A Construction of Small $(q-1)$-Regular Graphs of Girth 8 *

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Abstract

In this note we construct a new infinite family of $(q-1)$-regular graphs of girth 8 and order $2q(q-1)^2$ for all prime powers $q \geqslant 16$, which are the smallest known so far whenever $q-1$ is not a prime power or a prime power plus one itself.

Keywords: Cages, girth, Moore graphs, perfect dominating sets

1 Introduction

Throughout this note, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow the book by Bondy and Murty [11] for terminology and notation.

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Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *girth* of a graph G is the number $g = g(G)$ of edges in a smallest cycle. For every $v \in V$, $N_G(v)$ denotes the *neighbourhood* of v, that is, the set of all vertices adjacent to v. The *degree* of a vertex $v \in V$ is the cardinality of $N_G(v)$. Let $A \subset V(G)$, we denote by $N_G(A) = \cup_{a\in A} N_{G-A}(a)$ and by $N_G[A] = A \cup N_G(A)$. For $v, w \in V(G)$ denote by $d(v, w)$ the distance between v and w. Moreover, denote by $N^m(v) = \{w \in V(G) \mid d(v, w) = m\}$ and $N^m[v] = \{w \in V(G) \mid d(v, w) \leqslant m\}$ the m^{th} open and closed neighbourhood of v respectively.

A graph is called *regular* if all the vertices have the same degree. A (k, g)-*graph* is a k-regular graph with girth g. Erdős and Sachs [12] proved the existence of (k, g)-graphs for all values of k and g provided that $k \geqslant 2$. Since then most work carried out has focused on constructing a smallest one (cf. e.g. [1, 2, 3, 4, 5, 6, 7, 9, 13, 15, 18, 20, 21]). A (k, g)-*cage* is a k-regular graph with girth g having the smallest possible number of vertices. Cages have been intensively studied since they were introduced by Tutte [23] in 1947. More details about constructions of cages can be found in the survey by Exoo and Jajcay [14].

In this note we are interested in $(k, 8)$-cages. Counting the number of vertices in the distance partition with respect to an edge yields the following lower bound on the order of a $(k, 8)$-cage:

$$n_0(k, 8) = 2(1 + (k-1) + (k-1)^2 + (k-1)^3). \quad (1)$$

A $(k, 8)$-cage with $n_0(k, 8)$ vertices is called a Moore $(k, 8)$-*graph* (cf. [11]). These graphs have been constructed as the incidence graphs of generalized quadrangles of order $k-1$ (cf. [9]). All these objects are known to exist for all prime power values of $k-1$ (cf. e.g. [8, 16]), and no example is known when $k-1$ is not a prime power. Since they are incidence graphs, these cages are bipartite and have diameter 4.

A subset $U \subset V(G)$ is said to be *a perfect dominating set of* G if for each vertex $x \in V(G) \setminus U$, $|N_G(x) \cap U| = 1$ (cf. [17]). Note that if G is a $(k, 8)$-graph and U is a perfect dominating set of G, then $G-U$ is clearly a $(k-1)$-regular graph, of girth at least 8. Using classical generalized quadrangles, Beukemann and Metsch [10] proved that the cardinality of a perfect dominating set B of a Moore $(q+1, 8)$-graph, q a prime power, is at most $|B| \leqslant 2(2q^2 + 2q)$ and if q is even $|B| \leqslant 2(2q^2 + q + 1)$.

For $k = q + 1$ where $q \geqslant 2$ is a prime power, we find a perfect dominating set of cardinality $2(q^2 + 3q + 1)$ for all q (cf. Proposition 2). This result allows us to explicitly obtain q-regular graphs of girth 8 and order $2q(q^2-2)$ for any prime power q (cf. Definition 3 and Lemma 4). Finally, we prove the existence of a perfect dominating set of these q-regular graphs which allow us to construct a new infinite family of $(q-1)$-regular graphs of girth 8 and order $2q(q-1)^2$ for all prime powers q (cf. Theorem 5), which are the smallest known so far for $q \geqslant 16$ whenever $q-1$ is not a prime power or a prime power plus one itself. Previously, the smallest known $(q-1, 8)$-graphs, for q a prime power, were those of order $2q(q^2 - q - 1)$ which appeared in [7]. The first ten improved values appear in the following table in which $k = q-1$ is the degree of a $(k, 8)$–graph, and the other columns contain the old and the new upper bound on its order.

k	Bound in [7]	New bound		k	Bound in [7]	New bound
15	7648	7200		52	292030	286624
22	23230	22264		58	403678	396952
36	98494	95904		63	515968	508032
40	134398	131200		66	592414	583704
46	203134	198904		70	705598	695800

2 Construction of small $(q-1)$-regular graphs of girth 8

In this section we construct $(q-1)$-regular graphs of girth 8 with $2q(q-1)^2$ vertices, for every prime power $q \geqslant 4$. To this purpose we need the following coordinates for a Moore $(q+1, 8)$-cage Γ_q.

Definition 1. [19, 22] Let $\mathbb{F}_q$ be a finite field with $q \geqslant 2$ a prime power and ϱ a symbol not belonging to $\mathbb{F}_q$. Let $\Gamma_q = \Gamma_q[V_0, V_1]$ be a bipartite graph with vertex sets $V_i = \mathbb{F}_q^3 \cup \{(\varrho, b, c)_i, (\varrho, \varrho, c)_i : b, c \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_i\}$, $i = 0, 1$, and edge set defined as follows:

For all $a \in \mathbb{F}_q \cup \{\varrho\}$ and for all $b, c \in \mathbb{F}_q$:

$$N_{\Gamma_q}((a,b,c)_1) = \begin{cases} \{(w,\ aw+b,\ a^2w+2ab+c)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, a, c)_0\} & \text{if } a \in \mathbb{F}_q; \\ \{(c, b, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, c)_0\} & \text{if } a = \varrho. \end{cases}$$

$$N_{\Gamma_q}((\varrho,\varrho,c)_1) = \{(\varrho, c, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_0\}$$

$$N_{\Gamma_q}((\varrho,\varrho,\varrho)_1) = \{(\varrho, \varrho, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_0\}.$$

Or equivalently

For all $i \in \mathbb{F}_q \cup \{\varrho\}$ and for all $j, k \in \mathbb{F}_q$:

$$N_{\Gamma_q}((i,j,k)_0) = \begin{cases} \{(w,\ j-wi,\ w^2i-2wj+k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, j, i)_1\} & \text{if } i \in \mathbb{F}_q; \\ \{(j, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, j)_1\} & \text{if } i = \varrho. \end{cases}$$

$$N_{\Gamma_q}((\varrho,\varrho,k)_0) = \{(\varrho, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_1\};$$

$$N_{\Gamma_q}((\varrho,\varrho,\varrho)_0) = \{(\varrho, \varrho, w)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_1\}.$$

Note that ϱ is just a symbol not belonging to $\mathbb{F}_q$ and no arithmetical operation will be performed with it. Figure 1 shows a spanning tree of Γ_q with the vertices labelled according to Definition 1.

Proposition 2. *Let $q \geqslant 2$ be a prime power and let $\Gamma_q = \Gamma_q[V_0, V_1]$ be the Moore $(q+1, 8)$-graph with the coordinates as in Definition 1. Let $A = \{(\varrho, 0, c)_1 : c \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, 0)_1\}$ and let $x \in \mathbb{F}_q \setminus \{0\}$. Then the set*

$$N_{\Gamma_q}[A] \cup \left(\bigcap_{a \in A} N^2_{\Gamma_q}(a) \right) \cup N^2_{\Gamma_q}[(\varrho, \varrho, x)_1]$$

is a perfect dominating set of Γ_q of cardinality $2(q^2+3q+1)$.

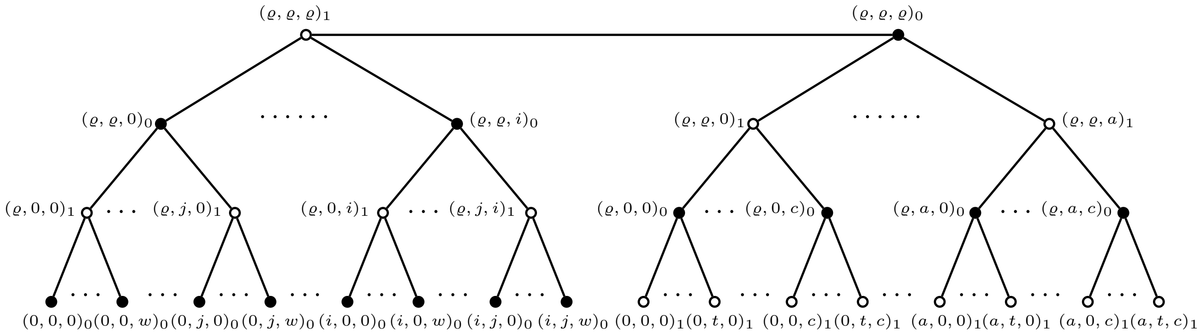

Figure 1: Spanning tree of Γ_q.

Proof. From Definition 1, it follows that $A = \{(\varrho, 0, c)_1 : c \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, 0)_1\}$ has cardinality $q+1$ and its elements are mutually at distance four. Then $|N_{\Gamma_q}[A]| = (q+1)^2 + q + 1$. By Definition 1, $N_{\Gamma_q}((\varrho, 0, c)_1) = \{(c, 0, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, c)_0\}$; and $N_{\Gamma_q}((\varrho, \varrho, 0)_1) = \{(\varrho, 0, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_0\}$. Then $(\varrho, \varrho, \varrho)_1 \in N^2_{\Gamma_q}((\varrho, 0, c)_1)) \cap N^2_{\Gamma_q}((\varrho, \varrho, 0)_1))$ for all $c \in \mathbb{F}_q$. Moreover, $N_{\Gamma_q}((c, 0, w)_0) = \{(a, -ac, a^2c + w)_1 : a \in \mathbb{F}_q\} \cup \{(\varrho, 0, c)_1\}$. Thus, for all $c_1, c_2, w_1, w_2 \in \mathbb{F}_q$, $c_1 \neq c_2$, we have $(a, -c_1a, a^2c_1 + w_1)_1 = (a, -c_2a, a^2c_2 + w_2)_1$ if and only if $a = 0$ and $w_1 = w_2$. Let $I_A = \bigcap_{a \in A} N^2_{\Gamma_q}(a)$. We conclude that $I_A = \{(\varrho, \varrho, \varrho)_1\} \cup \{(0, 0, w)_1 : w \in \mathbb{F}_q\}$ which implies that $|N_{\Gamma_q}[A]| + |I_A| = (q+1)^2 + 2(q+1)$.

Since $N^2_{\Gamma_q}[(\varrho, \varrho, x)_1] = \bigcup_{j \in \mathbb{F}_q} N_{\Gamma_q}[(\varrho, x, j)_0] \cup N_{\Gamma_q}[(\varrho, \varrho, \varrho)_0]$ we obtain that $(N_{\Gamma_q}[A] \cup I_A) \cap N^2_{\Gamma_q}[(\varrho, \varrho, x)_1] = \{(\varrho, \varrho, \varrho)_0, (\varrho, \varrho, 0)_1, (\varrho, \varrho, \varrho)_1\}$. Let $D = N_{\Gamma_q}[A] \cup I_A \cup N^2_{\Gamma_q}[(\varrho, \varrho, x)_1]$, then

$$\begin{aligned} |D| &= |N_{\Gamma_q}[A]| + |I_A| + |N^2_{\Gamma_q}[(\varrho, \varrho, x)_1]| - 3 \\ &= (q+1)^2 + 2(q+1) + 1 + (q+1) + q(q+1) - 3 \\ &= 2q^2 + 6q + 2. \end{aligned}$$

Let us prove that D is a perfect dominating set of Γ_q.

Let H denote the subgraph of Γ_q induced by D. Note that for $t, c \in \mathbb{F}_q$, the vertices $(x, t, c)_1 \in N^2_{\Gamma_q}((\varrho, \varrho, x)_1)$ have degree 2 in H because they are adjacent to the vertex $(\varrho, x, t)_0 \in N_{\Gamma_q}(\varrho, \varrho, x)_1$ and also to the vertex $(-x^{-1}t, 0, xt + z)_0 \in N_{\Gamma_q}(A)$. This implies that the vertices $(i, 0, j)_0 \in N_{\Gamma_q}(A)$, $i, j \in \mathbb{F}_q$, have degree 3 in H and, also that the diameter of H is 5. Moreover, for $k \in \mathbb{F}_q$, the vertices $(\varrho, \varrho, k)_0, (\varrho, 0, k)_0 \in D$ have degree 2 in H and the vertices $(\varrho, \varrho, j)_1 \in D$, $j \in \mathbb{F}_q \setminus \{0, x\}$ have degree 1 in H. All other vertices in D have degree $q+1$ in H.

Since the diameter of H is 5 and the girth is 8, $|N_{\Gamma_q}(v) \cap D| \leqslant 1$ for all $v \in V(\Gamma_q) \setminus D$, and also for all distinct $d, d' \in D$ we have $(N_{\Gamma_q}(d) \cap N_{\Gamma_q}(d')) \cap (V(\Gamma_q) \setminus D) = \emptyset$. Then, $|N_{\Gamma_q}(D) \cap (V(\Gamma_q) \setminus D)| = q^2(q-2) + 2q(q-1) + (q-2)q + q^2(q-1) = 2q^3 - 4q = |V(\Gamma_q) \setminus D|$. Hence $|N_{\Gamma_q}(v) \cap D| = 1$ for all $v \in V(\Gamma_q) \setminus D$. Thus D is a perfect dominating set of Γ_q. □

Definition 3. Let $q \geqslant 4$ be a prime power and let $x \in \mathbb{F}_q \setminus \{0,1\}$. Define G_q^x as the q-regular graph of order $2q(q^2-2)$ constructed by removing from Γ_q its perfect dominating set D given in Proposition 2.

Lemma 4. *The q-regular graph G_q^x in Definition 3 has girth exactly 8.*

Proof. The graph G_q^x, by Definition 3 is Γ_q minus a perfect dominating set D so it clearly has girth at least 8, and since it is bipartite its girth must be even. However, Moore's bound on the minimum number of vertices of a q-regular graph of girth 10 is $2\left(\sum_{i=0}^{4}(q-1)^4\right)$. Since the order of G_q^x is $2q(q^2-2) < 2\left(\sum_{i=0}^{4}(q-1)^4\right)$, for all $q \geqslant 2$, G_q^x must have girth exactly 8. □

Theorem 5. *Let $q \geqslant 5$ be a prime power and let G_q^x be the graph given in Definition 3. Let $R = N_{G_q^x}(\{(\varrho, j, k)_0 : j,k \in \mathbb{F}_q, j \neq 0,1,x\}) \cap N^5_{G_q^x}((\varrho,1,0)_0)$. Then, the set*

$$S := \bigcup_{j \in \mathbb{F}_q} N_{G_q^x}[(\varrho,1,j)_0] \cup N_{G_q^x}[R]$$

is a perfect dominating set in G_q^x of cardinality $4q^2-6q$. Hence, $G_q^x - S$ is a $(q-1)$-regular graph of girth 8 and order $2q(q-1)^2$.

Proof. Once $x \in \mathbb{F}_q \setminus \{0,1\}$ has been chosen to define G_q^x, to simplify notation, we will denote G_q^x by G_q throughout the proof. Denote by $P = \{(\varrho,j,k)_0 : j,k \in \mathbb{F}_q, j \neq 0,1,x\}$, then $R = N_{G_q}(P) \cap N^5_{G_q}((\varrho,1,0)_0)$. Note that $d_{G_q}((\varrho,1,0)_0,(\varrho,j,k)_0) = 4$, because according to Definition 1, G_q contains the following paths of length four (see Figure 2): $(\varrho,1,0)_0\ (1,b,0)_1 (w,w+b,w+2b)_0\ (j,t,k)_1\ (\varrho,j,k)_0$, for all $b,j,t \in \mathbb{F}_q$ such that $b+w \neq 0$ due to the vertices $(j,0,k)_0$ with second coordinate zero having been removed from Γ_q to obtain G_q.

By Definition 1 we have $w+b = jw+t$ and $w+2b = j^2w + 2jt + k$. If $w+b=0$, then $-w = b = tj^{-1}$ and $b = jt+k$ yielding that $t = (1-j^2)^{-1}jk$. This implies that $(j,(1-j^2)^{-1}jk,k)_1 \in R$ is the unique neighbor in R of $(\varrho,j,k)_0 \in P$. Therefore every $(\varrho,j,k)_0 \in P$ has a unique neighbor $(j,t,k)_1 \in R$ leading to:

$$|R| = |P| = q(q-3). \tag{2}$$

Thus, every $v \in N_{G_q}(R) \setminus P$ has at most $|R|/q = q-3$ neighbors in R because for each j the vertices from the set $\{(\varrho,j,k)_0 : k \in \mathbb{F}_q\} \subset P$ are mutually at distance 6 (they were the q neighbors in Γ_q of the removed vertex $(\varrho,\varrho,j)_1$). Furthermore, every $v \in N_{G_q}(R) \setminus P$ has at most one neighbor in $N^5_{G_q}((\varrho,1,0)_0) \setminus R$ because the vertices $\{(\varrho,1,j)_0 : j \in \mathbb{F}_q, j \neq 0\}$ are mutually at distance 6. Therefore every $v \in N_{G_q}(R) \setminus P$ has at least two neighbors in $N^3_{G_q}((\varrho,1,0)_0)$. Thus denoting $K = N_{G_q}(N_{G_q}(R) \setminus P) \cap N^3_{G_q}((\varrho,1,0)_0)$ we have

$$|K| \geqslant 2|N_{G_q}(R) \setminus P|. \tag{3}$$

Moreover, observe that $(N_{G_q}(P) \setminus R) \cap K = \emptyset$ because these two sets are at distance four (see Figure 2). Since the elements of P are mutually at distance at least 4 we obtain that $|N_{G_q}(P) \setminus R| = q|P| - |R| = (q-1)|P|$. Hence by (2)

$$|N^3_{G_q}((\varrho,1,0)_0)| \geqslant |N_{G_q}(P) \setminus R| + |K| = (q-1)|P| + |K| = (q-1)q(q-3) + |K|.$$

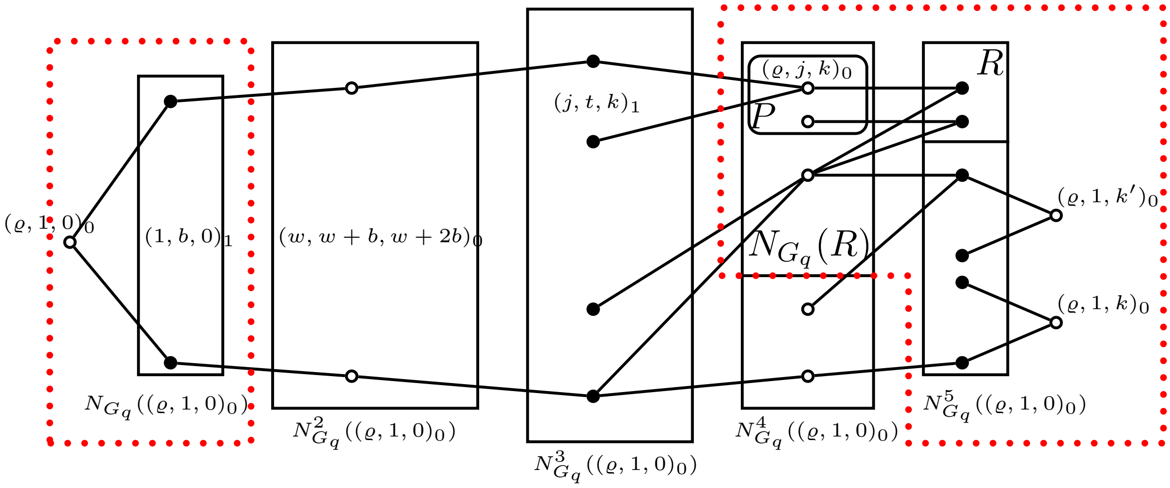

Figure 2: Structure of the graph G_q. The perfect dominating set lies inside the dotted box.

Since $|N^3_{G_q}((\varrho,1,0)_0)| = q(q-1)^2$ we obtain that $|K| \leqslant 2q(q-1)$ yielding by (3) that $|N_{G_q}(R) \setminus P| \leqslant q(q-1)$. As P contains at least q elements mutually at distance 6, R contains at least q elements mutually at distance 4. Thus we have $|N_{G_q}(R) \setminus P| \geqslant q^2 - q$. Therefore $|N_{G_q}(R) \setminus P| = q^2 - q$ and all the above inequalities are actually equalities. Thus by (2) we get

$$|N_{G_q}(R)| = q^2 - q + |P| = 2q(q-2) \tag{4}$$

and every $v \in N_{G_q}(R) \setminus P$ has exactly 1 neighbor in $N^5_{G_q}((\varrho,1,0)_0) \setminus R$. Therefore we have

$$\begin{aligned} \left|N^4_{G_q}((\varrho,1,0)_0) \setminus N_{G_q}(R)\right| &= \left| \bigcup_{j \in \mathbb{F}_q \setminus \{0\}} \left(N^2_{G_q}((\varrho,1,j)_0) \cup P\right) \setminus N_{G_q}(R)\right| \\ &= q(q-1)^2 + q(q-3) - 2q(q-2) \\ &= q(q-1)(q-2). \end{aligned}$$

Let us denote by $E[A,B]$ the set of edges between any two sets of vertices A and B. Then $|E[N^3_{G_q}((\varrho,1,0)_0), N^4_{G_q}((\varrho,1,0)_0)]| = q(q-1)^3$ and $|E[N^3_{G_q}((\varrho,1,0)_0), N^4_{G_q}((\varrho,1,0)_0) \setminus N_{G_q}(R)]| = q(q-1)^2(q-2)$. Therefore,

$$|E[N^3_{G_q}((\varrho,1,0)_0), N_{G_q}(R)]| = q(q-1)^3 - q(q-1)^2(q-2) = q(q-1)^2 = |N^3_{G_q}((\varrho,1,0)_0)|,$$

which implies that every $v \in N^3_{G_q}((\varrho,1,0)_0)$ has exactly one neighbor in $N_{G_q}(R)$. It follows that $S = \bigcup_{j \in \mathbb{F}_q} N_{G_q}[(\varrho,1,j)_0] \cup N_{G_q}[R]$ is a perfect dominating set of G_q. Furthermore, by (2) and (4), $|S| = q^2 + q + q(3q-7) = 4q^2 - 6q$. Therefore a $(q-1)$-regular graph of girth 8 can be obtained by deleting from G_q the perfect dominating set S, see Figure 2. This graph has order $2q(q^2-2) - 2q(2q-3) = 2q(q-1)^2$.

Finally, as in the proof of Lemma 4, recall that $G_q - S$ must have even girth since it is bipartite, and that the minimum number of vertices of a $(q-1)$-regular graph of girth 10 is $2\left(\sum_{i=0}^{4}(q-2)^4\right)$. The order of $G_q - S$ is $2q(q-1)^2 < 2\left(\sum_{i=0}^{4}(q-2)^4\right)$, for all $q \geqslant 5$, a in the hypothesis. Therefore, $G_q - S$ has girth 8. □

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Corrigendum – Added May 18, 2021

With respect to the original version of the paper, we actually construct a new infinite family of $(q-1)$-regular graphs of girth 8 and order $2q(q^2-q-4)$ for all prime powers $q \geqslant 5$, which are the smallest known so far whenever $q-1$ is not a prime power or a prime power plus one itself.

The main difference is that we can no longer prove the existence of a perfect dominating set of the q-regular graphs in Definition 3 and Lemma 4 (the previous one was wrong for $q \geqslant 7$, it worked only for $q = 5$). Therefore, we cannot construct the infinite family of $(q-1)$-regular graphs of girth 8 and order $2q(q-1)^2$ for all prime powers q as originally stated in Theorem 5.

Instead, we use the following definition: if G is k-regular, a subset $S \subset V(G)$ is *a quasi-perfect dominating set of* G if $G - S$ is $(k, k-1)$–regular and the set of vertices of degree k in $G - S$ is either empty or it induces a perfect matching. We prove the existence of a quasi-perfect dominating set of the previously constructed q-regular graphs which allows us to construct a new infinite family of $(q-1)$-regular graphs of girth 8 and order $2q(q^2-q-4)$ for all odd prime powers q (cf. Theorem 5). These are the smallest $(q-1)$-regular graphs of girth 8 known so far for $q \geqslant 16$ whenever $q-1$ is not a prime power or a prime power plus one itself. Previously, the smallest known $(q-1, 8)$-graphs, for q a prime power, were those of order $2q(q^2-q-1)$ which appeared in [7].

In this corrigendum we construct $(q-1)$-regular graphs of girth 8 with $2q(q^2-q-4)$ vertices, for every prime power $q \geqslant 5$. To this purpose we use the coordinates for a Moore $(q+1, 8)$-cage Γ_q stated in Definition 1.

In definition 3 we no longer need $x \in \mathbb{F}_q \setminus \{0, 1\}$, but it is enough to have $x \in \mathbb{F}_q \setminus \{0\}$. We rewrite it for completion. The proof that it has girth 8 remains the same.

Definition 3. Let $q \geqslant 4$ be a prime power and let $x \in \mathbb{F}_q \setminus \{0\}$. Define G_q^x as the q-regular graph of order $2q(q^2-2)$ constructed by removing from Γ_q its perfect dominating set D given in Proposition 2.

Lemma 4. *The q-regular graph G_q^x in Definition 3 has girth exactly 8.*

What follows are the new and corrected results:

Theorem 5. *Let $q \geqslant 5$ be an odd prime power and let G_q^x be the graph given in Definition 3 for some $x \in \mathbb{F}_q \setminus \{0, 1, -1\}$. It is also required that if $q \equiv 1 \pmod 4$ and $q > 5$, then x must satisfy that $x^2 + 1 = 0$. Let $R = \{(1, b, -2b)_1, (-1, b, 2b)_1 : b \in \mathbb{F}_q\}$, and let $y \in \mathbb{F}_q \setminus \{0, 1, -1, x\}$; it is also required that if $q \equiv 1 \pmod 4$ and $q > 5$, then $y = x^{-1}$. Then, the set*

$$S := \bigcup_{j \in \mathbb{F}_q} N_{G_q^x}[(\varrho, y, j)_0] \cup N_{G_q^x}[R]$$

is a quasi-perfect dominating set in G_q^x of cardinality $2q^2+4q$. The set of edges $M = \{\{(a,b,-b(a+a^{-1}))_1, (\varrho, a, -b(a+a^{-1}))_0\} : a,b \in \mathbb{F}_q, a \notin \{0,1,-1,x,y\}\}$ is empty if $q=5$ or a matching if $q \geqslant 7$ and $G_q^x - S - M$ is a $(q-1)$-regular graph of girth 8 and order $2q(q^2-q-4)$.

Proof. To simplify notation, we will denote G_q^x by G throughout the proof. Observe that $N_G(\{(1,b,-2b)_1 : b \in \mathbb{F}_q\} = \{(j,t,j)_0 : j,t \in \mathbb{F}_q, t \neq 0\} \cup \{(\varrho,1,c)_0 : c \in \mathbb{F}_q\}$, because recall that the vertices $(j,0,k)_0$ with second coordinate zero have been removed from Γ_q to obtain G. Similarly, $N_G(\{(-1,b,2b)_1 : b \in \mathbb{F}_q\} = \{(j,t,j)_0 : j,t \in \mathbb{F}_q, t \neq 0\} \cup \{(\varrho,-1,c)_0 : c \in \mathbb{F}_q\}$, yielding that $|N_G[R]| = 2q+q(q-1)+2q = q^2+3q$. Moreover, since $y \in \mathbb{F}_q \setminus \{0,1,-1,x\}$, it follows that the set $\bigcup_{j\in\mathbb{F}_q} N_G[(\varrho,y,j)_0] = \{(y,b,c)_1 : b,c \in \mathbb{F}_q\}$ is disjoint with $N_G[R]$. Therefore $|S| = 2q^2+4q$. Let us show that S is a quasi-perfect dominating set. Observe that $(j,t,k)_0 \in V(G)$, $j,t,k \in \mathbb{F}_q, t \neq 0$, is adjacent to exactly one vertex $(y, t-yj, y^2j-2yt+k)_1$ starting by y. This implies that, once removed from G the set $\bigcup_{j\in\mathbb{F}_q} N_G[(\varrho,y,j)_0]$, all the vertices $(j,t,k)_0$ will have degree $q-1$ and the vertices $(\varrho,a,c)_0$, $a \in \mathbb{F}_q \setminus \{0,x,y\}$, $c \in \mathbb{F}_q$ will have degree q. Thus, once the points $(j,t,j)_0$, $j,t \in \mathbb{F}_q, t \neq 0$, have been removed from G, only the vertices $(a,b,c)_1$ adjacent to vertices $(j,0,j)_0$ in Γ_q will remain with degree q in $G-S$. Therefore, the vertices of the set $M = \{(a,b,-b(a+a^{-1}))_1, (\varrho,a,c)_0 : a,b,c \in \mathbb{F}_q, a \notin \{0,1,-1,x,y\}\}$ have degree q in $G-S$. Suppose $q \equiv 3 \pmod 4$, we know that -1 has no square root in $\mathbb{F}_q$. This implies that $a+a^{-1} \neq 0$ for all $a \in \mathbb{F}_q \setminus \{0,1,-1,x,y\}$ and for every $b \in \mathbb{F}_q$ there is a unique value $-b(a+a^{-1})$ associated to a. Since $(a,b,-b(a+a^{-1}))_1$ is adjacent to $(\varrho,a,-b(a+a^{-1})_0$, it follows that the subgraph induced by M is a matching. Finally, suppose that $q \equiv 1 \pmod 4$, we know that -1 has two square roots in $\mathbb{F}_q$, which are x and x^{-1} because by hypothesis $x^2+1=0$. Hence $a+a^{-1} \neq 0$ for all $a \in \mathbb{F}_q \setminus \{0,1,-1,x,y\}$ and we proceed as before. We conclude that S is a quasi-perfect dominating set. □

Theorem 6. *Let $q \geqslant 8$ be an even prime power and $x \in \mathbb{F}_q \setminus \{0,1\}$. Let G_q^x be the graph given in Definition 3. Let $m \in \mathbb{F}_q \setminus \{0,1,x\}$ and let $R = \{(m,b,(m+m^{-1})b)_1, (m^{-1},b,(m+m^{-1})b)_1 : b \in \mathbb{F}_q\}$. Then, the set*

$$S := \{N_{G_q^x}[(\varrho,1,c)_0] : c \in \mathbb{F}_q\} \cup N_{G_q^x}[R]$$

is a quasi-perfect dominating set in G_q^x of cardinality $2q^2-4q$. The set of edges $M = \{\{(\varrho,a,(a+a^{-1})b)_0, (a,b,(a+a^{-1})b)_1\} : a \in \mathbb{F}_q, a \neq 0,1,x,m,m^{-1}\}$ satisfies for $q \geqslant 8$ that $G_q^x - S - M$ is a $(q-1)$-regular graph of girth 8 and order $2q(q^2-q-4)$.

Proof. Once $x \in \mathbb{F}_q \setminus \{0,1\}$ has been chosen to define G_q^x, to simplify notation, we will denote G_q^x by G throughout the proof. We have $d_G((\varrho,a,c)_0, (\varrho,a,c')_0) = 6$, for all $a \in \mathbb{F}_q$, $a \neq 0,x$, and two distinct $c,c' \in \mathbb{F}_q$ (these two vertices were neighbors in Γ_q of vertex $(\varrho,\varrho,a)_1$ eliminated from Γ_q to obtain G). Hence, after the deletion of $A = \{N_{G_q^x}[(\varrho,1,c)_0] : c \in \mathbb{F}_q\}$ the resulting graph H has degrees $q-1$ and q and every vertex of the form $(i,j,k)_0$ with $i \in \mathbb{F}_q$ has degree $q-1$. Indeed, it is enough to note that $(i,j,k)_0, (1,i+j,i+k)_1, (\varrho,1,i+k)_0$ is a path of length two. Also observe that $|A| = q(q+1)$.

Let $m \in \mathbb{F}_q \setminus \{0,1,x\}$, then $m \neq m^{-1}$. Consider the sets: $W_1 = \{(m, b, (m+m^{-1})b)_1 : b \in \mathbb{F}_q\}$ and $W_2 = \{(m^{-1}, b, (m+m^{-1})b)_1 : b \in \mathbb{F}_q\}$. Let us show that $N_G(W_1) \setminus \{(\varrho, m, i)_0 : i \in \mathbb{F}_q\} = N_G(W_2) \setminus \{(\varrho, m^{-1}, k)_0 : k \in \mathbb{F}_q\}$. It is enough to show that if $\ell_1 \in W_1$ and $\ell_2 \in W_2$ we have $d_G(\ell_1, \ell_2) = 2$. Indeed, $\ell_1 = (m, b, (m+m^{-1})b)_1, (w, mw+b, m^2w+(m+m^{-1})b)_0, (m^{-1}, c, (m+m^{-1})c)_1 = \ell_2$ is a path of length two, with $c = (m+m^{-1})w + b$. Therefore, $N_G(W_1) \setminus \{(\varrho, m, i)_0 : i \in \mathbb{F}_q\} = N_{G_q}(W_2) \setminus \{(\varrho, m^{-1}, k)_0 : k \in \mathbb{F}_q\}$ as claimed. Then, the cardinality of $N_G[R]$ is $|N_G[W_1]| + |W_2| + |\{(\varrho, m^{-1}, k)_0 : k \in \mathbb{F}_q\}| = q^2 + 3q$.

Let $v \notin N_G[R]$ be such that $v = (r, s, t)_1$ with $r \in \mathbb{F}_q$, $r \neq 1$, and suppose that $|N_G(v) \cap N_G[R]| \geqslant 1$. If $r = m$ and $(m, b, (m+m^{-1})b)_1, (\varrho, m, (m+m^{-1})b)_0, (m, s, t)_1$ is a path of length 2, then $t = (m+m^{-1})b$. Analogously, if $r = m^{-1}$, then $t = (m+m^{-1})b$. If $t \neq (m+m^{-1})b$, then $d_G((m, b, (m+m^{-1})b)_1, (m, s, t)_1) = 4$ because $d_G((\varrho, m, (m+m^{-1})b)_0, (\varrho, m, t)_0) = 6$ and $(m, b, (m+m^{-1})b)_1 \sim (\varrho, m, (m+m^{-1})b)_0$, $(m, s, t)_1 \sim (\varrho, m, t)_0$. Hence, any path of length two $(m, b, (m+m^{-1})b)_1, (w, wm+b, wm^2+(m+m^{-1})b)_0, (r, s, t)_1$ satisfies that $r \neq m$, $w \neq m^{-1}b$ because the elements $(i, 0, k)_0$ are not in G and,

$mw + b = rw + s$, or equivalently $w(m+r) = b + s$ and

$m^2w + (m+m^{-1})b = r^2w + t$, or equivalently $w(m^2+r^2) = (m+m^{-1})b + t$.

Since q is even, $(m^2+r^2) = (m+r)^2$, and it follows that $(m+r)(b+s) = (m+m^{-1})b+t$, which implies that $t = (m^{-1}+r)b + (m+r)s$.

Analogously, if $(m^{-1}, b, (m+m^{-1})b)_1, (w, wm^{-1}+b, wm^{-2}+(m+m^{-1})b)_0, (r, s, t)_1$ is a path of length 2, then $r \neq m^{-1}$ and $t = (m+r)b + (m^{-1}+r)s$.

Suppose by contradiction that $|N_G(v) \cap N_G[R]| \geqslant 2$, which implies that there is a path $\ell_1, p_1, v, p_2, \ell_2$ of length four with $p_i \in N(R)$ and $\ell_i \in R$, $i = 1, 2$. This means that $\ell_1, \ell_2 \in W_1$ or $\ell_1, \ell_2 \in W_2$, because $d_G(\ell_1, \ell_2) = 4$. Suppose that both $\ell_1, \ell_2 \in W_1$, (the case $\ell_1, \ell_2 \in W_2$ is analogous) then $\ell_i = (m, b_i, (m+m^{-1})b_i)_1$, $i = 1, 2$, with $b_1 \neq b_2$ because we are assuming that $\ell_1 \neq \ell_2$. Then $v = (r, s, t)$ with either $t = (m+m^{-1})b_1 = (m+m^{-1})b_2$ or with $t = (m^{-1}+r)b_1 + (m+r)s = (m^{-1}+r)b_2 + (m+r)s$. In the former case $b_1 = b_2$ which is a contradiction. In the later case, $(m^{-1}+r)(b_1+b_2) = 0$. Since $b_1 \neq b_2$ it follows that $r = m^{-1}$ which is a contradiction. Then $|N_G(v) \cap N_G[R]| = 1$.

Let $H' = G - (\{N_{G_q^x}[(\varrho, 1, c)_0] : c \in \mathbb{F}_q\} \cup N_{G_q^x}[R])$. The degree of $(\varrho, a, c)_0$ with $a, c \in \mathbb{F}_q$, $a \neq 0, 1, x, m, m^{-1}$, is q in H'. And every element $(a, b, c)_1$ such that $d_G(\ell, (a, b, c)_1) = 4$ for some $\ell \in R$ and $d_G((\varrho, 1, c)_0, (a, b, c)_1) = 5$ for some $c \in \mathbb{F}_q$ has also degree q in H'.

For all $b, w \in \mathbb{F}_q$ we have the following shortest paths:

$$(\varrho, 1, 0)_0, (1, b, 0)_1 (w, w+b, w)_0, (a, w+b+aw, a^2w+w)_1, (\varrho, a, a^2w+w)_0, (a, aw, aw(a+a^{-1}))_1$$

$$(m, b, (m+m^{-1})b)_1, (0, b, (m+m^{-1})b)_0, (\varrho, b, 0)_1, (0, b, (a+a^{-1})b)_0, (a, b, (a+a^{-1})b)_1$$

Therefore the elements $(a, b, (a+a^{-1})b)_1$ with $a \neq 0, 1, x, m, m^{-1}$ have degree q in H'. Since $(a, b, (a+a^{-1})b)_1$ is adjacent to $(\varrho, a, c)_0$ by deleting edges $M = \{\{(\varrho, a, a+a^{-1})b)_0, (a, b, (a+a^{-1})b)_1\} : a \in \mathbb{F}_q, a \neq 0, 1, x, m, m^{-1}\}$ from H' we obtain a $(q-1)$-regular graph of order $2q(q^2-q-4)$ and the result holds. □