Proof of a conjecture of Amdeberhan and Moll on a divisibility property of binomial coefficients

Quan-Hui Yang*

School of Mathematics and Statistics
Nanjing University of Information Science and Technology
Nanjing, P.R. China

yangquanhui01@163.com

Submitted: Feb 10, 2014; Accepted: Dec 21, 2014; Published: Jan 9, 2015
Mathematics Subject Classifications: 05A10, 11B65

Abstract

Let a, b and n be positive integers with $a > b$. In this note, we prove that

$$(2bn+1)(2bn+3)\binom{2bn}{bn}\Bigg|3(a-b)(3a-b)\binom{2an}{an}\binom{an}{bn}.$$

This confirms a recent conjecture of Amdeberhan and Moll.

Keywords: binomial coefficients; p-adic order; divisibility properties

1 Introduction

In 2009, Bober [1] determined all cases such that

$$\frac{(a_1n)!\cdots(a_kn)!}{(b_1n)!\cdots(b_{k+1}n)!}\in\mathbb{Z},$$

where $a_s \neq b_t$ for all s, t, $\sum a_s = \sum b_t$ and $\gcd(a_1, \ldots, a_k, b_1, \ldots, b_{k+1}) = 1$.

Recently, Z.-W. Sun [12, 13] studied divisibility properties of binomial coefficients and obtained some interesting results. For example,

$$2(2n+1)\binom{2n}{n}\Bigg|\binom{6n}{3n}\binom{3n}{n},$$

*Supported by the National Natural Science Foundation of China, Grant No. 11371195 and the Startup Foundation for Introducing Talent of NUIST, Grant No. 2014r029.

$$(10n+1)\binom{3n}{n}\bigg|\binom{15n}{5n}\binom{5n-1}{n-1}.$$

Later, Guo and Krattenthaler (see [5, 8]) obtained some similar divisibility results. Related results appear in [2]-[4] and [7]-[11].

Introduce the notation

$$S_n=\frac{\binom{6n}{3n}\binom{3n}{n}}{2(2n+1)\binom{2n}{n}} \quad \text{and} \quad t_n=\frac{\binom{15n}{5n}\binom{5n-1}{n-1}}{(10n+1)\binom{3n}{n}}.$$

In [6], Guo proved the conjectures due to Z.W. Sun [12, 13].

Theorem A. ([12, Conjecture 3(i)].) *Let n be a positive integer. Then*

$$3S_n\equiv 0 \pmod{2n+3}.$$

Theorem B. ([13, Conjecture 1.3].) *Let n be a positive integer. Then*

$$21t_n\equiv 0 \pmod{10n+3}.$$

Recently, T. Amdeberhan and V. H. Moll proposed a conjecture related to Theorems A and B, which was only presented as Conjecture 7.1 in Guo's paper [6] by private communication.

This notes provides a proof of this conjecture.

Theorem 1. *Let a, b and n be positive integers with $a>b$. Then*

$$(2bn+1)(2bn+3)\binom{2bn}{bn}\bigg|3(a-b)(3a-b)\binom{2an}{an}\binom{an}{bn}.$$

Remark 2. Theorem A is the special case $a=3$, $b=1$ of Theorem 1.

2 Proofs

For a real number z, denote the greatest integer not exceeding z by $\lfloor z\rfloor$ and $\{z\}$ denotes the fractional part of z. For an integer n and a prime p, write $p^k\|n$ if $p^k|n$ and $p^{k+1}\nmid n$. The integer k above is denoted by $\nu_p(n)$.

It is well known that

$$\nu_p(n!)=\sum_{i=1}^{\infty}\left\lfloor\frac{n}{p^i}\right\rfloor. \tag{1}$$

The proof of Theorem 1 begins with a preliminary result.

Lemma 3. *Let x and y be two real numbers. Then*

$$\lfloor 2x\rfloor+\lfloor y\rfloor\geqslant\lfloor x\rfloor+\lfloor x-y\rfloor+\lfloor 2y\rfloor.$$

Proof. The identity $2x+y=x+(x-y)+2y$ shows that it suffices to $\{2x\}+\{y\}\leqslant\{x\}+\{x-y\}+\{2y\}$. The proof now follows by comparing $\{x\}$ and $\{y\}$ to 1/2. The details are left to the reader. □

Proof of Theorem 1. Let

$$T(a,b,n):=\binom{2an}{an}\binom{an}{bn}\Big/\binom{2bn}{bn}=\frac{(2an)!(bn)!}{(an)!(an-bn)!(2bn)!}.$$

By (1), for any prime p,

$$\nu_p(T(a,b,n))=\sum_{i=1}^{\infty}\left(\left\lfloor\frac{2an}{p^i}\right\rfloor+\left\lfloor\frac{bn}{p^i}\right\rfloor-\left\lfloor\frac{an}{p^i}\right\rfloor-\left\lfloor\frac{an-bn}{p^i}\right\rfloor-\left\lfloor\frac{2bn}{p^i}\right\rfloor\right).$$

Lemma 3 shows that each term of $\nu_p(T(a,b,n))$ is nonnegative. Hence $\nu_p(T(a,b,n))\geqslant 0$. Therefore, $T(a,b,n)\in\mathbb{Z}$.

Since $\gcd(2bn+1,2bn+3)=1$, it suffices to prove that

$$2bn+1|3(a-b)(3a-b)T(a,b,n)$$

and

$$2bn+3|3(a-b)(3a-b)T(a,b,n).$$

The second statement is established here. The proof of the first statement is similar and the details are omitted. Suppose that $p^\alpha\|2bn+3$ with $\alpha\geqslant 1$. It is shown that

$$p^\alpha|3(a-b)(3a-b)T(a,b,n). \tag{2}$$

Let $p^\beta\|a-b$ and $p^\gamma\|3a-b$ with $\beta\geqslant 0$ and $\gamma\geqslant 0$. Write $\tau=\max\{\beta,\gamma\}$. If $\alpha\leqslant\tau$, then (2) clearly holds. Now we assume $\alpha>\tau$.

Suppose that $p\geqslant 5$. The statement

$$\left\lfloor\frac{2an}{p^i}\right\rfloor+\left\lfloor\frac{bn}{p^i}\right\rfloor-\left\lfloor\frac{an}{p^i}\right\rfloor-\left\lfloor\frac{an-bn}{p^i}\right\rfloor-\left\lfloor\frac{2bn}{p^i}\right\rfloor=1$$

is established for $i=\tau+1,\tau+2,\dots,\alpha$. This is proven next. Noting that $p|2bn+3$ and $p\geqslant 5$, it follows that $\gcd(p,n)=1$.

Observe that $p^\alpha\|2bn+3$, it follows that $2bn\equiv p^\alpha-3\pmod{p^\alpha}$ and $bn\equiv(p^\alpha-3)/2\pmod{p^\alpha}$.

Take $i\in\{\tau+1,\tau+2,\dots,\alpha\}$. Then $2bn\equiv p^i-3\pmod{p^i}$ and $bn\equiv(p^i-3)/2\pmod{p^i}$. Now we divide into several cases according to the value of $an\pmod{p^i}$.

Case 1. $an\equiv t\pmod{p^i}$ with $0\leqslant t<(p^i-3)/2$. It follows that $2an\equiv 2t\pmod{p^i}$ and $0\leqslant 2t<p^i-3$. Also

$$an-bn\equiv t-(p^i-3)/2+p^i\pmod{p^i},$$

where $0 \leqslant t - (p^i - 3)/2 + p^i < p^i$. Hence

$$\begin{aligned}
&\left\lfloor \frac{2an}{p^i} \right\rfloor + \left\lfloor \frac{bn}{p^i} \right\rfloor - \left\lfloor \frac{an}{p^i} \right\rfloor - \left\lfloor \frac{an-bn}{p^i} \right\rfloor - \left\lfloor \frac{2bn}{p^i} \right\rfloor \\
= \; & \frac{2an-2t}{p^i} + \frac{bn-(p^i-3)/2}{p^i} - \frac{an-t}{p^i} \\
& - \left(\frac{an-bn-(t-(p^i-3)/2+p^i)}{p^i} \right) - \frac{2bn-(p^i-3)}{p^i} \\
= \; & 1.
\end{aligned}$$

Case 2. $an \equiv (p^i-3)/2 \pmod{p^i}$. Then, $an - bn \equiv 0 \pmod{p^i}$. Since $\gcd(p,n)=1$, it follows that $p^i | a-b$. However, $p^\beta \| a-b$ and $\beta \leqslant \tau < i$. This is a contradiction.

Case 3. $an \equiv (p^i-1)/2 \pmod{p^i}$. It follows that

$$3an - bn \equiv \frac{3(p^i-1)}{2} - \frac{(p^i-3)}{2} \equiv 0 \pmod{p^i}.$$

The fact that $\gcd(p,n)=1$ implies $p^i | 3a-b$. This contradicts $p^\gamma \| 3a-b$ and $\gamma < i$.

Case 4. $an \equiv t \pmod{p^i}$ with $(p^i+1)/2 \leqslant t < p^i$. Then

$$2an \equiv 2t - p^i \pmod{p^i}, \quad 0 \leqslant 2t - p^i < p^i,$$

and

$$an - bn \equiv t - (p^i-3)/2 \pmod{p^i}, \quad 0 \leqslant t - (p^i-3)/2 < p^i.$$

Hence

$$\begin{aligned}
&\left\lfloor \frac{2an}{p^i} \right\rfloor + \left\lfloor \frac{bn}{p^i} \right\rfloor - \left\lfloor \frac{an}{p^i} \right\rfloor - \left\lfloor \frac{an-bn}{p^i} \right\rfloor - \left\lfloor \frac{2bn}{p^i} \right\rfloor \\
= \; & \frac{2an-(2t-p^i)}{p^i} + \frac{bn-(p^i-3)/2}{p^i} - \frac{an-t}{p^i} \\
& - \left(\frac{an-bn-(t-(p^i-3)/2)}{p^i} \right) - \frac{2bn-(p^i-3)}{p^i} \\
= \; & 1.
\end{aligned}$$

Therefore, $\nu_p(T(a,b,n)) \geqslant \alpha - \tau$, and this implies

$$\nu_p(3(a-b)(3a-b)T(a,b,n)) \geqslant \alpha.$$

The proof of (2), for $p \geqslant 5$, is complete.

Now assume $p=3$. If $9|n$, then $3|2bn+3$ and $9 \nmid 2bn+3$. It follows that $\alpha=1$, and then (2) clearly holds. If $9 \nmid n$, then the proof of the case $p \geqslant 5$ applies to this situation. In Case 2, $an-bn \equiv 0 \pmod{3^i}$ gives $3^{i-1}|a-b$. In Case 3, $3^{i-1}|3a-b$. Thus, if $i \geqslant \tau+2$, then $i-1 \geqslant \tau+1$. It is a contradiction in both cases. Hence

$$\left\lfloor \frac{2an}{3^i} \right\rfloor + \left\lfloor \frac{bn}{3^i} \right\rfloor - \left\lfloor \frac{an}{3^i} \right\rfloor - \left\lfloor \frac{an-bn}{3^i} \right\rfloor - \left\lfloor \frac{2bn}{3^i} \right\rfloor = 1$$

for $i = \tau+2, \tau+3, \ldots, \alpha$. It follows that $\nu_3(T(a,b,n)) \geqslant \alpha-\tau-1$, and then

$$\nu_3(3(a-b)(3a-b)T(a,b,n)) \geqslant \alpha.$$

That is, (2) also holds. Hence, $2bn+3|3(a-b)(3a-b)T(a,b,n)$.

Therefore,

$$(2bn+1)(2bn+3)\binom{2bn}{bn}\bigg|3(a-b)(3a-b)\binom{2an}{an}\binom{an}{bn}.$$

This completes the proof of Theorem 1.

□

Acknowledgements

We are grateful to the anonymous referee for carefully reading our original manuscript and giving many detailed comments.

References

[1] J. W. Bober. Factorial ratios, hypergeometric series, and a family of step functions. *J. Lond. Math. Soc.*, 79:422–444, 2009.

[2] N. J. Calkin. Factors of sums of powers of binomial coefficients. *Acta Arith.*, 86:17–26, 1998.

[3] H.-Q. Cao, H. Pan. Factors of alternating binomial sums. *J. Adv. in Appl. Math.*, 45:96–107, 2010.

[4] N. J. Fine. Binomial coefficients modulo a prime. *Amer. Math. Monthly*, 54:589–592, 1947.

[5] V. J. W. Guo. Proof of Sun's conjecture on the divisibility of certain binomial sums. *Electron. J. Combin.*, 20(4):#P4.20, 2013.

[6] V. J. W. Guo. Proof of two divisibility properties of binomial coefficients conjectured by Z.-W. Sun. *Electron. J. Combin.*, 21(2):#P2.54, 2014.

[7] V. J. W. Guo, F. Jouhet and J. Zeng, Factors of alternating sums of products of binomial and q-binomial coefficients. *Acta Arith.*, 127:17–31, 2007.

[8] V. J. W. Guo, C. Krattenthaler. Some divisibility properties of binomial and q-binomial coefficients. *J. Number Theory*, 135:167–184, 2014.

[9] V. J. W. Guo, J. Zeng. Factors of binomial sums from the Catalan triangle. *J. Number Theory*, 130:172–186, 2010.

[10] V. J. W. Guo, J. Zeng. Factors of sums and alternating sums involving binomial coefficients and powers of integers. *Int. J. Number Theory*, 7:1959–1976, 2011.

[11] M. Razpet. On divisibility of binomial coefficients. *Discrete Math.*, 135:377–379, 1994.

[12] Z.-W. Sun. On divisibility of binomial coefficients. *J. Austral. Math. Soc.*, 93:189–201, 2012.

[13] Z.-W. Sun. Products and sums divisible by central binomial coefficients. *Electron. J. Combin.*, 20(1):#P1.9, 2013.