partitions P, Q, we make the following simple observations:

- P and Q cannot intersect in any singleton $\{x\}$ where $x \in \mu(P) \triangle \mu(Q)$ (here the operation $\triangle$ denotes the symmetric difference of two sets).
- P and Q must intersect in every singleton $\{x\}$ where $x \in [n] - (\mu(P) \cup \mu(Q))$.

Let

$$\mathcal{F}_{2t} = \{P \in \mathcal{B}(n) \ : \ |\mu(P)| \leqslant t\}\,;$$
$$\mathcal{F}_{2t+1}(i_0) = \{P \in \mathcal{B}(n) \ : \ |\mu(P) \cap ([n] \setminus \{i_0\})| \leqslant t\}\,.$$

It can be readily verified that $\mathcal{F}_{2t} \in I(n, n-2t)$ and $\mathcal{F}_{2t+1}(i_0) \in I(n, n-2t-1)$. Moreover,

$$|\mathcal{F}_{2t}| = \sum_{i=0}^{t} \tilde{B}_i \binom{n}{i}, \tag{1}$$

$$|\mathcal{F}_{2t+1}(i_0)| = \sum_{i=0}^{t} \tilde{B}_i \binom{n}{i} + \tilde{B}_{t+1} \binom{n-1}{t}. \tag{2}$$

In this paper, we will prove the following theorem.

Theorem 3. *For $r \geqslant 3$, there exists an $n_0 = n_0(r)$ such that for all $n \geqslant n_0$, if $\mathcal{A} \subseteq \mathcal{B}(n)$ is $(n-r)$-intersecting, then*

$$|\mathcal{A}| \leqslant \begin{cases} |\mathcal{F}_{2t}|, & \text{if } r = 2t; \\ |\mathcal{F}_{2t+1}(1)|, & \text{if } r = 2t+1. \end{cases}$$

Moreover, equality holds if and only if

$$\mathcal{A} = \begin{cases} \mathcal{F}_{2t}, & \text{if } r = 2t; \\ \mathcal{F}_{2t+1}(i_0), & \text{if } r = 2t+1, \end{cases}$$

for some $i_0 \in [n]$.

Note that Theorem 3 can be considered as an analogue of Theorem 2 for set partitions. Let $A_0 = \{\{x\} \ : \ x \in [n]\}$ and $A_1 = \{\{x\} \ : \ x \in [n] \setminus \{1,2\}\} \cup \{\{1,2\}\}$. Then $\mathcal{F}_2 = \{A_0\}$ and $\{A_0, A_1\} \in I(n, n-2)$. So, $|\{A_0, A_1\}| = 2 > |\mathcal{F}_2| = 1$. This explains why $r \geqslant 3$ is required in Theorem 3.

2 Splitting operation

In this section, we summarize some important results regarding the splitting operation for intersecting family of set partitions. We refer the reader to [27] for proofs which are omitted here.

Let $i, j \in [n]$, $i \neq j$, and $P \in \mathcal{B}(n)$. Denote by $P_{[i]}$ the block of P which contains i. We define the (i, j)-*split* of P to be the following set partition:

$$s_{ij}(P) = \begin{cases} P \setminus \{P_{[i]}\} \cup \{\{i\}, P_{[i]} \setminus \{i\}\} & \text{if } j \in P_{[i]}, \\ P & \text{otherwise.} \end{cases}$$

For a family $\mathcal{A} \subseteq \mathcal{B}(n)$, let $s_{ij}(\mathcal{A}) = \{s_{ij}(P) : P \in \mathcal{A}\}$. Any family $\mathcal{A}$ of set partitions can be decomposed with respect to given i, $j \in [n]$ as follows:

$$\mathcal{A} \;\; = \;\; (\mathcal{A} \setminus \mathcal{A}_{ij}) \cup \mathcal{A}_{ij},$$

where $\mathcal{A}_{ij} = \{P \in \mathcal{A} : s_{ij}(P) \notin \mathcal{A}\}$. Define the (i, j)-*splitting* of $\mathcal{A}$ to be the family

$$S_{ij}(\mathcal{A}) = (\mathcal{A} \setminus \mathcal{A}_{ij}) \cup s_{ij}(\mathcal{A}_{ij}).$$

Surprisingly, it turns out that for any $\mathcal{A} \in I(n, m)$, splitting operations preserve the size and the intersecting property.

Lemma 4 ([27], Proposition 3.2)**.** *Let $\mathcal{A} \in I(n, m)$. Then $S_{ij}(\mathcal{A}) \in I(n, m)$ and $|S_{ij}(\mathcal{A})| = |\mathcal{A}|$.*

A family $\mathcal{A}$ of set partitions is *compressed* if for any $i, j \in [n]$, $i \neq j$, we have $S_{ij}(\mathcal{A}) = \mathcal{A}$.

Lemma 5 ([27], Proposition 3.3)**.** *Given a family $\mathcal{A} \in I(n, t)$, by repeatedly applying the splitting operations, we eventually obtain a compressed family $\mathcal{A}^* \in I(n, t)$ with $|\mathcal{A}^*| = |\mathcal{A}|$.*

Lemma 6. *Let a, b be positive integers with $a + b \leqslant n$. Let $P, Q \in \mathcal{B}(n)$ be such that $|P \cap Q| \geqslant n - a$. If $|\tau(P) \setminus \mu(Q)| \leqslant n - a - b$, then P and Q have at least b blocks of size at least 2 in common and $|\mu(P) \cap \mu(Q)| \geqslant 2b$.*

Proof. Since $|\tau(P) \setminus \mu(Q)| \leqslant n - a - b$, P and Q have at most $n - a - b$ singletons in common. Now, $|P \cap Q| \geqslant n - a$ means that P and Q have at least $n - a$ blocks in common. Therefore, P and Q must have at least b blocks of size at least 2 in common. Let $W_1, \ldots, W_b \in P \cap Q$ with $|W_i| \geqslant 2$ for all i. Then $\bigcup_{i=1}^{b} W_i \subseteq \mu(P) \cap \mu(Q)$ and $W_i \cap W_j = \varnothing$ for $i \neq j$. This implies that $2b \leqslant \sum_{i=1}^{b} |W_i| \leqslant |\mu(P) \cap \mu(Q)|$. □

Lemma 7. *If $\mathcal{A} \in I(n, n - r)$, then $\max_{P \in \mathcal{A}} |\mu(P)| \leqslant 2r$.*

Proof. Suppose $\max_{P \in \mathcal{A}} |\mu(P)| = 2r + s$ where $s \geqslant 1$. Let $P_0 \in \mathcal{A}$ with $|\mu(P_0)| = 2r + s$. Then $|\tau(P_0)| = n - 2r - s$. Note that $|P_0| \geqslant n - r$ for $\mathcal{A} \in I(n, n - r)$. By Lemma 6 (take $Q = P = P_0$ with $a = r$, $b = r + s$), we have $2r + s = |\mu(P_0)| \geqslant 2(r + s)$. Thus, we have $s \leqslant 0$, a contradiction. Hence, the lemma follows. □

The following theorem says that the family $\mathcal{F}_{2t+1}(i_0)$ is preserved when 'undoing' the splitting operations.

Theorem 8. *If* $t \geqslant 1$, $n \geqslant 5t+3$, $\mathcal{A} \in I(n, n-2t-1)$ *and* $S_{ij}(\mathcal{A}) = \mathcal{F}_{2t+1}(i_0)$, *then* $\mathcal{A} = \mathcal{F}_{2t+1}(i_0)$.

Proof. Suppose $\mathcal{A} \nsubseteq \mathcal{F}_{2t+1}(i_0)$. Then $\max_{P\in\mathcal{A}} |\mu(P) \cap ([n] \setminus \{i_0\})| = t+s$ with $s \geqslant 1$. Let $P_0 \in \mathcal{A}$ with $|\mu(P_0) \cap ([n] \setminus \{i_0\})| = t+s$. Then $|\mu(P_0)| = t+1+s$ or $t+s$, depending on whether $i_0 \in \mu(P_0)$ or not. By Lemma 7, $|\mu(P_0)| \leqslant 4t+2$. Since $n \geqslant 5t+3$, there is a t-set $T \subseteq [n] \setminus (\mu(P_0) \cup \{i_0\})$. Let $A_1 = \{\{x\} \,:\, x \in [n] \setminus (T \cup \{i_0\})\} \cup \{T \cup \{i_0\}\}$. Then $A_1 \in \mathcal{F}_{2t+1}(i_0) = S_{ij}(\mathcal{A})$.

Now, $|\tau(P_0) \setminus (T \cup \{i_0\})| = n-2t-1-s$, $|T \cup \{i_0\}| = t+1 \geqslant 2$ and $(T \cup \{i_0\}) \nsubseteq \mu(P_0)$. The only block in A_1 that has size greater than one is $T \cup \{i_0\}$. Since $(T \cup \{i_0\}) \nsubseteq \mu(P_0)$, $T \cup \{i_0\} \notin P_0$. So, P_0 and A_1 have singletons in common only. Note that the number of singletons that P_0 and A_1 have in common is exactly $|\tau(P_0) \setminus (T \cup \{i_0\})| = n-2t-1-s$. Thus, $|P_0 \cap A_1| \leqslant n-2t-1-s \leqslant n-2t-2$. This means that $A_1 \notin \mathcal{A}$ and $A_1 = s_{ij}(C_1)$ for some $C_1 \in \mathcal{A}$.

Now, there are two possibilities for C_1 depending on whether j is in $T \cup \{i_0\}$ or not:

(i) $C_1 = \{\{x\} \,:\, x \in [n] \setminus (T \cup \{i, i_0\})\} \cup \{T \cup \{i, i_0\}\}$ and $j \in T \cup \{i_0\}$.

(ii) $C_1 = \{\{x\} \,:\, x \in [n] \setminus (T \cup \{i, i_0, j\})\} \cup \{(T \cup \{i_0\}), \{i, j\}\}$.

If (i) holds, then $(T \cup \{i, i_0\}) \notin P_0$ since $(T \cup \{i_0\}) \nsubseteq \mu(P_0)$, and $|\tau(P_0) \setminus (T \cup \{i, i_0\})| \leqslant |\tau(P_0) \setminus (T \cup \{i_0\})| = n-2t-1-s$. Thus, $|P_0 \cap C_1| \leqslant n-2t-1-s \leqslant n-2t-2$, a contradiction.

Suppose (ii) holds. The number of singletons that P_0 and C_1 have in common is at most $|\tau(P_0) \setminus (T \cup \{i, i_0, j\})| \leqslant |\tau(P_0) \setminus (T \cup \{i_0\})| \leqslant n-2t-1-s$. Recall that $T \cup \{i_0\} \notin P_0$. If $\{i, j\} \notin P_0$, then $|P_0 \cap C_1| \leqslant n-2t-1-s \leqslant n-2t-2$, a contradiction. If $\{i, j\} \in P_0$, then $|P_0 \cap C_1| \leqslant n-2t-s$. Since $|P_0 \cap C_1| \geqslant n-2t-1$, we must have $s = 1$, $|P_0 \cap C_1| = n-2t-1$, and $C_1 = \{\{x\} \,:\, x \in [n] \setminus (T \cup \{i, i_0, j\})\} \cup \{(T \cup \{i_0\}), \{i, j\}\}$. But then $|\mu(C_1) \cap ([n] \setminus \{i_0\})| = |T \cup \{i, j\}| = t+2 > t+1 = |\mu(P_0) \cap ([n] \setminus \{i_0\})|$, contradicting the choice of P_0. Thus, $\mathcal{A} \subseteq \mathcal{F}_{2t+1}(i_0)$. By Lemma 4, $|\mathcal{A}| = |S_{ij}(\mathcal{A})| = |\mathcal{F}_{2t+1}(i_0)|$. Hence, $\mathcal{A} = \mathcal{F}_{2t+1}(i_0)$. □

3 Main result

Lemma 9. *Let* $t \geqslant 1$, $\mathcal{A} \subseteq \mathcal{B}(n)$ *and* $W \subseteq [n]$. *Suppose that* $|W| \leqslant q$ *and* $|\mu(P) \setminus W| \leqslant t-1$ *for all* $P \in \mathcal{A}$. *Then there exists an* $n_0 = n_0(q, t)$ *such that for all* $n \geqslant n_0$,

$$|\mathcal{A}| < n^{t-0.5}.$$

Proof. Note that for each $P \in \mathcal{A}$,

$$\mu(P) = C_1 \cup C_2,$$

where $C_1 \subseteq [n] \setminus W$, $|C_1| \leqslant t-1$ and $C_2 \subseteq W$. The number of such C_1 is at most

$$\sum_{i=0}^{t-1} \binom{n-|W|}{i} \leqslant \sum_{i=0}^{t-1} \binom{n}{i},$$

and the number of such C_2 is at most $2^{|W|} \leqslant 2^q$. Furthermore, $|\mu(P)| = |\mu(P) \setminus W| + |\mu(P) \cap W| \leqslant t-1+q$. Therefore the number of $Q \in \mathcal{A}$ with $\mu(Q) = \mu(P)$ is at most $\tilde{B}_{|\mu(P)|} \leqslant \tilde{B}_{t-1+q}$, where B_m is the number of singleton-free set partitions of $[m]$. Thus

$$|\mathcal{A}| \leqslant \tilde{B}_{t-1+q} 2^q \sum_{i=0}^{t-1} \binom{n}{i}.$$

If $t=1$, then $|\mathcal{A}| \leqslant \tilde{B}_q 2^q < n^{0.5}$ provided that $n > (B_q 2^q)^2$. Suppose $t \geqslant 2$. Then

$$\begin{aligned}
|\mathcal{A}| &\leqslant \tilde{B}_{t-1+q} 2^q \left(1 + \sum_{i=1}^{t-1} \frac{n^i}{i!} \prod_{j=1}^{i-1} \left(1 - \frac{j}{n}\right)\right) \\
&< \tilde{B}_{t-1+q} 2^q \left(1 + \sum_{i=1}^{t-1} n^{t-1}\right) \\
&= \tilde{B}_{t-1+q} 2^q t n^{t-1} < n^{t-0.5},
\end{aligned}$$

provided that $n \geqslant (\tilde{B}_{t-1+q} 2^q t)^2$. This completes the proof of the lemma. □

Lemma 10. *For $t \geqslant 2$, there exists an $n_0 = n_0(t)$ such that for all $n \geqslant n_0$, if $\mathcal{A} \in I(n, n-2t)$, then*

$$|\mathcal{A}| \leqslant |\mathcal{F}_{2t}|.$$

Moreover, equality holds if and only if $\mathcal{A} = \mathcal{F}_{2t}$.

Proof. Suppose $\mathcal{A} \not\subseteq \mathcal{F}_{2t}$. Then $\max_{P \in \mathcal{A}} |\mu(P)| = t+s$ with $s \geqslant 1$. Let $P_0 \in \mathcal{A}$ with $|\mu(P_0)| = t+s$. By Lemma 7, $\max_{P \in \mathcal{A}} |\mu(P)| \leqslant 4t$.

Claim*. $|\mu(P) \setminus \mu(P_0)| \leqslant t-1$ for all $P \in \mathcal{A}$.

Suppose there is a $Q \in \mathcal{A}$ with $|\mu(Q) \setminus \mu(P_0)| \geqslant t$. Then $|\tau(P_0) \setminus \mu(Q)| \leqslant n-2t-s$. Since $|P_0 \cap Q| \geqslant n-2t$, by Lemma 6, $|\mu(P_0) \cap \mu(Q)| \geqslant 2s$. Therefore $|\mu(Q)| = |\mu(Q) \setminus \mu(P_0)| + |\mu(P_0) \cap \mu(Q)| \geqslant t+2s$. On the other hand, $|\mu(Q)| \leqslant |\mu(P_0)| = t+s$ by the choice of P_0. This implies that $s \leqslant 0$, a contradiction. Hence, the claim follows.

By Claim* and Lemma 9 (take $W = \mu(P_0)$ and $q = 4t$), $|\mathcal{A}| < n^{t-0.5}$. Note that $\tilde{B}_t \geqslant \tilde{B}_2 = 1$ for $t \geqslant 2$. So, by equation (1),

$$|\mathcal{F}_{2t}| = \sum_{i=0}^{t} \tilde{B}_i \binom{n}{i} \geqslant \tilde{B}_t \binom{n}{t} \geqslant \frac{1}{t!} \prod_{j=0}^{t-1} (n-j) \geqslant \frac{n^t}{t! 2^{t-1}} > n^{t-0.5},$$

provided that $n \geqslant \max\left((t! 2^{t-1})^2, 2t-2\right)$. Thus, $|\mathcal{A}| < |\mathcal{F}_{2t}|$.

Suppose $\mathcal{A} \subseteq \mathcal{F}_{2t}$. Then $|\mathcal{A}| \leqslant |\mathcal{F}_{2t}|$ and equality holds if and only if $\mathcal{A} = \mathcal{F}_{2t}$. □

Lemma 11. *For $t \geqslant 1$, there exists an $n_0 = n_0(t)$ such that for all $n \geqslant n_0$, if $\mathcal{A} \in I(n, n-2t-1)$ and $\mathcal{A}$ is compressed, then*

$$|\mathcal{A}| \leqslant |\mathcal{F}_{2t+1}(1)|.$$

Moreover, equality holds if and only if $\mathcal{A} = \mathcal{F}_{2t+1}(i_0)$ for some $i_0 \in [n]$.

Proof. Since $t \geqslant 1$, $\tilde{B}_{t+1} \geqslant \tilde{B}_2 = 1$. Therefore, by equation (2), for all $a \in [n]$,

$$\begin{aligned}|\mathcal{F}_{2t+1}(a)| &= \sum_{i=0}^{t} \tilde{B}_i \binom{n}{i} + \tilde{B}_{t+1}\binom{n-1}{t} \\ &\geqslant \tilde{B}_t \binom{n}{t} + \binom{n-1}{t} \\ &= \tilde{B}_t \binom{n}{t} + \frac{1}{t!}\prod_{j=0}^{t-1}(n-1-j) \\ &\geqslant \tilde{B}_t \binom{n}{t} + \frac{n^t}{t!2^t}, \qquad (3)\end{aligned}$$

provided that $n \geqslant 2t$.

Suppose $\max_{P\in\mathcal{A}} |\mu(P)| \leqslant t$. Then $|\mu(P) \cap ([n] \setminus \{1\})| \leqslant t$ for all $P \in \mathcal{A}$. Hence, $\mathcal{A} \subseteq \mathcal{F}_{2t+1}(1)$ and the lemma follows.

Suppose $\max_{P\in\mathcal{A}} |\mu(P)| = t + s$ with $s \geqslant 1$. Let $P_0 \in \mathcal{A}$ with $|\mu(P_0)| = t + s$. By Lemma 7, $\max_{P\in\mathcal{A}} |\mu(P)| \leqslant 4t + 2$.

Claim**. If $s \geqslant 2$, then $|\mu(P) \setminus \mu(P_0)| \leqslant t - 1$ for all $P \in \mathcal{A}$.

Suppose there is a $Q \in \mathcal{A}$ with $|\mu(Q) \setminus \mu(P_0)| \geqslant t$. Then $|\tau(P_0) \setminus \mu(Q)| \leqslant n - 2t - s = n - 2t - 1 - (s-1)$. Since $|P_0 \cap Q| \geqslant n - 2t - 1$, by Lemma 6, P_0 and Q have at least $(s-1)$ blocks of size at least 2 in common and $|\mu(P_0) \cap \mu(Q)| \geqslant 2(s-1)$. Therefore $|\mu(Q)| = |\mu(Q) \setminus \mu(P_0)| + |\mu(P_0) \cap \mu(Q)| \geqslant t + 2(s-1)$. On the other hand, $|\mu(Q)| \leqslant |\mu(P_0)| = t + s$ by the choice of P_0. This implies that $s \leqslant 2$. Since $s \geqslant 2$, we must have $s = 2$, $|\mu(Q)| = |\mu(P_0)| = t+2$ and P_0 and Q have exactly one block of size 2 in common, say $\{i, j\}$. Since $\mathcal{A}$ is compressed, $s_{ij}(Q) \in \mathcal{A}$. Note that $\mu(s_{ij}(Q)) = \mu(Q) \setminus \mu(P_0)$. So, $|\mu(s_{ij}(Q)) \setminus \mu(P_0)| = t$ and $|\tau(P_0) \setminus \mu(s_{ij}(Q))| = n - 2t - 2 = n - 2t - 1 - 1$. Since $|P_0 \cap s_{ij}(Q)| \geqslant n - 2t - 1$, by Lemma 6, $|\mu(P_0) \cap \mu(s_{ij}(Q))| \geqslant 2$. This contradicts that $\mu(s_{ij}(Q)) = \mu(Q) \setminus \mu(P_0)$. Hence, the claim follows.

Suppose $s \geqslant 2$. By Claim** and Lemma 9 (take $W = \mu(P_0)$ and $q = 4t + 2$), $|\mathcal{A}| < n^{t-0.5}$ for sufficiently large n. It then follows from equation (3) that

$$|\mathcal{A}| < n^{t-0.5} < \frac{n^t}{t!2^t} \leqslant |\mathcal{F}_{2t+1}(1)|,$$

if $n \geqslant (t!2^t)^2$.

Suppose $s = 1$. Then $\mu(P) \leqslant t+1$ for all $P \in \mathcal{A}$. Let $P_0, P_1, \ldots, P_m \in \mathcal{A}$ be such that for all $1 \leqslant i \leqslant m$, we have

(i) $|\mu(P_i)| = t+1$, and

(ii) $|\mu(P_i) \setminus (\bigcup_{j=0}^{i-1} \mu(P_j))| = t$.

We may assume that m is the largest integer in the sense that there is no $R \in \mathcal{A}$ with $|\mu(R)| = t+1$ and $|\mu(R) \setminus (\bigcup_{j=0}^{m} \mu(P_j))| = t$.

If there is a $Q \in \mathcal{A}$ with $|\mu(Q) \setminus (\bigcup_{j=0}^{m} \mu(P_j))| \geqslant t+1$, then $|\mu(Q)| = t+1$ and $\mu(Q) \cap \mu(P_0) = \varnothing$. So, $|P_0 \cap Q| = |\tau(P_0) \setminus \mu(Q)| = n-2t-2 < n-2t-1$, a contradiction. Thus, $|\mu(P) \setminus (\bigcup_{j=0}^{m} \mu(P_j))| \leqslant t$ for all $P \in \mathcal{A}$, and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ where

$$\mathcal{A}_1 = \left\{ P \in \mathcal{A} \ : \ \left| \mu(P) \setminus \left(\bigcup_{j=0}^{m} \mu(P_j) \right) \right| \leqslant t-1 \right\},$$

$$\mathcal{A}_2 = \left\{ P \in \mathcal{A} \ : \ \left| \mu(P) \setminus \left(\bigcup_{j=0}^{m} \mu(P_j) \right) \right| = t \text{ and } |\mu(P)| = t \right\}.$$

Suppose $m \leqslant t$. Then $\left| \bigcup_{j=0}^{m} \mu(P_j) \right| \leqslant \sum_{j=0}^{m} |\mu(P_j)| \leqslant (t+1)^2$. By Lemma 9 (take $W = \bigcup_{j=0}^{m} \mu(P_j)$ and $q = (t+1)^2$), $|\mathcal{A}_1| < n^{t-0.5}$ for sufficiently large n. Note that the number of $\mu(R)$ with $R \in \mathcal{A}$ and $|\mu(R)| = t$ is at most $\binom{n}{t}$ and the number of $Q \in \mathcal{A}$ with $\mu(Q) = \mu(R)$ is at most $\tilde{B}_t$. Thus,

$$|\mathcal{A}_2| \leqslant \tilde{B}_t \binom{n}{t}.$$

It then follows from equation (3) that

$$|\mathcal{A}| \leqslant |\mathcal{A}_1| + |\mathcal{A}_2| < n^{t-0.5} + \tilde{B}_t \binom{n}{t} < \frac{n^t}{t!2^t} + \tilde{B}_t \binom{n}{t} \leqslant |\mathcal{F}_{2t+1}(1)|,$$

if $n \geqslant (t!2^t)^2$.

Suppose $m \geqslant t+1$.

Claim***. There is a $i_0 \in [n]$ with $i_0 \in P_i$ for $i = 0, 1, 2, \ldots, t+1$.

Note that if $\mu(P_i) \cap \mu(P_j) = \varnothing$ for $i \neq j$, then $|P_i \cap P_j| = |\tau(P_i) \setminus \mu(P_j)| = n-2t-2 < n-2t-1$, a contradiction. So, $\mu(P_i) \cap \mu(P_j) \neq \varnothing$ for $i \neq j$. By properties (i) and (ii), we may conclude that $|\mu(P_i) \cap \mu(P_j)| = 1$ for all i, j with $i \neq j$.

Let $\mu(P_1) \cap \mu(P_0) = \{i_0\}$, $\mu(P_i) \cap \mu(P_0) = \{j_1\}$ and $\mu(P_i) \cap \mu(P_1) = \{j_2\}$ where $2 \leqslant i \leqslant t+1$. Since $|\mu(P_i)| = t+1$ and $|\mu(P_i) \setminus (\bigcup_{j=0}^{i-1} \mu(P_j))| = t$, $j_1 = j_2 \in \mu(P_1) \cap \mu(P_0) = \{i_0\}$. Thus, $i_0 \in P_i$ for $i = 0, 1, 2, \ldots, t+1$. This completes the proof of the claim.

By Claim***,

$$\mu(P_i) = W_i \cup \{i_0\},$$

for $i = 0, 1, \ldots, t+1$ and $W_i \cap W_j = \varnothing$ for $i \neq j$. Suppose $\mathcal{A} \not\subseteq \mathcal{F}_{2t+1}(i_0)$. Then there is a $Q \in \mathcal{A}$ with $|\mu(Q) \cap ([n] \setminus \{i_0\})| = t+1$, i.e., $|\mu(Q)| = t+1$ and $i_0 \notin \mu(Q)$. Note that $\mu(Q) \cap \mu(P_i) \neq \varnothing$ for all i, for otherwise, $|Q \cap P_i| = |\tau(Q) \setminus \mu(P_i)| = n-2t-2 < n-2t-1$. Therefore $\mu(Q) \cap W_i \neq \varnothing$. Since $W_i \cap W_j = \varnothing$ for $i \neq j$, $\mu(Q)$ will have at least $t+2$ elements, a contradiction. Hence, $\mathcal{A} \subseteq \mathcal{F}_{2t+1}(i_0)$, $|\mathcal{A}| \leqslant |\mathcal{F}_{2t+1}(i_0)|$ and equality holds if and only if $\mathcal{A} = \mathcal{F}_{2t+1}(i_0)$.

This completes the proof of the lemma. □

Proof of Theorem 3. If $r = 2t$, then the theorem follows from Lemma 10. Suppose $r = 2t+1$. By repeatedly applying the splitting operations, we eventually obtain a compressed family $\mathcal{A}^* \in I(n, n-2t-1)$ with $|\mathcal{A}^*| = |\mathcal{A}|$ (Lemma 5). It then follows from Lemma 11 that $|\mathcal{A}| = |\mathcal{A}^*| \leqslant |\mathcal{F}_{2t+1}(1)|$ and equality holds if and only if $\mathcal{A}^* = \mathcal{F}_{2t+1}(i_0)$ for some $i_0 \in [n]$. By Theorem 8, we may conclude that $\mathcal{A}^* = \mathcal{F}_{2t+1}(i_0)$ implies that $\mathcal{A} = \mathcal{F}_{2t+1}(i_0)$. This completes the proof of Theorem 3. □

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