Sprague-Grundy Values of the $\mathcal{R}$-Wythoff Game

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Abstract

We examine the Sprague-Grundy values of the game of $\mathcal{R}$-Wythoff, a restriction of Wythoff's game introduced by Ho, where each move is either to remove a positive number of tokens from the larger pile or to remove the same number of tokens from both piles. Ho showed that the P-positions of $\mathcal{R}$-Wythoff agree with those of Wythoff's game, and found all positions of Sprague-Grundy value 1. We describe all the positions of Sprague-Grundy value 2 and 3, and also conjecture a general form of the positions of Sprague-Grundy value g.

Keywords: Wythoff's Game; Sprague-Grundy values

1 Introduction

Wythoff's Game is a two-player impartial game played with two piles of tokens. Players alternate turns and for each move a player can remove either a positive number of tokens from one pile, or the same positive number of tokens from both piles. The last player to move wins.

$\mathcal{R}$-Wythoff is a restriction of Wythoff's game introduced by Ho [3] where each move is either to remove a positive number of tokens from the larger pile or to remove the same number of tokens from both piles.

From here on, we assume that both players play optimally - that is, every move leads to the best possible outcome for that player regardless of his opponent's responses. An N-position of the game is one where the next player to move wins, and a P-position is one where the previous player wins. A generalization of these concepts is given by the Sprague-Grundy function $\mathcal{G}$, defined as follows:

- The terminal position has Sprague-Grundy value 0.

- Let $\mathbb{N}_0$ be the set of non-negative integers. Given a finite subset $S \in \mathbb{N}_0$, the *minimal excludant* of S is $\text{mex}(S) = \min(\mathbb{N}_0 \setminus S)$, or the smallest non-negative integer not in S. The *Sprague-Grundy value* of a position p is defined recursively as $\mathcal{G}(p) = \text{mex}\{\mathcal{G}(q) : q \in F\}$, where F is the set of all positions reachable in one move from p.

This function generalizes P- and N- positions because the P-positions of any game are exactly the positions with Sprague-Grundy value 0. Additionally, knowing the Sprague-Grundy function of individual combinatorial games allows fast calculation of the Sprague-Grundy function, and hence winning strategy, of the sum of these games.

1.1 Previous Results

Wythoff gave a simple closed form for the P-positions of his game.

Theorem 1 ([6]). *The P-positions of Wythoff's game are $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor)$ and $(\lfloor \phi^2 n \rfloor, \lfloor \phi n \rfloor)$ for $n \geqslant 0$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.*

For the game of $\mathcal{R}$-Wythoff, Ho proved the remarkable fact that the positions of Sprague-Grundy value 0 are exactly the same set as those of Wythoff's game. He additionally showed that the positions of Sprague-Grundy value 1 are exactly the translations of the P-positions by -1 in both dimensions, with finite exceptions [3].

However, aside from these positions with Sprague-Grundy value 0, the Sprague-Grundy values of Wythoff's game are quite chaotic; for example, no polylogarithmic algorithm has been found to determine the Sprague-Grundy value of a given position [5]. Some authors analyzed positions of a *fixed* Sprague-Grundy value. Blass and Fraenkel looked at all positions of Sprague-Grundy value 1 [1], and Nivasch analyzed positions of value g for an arbitrary fixed value[5].

Here we analyze the set of positions of a fixed Sprague-Grundy value for the game $\mathcal{R}$-Wythoff. We determine all positions of Sprague-Grundy value 2 and 3, and conjecture that for any constant g, the set of all positions having Sprague-Grundy value g has a form similar to that of $2-$ and $3-$ positions. If this conjecture is true, then all positions of value g can be characterized after finite computation, in contrast to existing results on Wythoff's game.

1.2 Notation

Let (a, b) represent a position in $\mathcal{R}$-Wythoff. If $\mathcal{G}(a, b) = g$, we call (a, b) a *g-position.*

A position q is a *follower* of p if $p \to q$ is a valid move. If $\mathcal{G}(q) = g$ we will sometimes call q a g-follower of p.

When plotting values of $\mathcal{G}$ we will use the following graphical representation. The first coordinate is plotted vertically, increasing upwards, and the second coordinate is plotted horizontally (see Figure 1). Consequently we call *row r* the set of points (r, x) for $x \geqslant 0$ and *column c* the set of points (x, c) for $x \geqslant 0$. Also, we call *diagonal d* the set of points $(x, x + d)$ for $x \geqslant 0$. In general, when talking about a position (a, b) we assume $a \leqslant b$.

9	9	9	9	5	9	1	9	5	9	10
8	8	8	8	8	8	2	8	6	7	9
7	7	7	7	7	0	7	7	8	6	5
6	6	6	6	1	1	4	5	7	8	9
5	5	5	5	0	5	6	4	7	2	1
4	4	4	4	2	3	5	1	0	8	9
3	3	3	3	4	2	0	1	7	8	5
2	2	0	1	3	4	5	6	7	8	9
1	1	2	0	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
a/b	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Figure 1: Sprague-Grundy Values of R-Wythoff

Let $T_g = ((a_0^g, b_0^g), (a_1^g, b_1^g), \ldots)$ denote the sequence of g-values having $a_n^g \leqslant b_n^g$, in increasing order of first coordinate. For convenience let $p_n^g = (a_n^g, b_n^g)$ and $d_n^g = b_n^g - a_n^g$. When it is clear that we are talking about a specific g, the superscript will sometimes be dropped.

2 Computing Sprague-Grundy Values

2.1 Computing g-positions

We first present an algorithm, based off one of Blass and Frankel's for Wythoff's game [1], which computes the sequence T_g for any positive integer g. Suppose we have already know T_h for $0 \leqslant h < g$. Now suppose we have used Algorithm $\mathcal{R}$WSG to compute p_i^g for $i = 0, \ldots, k-1$. We run $\mathcal{R}$WSG again to compute p_k^g. Intuitively, it is greedily putting p_k^g into the smallest viable row and then the earliest column that does not lie on the same diagonal as an existing g-position.

Algorithm $\mathcal{R}$WSG

1. $p \leftarrow \text{mex}\{a_i^g, b_i^g \,:\, 0 \leqslant i < k\}$

2. $d \leftarrow$ smallest non-negative integer such that

   (i) $d \notin \{d_i^g \,:\, 0 \leqslant i < k\}$ and

   (ii) $(p, p+d) \notin T_h$ for $0 \leqslant h < g$

3. $(a_k^g, b_k^g) \leftarrow (p, p+d)$

We will now prove the correctness of $\mathcal{R}$WSG on computing T_g given that $T_0, \ldots, T_{g-1}$ are known.

Theorem 2. *The sequence determined by Algorithm $\mathcal{R}$WSG is exactly the sequence T_g of positions with Sprague-Grundy value g.*

Proof. We proceed by induction on k, where we have used the algorithm to compute p_k^g. For $k = 0$ the algorithm computes p_0^g correctly as $(0, g)$, since $p_0^h = (0, h)$ for $0 \leqslant h < g$. Now suppose we have correctly computed $p_0^g, \ldots, p_{k-1}^g$. Let $(p, p + d)$ be the position $\mathcal{R}$WSG computes next; we must show it is p_k^g.

First, note that $a_k^g \geqslant p$. This follows from the fact that $a_k^g \neq a_i^g$ and $a_k^g \neq b_i^g$ for any $0 \leqslant i < k$.

Next, each of the positions $(p, p + i) : 0 \leqslant i < d$ must have not satisfied either 2(i) or 2(ii), or the algorithm would have selected position $(p, p + i)$ instead of $(p, p + d)$. Thus each of these positions either lies in T_h for some $h < g$, or has a g-position as a follower along a diagonal. In particular, $\mathcal{G}(p, p + i) \neq g$ for all $0 \leqslant i < d$.

Now by Step 2(ii), $(p, p + d) \notin T_0, \ldots, T_{g-1}$, so $\mathcal{G}(p, p + d) \geqslant g$.

Finally, we deduce that $\mathcal{G}(p, p + d) \leqslant g$. Otherwise, $(p, p + d)$ has a follower that is a g-position from the definition of mex. But we just observed that no position $(p, p + i)$: $0 \leqslant i < d$ is a g-position, so it has no g-position left of it along its row. Also, by Step 2(i) it has no g-position on its diagonal. So $(p, p+d)$ does not have a g-follower, implying that $\mathcal{G}(p, p + d) \leqslant g$.

We conclude that $\mathcal{G}(p, p + d) = g$, and since $a_k^g \geqslant p$ we must have $p_k^g = (a_k^g, b_k^g) = (p, p + d)$. □

In practice, when using $\mathcal{R}$WSG to compute T_g it suffices to pick a large upper bound U and compute T_h for $0 \leqslant h < g$ up to all $a_i^h \leqslant U$, and then compute T_g up to $a_i^g \leqslant U$.

2.2 Alternate Proofs of Previous Results

As a direct consequence of Step 1 of this algorithm, we have an alternate proof of two theorems of Ho.

Theorem 3 (Theorem 7 of [3]). *For integers a and c, there exists an integer b such that $\mathcal{G}(a, b) = c$.*

Theorem 4 (Theorem 8 of [3]). *For nonnegative integers a and c, there exists a unique b such that $\mathcal{G}(b, a + b) = c$.*

Proof. Uniqueness is immediate since no g-position can be a follower of another. For existence, it suffices to show that the value a is chosen in Step 2 at some iteration in $\mathcal{R}$WSG. Step 2 will only choose a value greater than a if a fails (i) or (ii). If it fails (i), then some g-position already lies on diagonal a and we are done. It can only fail (ii) a finite number of times, once for each $h < g$. Therefore Step 2 will set $d = a$ after a finite number of iterations. □

3 Characterization of g-Positions

Ho determined the positions of Sprague-Grundy value 0 and 1.

Theorem 5 (Theorem 2 of [3]). *The position (a, b) with $a \leqslant b$ is a P-position if and only if it is of the form $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor)$ for $n \geqslant 0$.*

Theorem 6 (Theorem 4 of [3]). *The position* (a, b) *with* $a \leqslant b$ *has Sprague-Grundy value* 1 *if and only if* (a, b) *is an element of the set*

$$\{(2,2),(4,6),(\lfloor \phi n \rfloor - 1, \lfloor \phi^2 n \rfloor - 1) \mid n \geqslant 1, n \neq 2\}$$

3.1 Positions of Sprague-Grundy value 2 and 3

We begin by giving all positions of Sprague-Grundy value 2. Recall that p_n^0 denotes the nth 0-position and for shorthand define $p_n^0 - (x, y) = (\lfloor \phi n \rfloor - x, \lfloor \phi^2 n \rfloor - y)$; that is, subtracting positions is done by coordinate-wise subtraction.

Theorem 7. *The first* 10 *values of* T_2 *are* $(0,2),(1,1),(3,4),(5,8),(6,11),(7,11),$ $(9,16),(10,16),(12,21),(13,21)$.

Define the sequences

$$\begin{aligned}
(m_k^1)_{k\geqslant 0} &: m_0^1 = 10, m_1^1 = 17, m_{k+2}^1 = m_{k+1}^1 + m_k^1 \\
(m_k^2)_{k\geqslant 0} &: m_0^2 = 11, m_1^2 = 18, m_{k+2}^2 = m_{k+1}^2 + m_k^2 \\
(m_k^3)_{k\geqslant 0} &: m_0^3 = 12, m_1^3 = 20, m_{k+2}^3 = m_{k+1}^3 + m_k^3 \\
(m_k^4)_{k\geqslant 0} &: m_0^4 = 15, m_1^4 = 24, m_{k+2}^4 = m_{k+1}^4 + m_k^4
\end{aligned}$$

For $n \geqslant 10$, $p_n^2 =$

$$\begin{cases}
p_n^0 - (2,2) & \text{if } n = m_k^1,\ k \text{ even} \\
p_n^0 - (4,4) & \text{if } n = m_k^1,\ k \text{ odd} \\
p_n^0 - (2,2) & \text{if } n = m_k^2,\ k \text{ even} \\
p_n^0 - (4,4) & \text{if } n = m_k^2,\ k \text{ odd} \\
p_n^0 - (2,2) & \text{if } n = m_k^3,\ k \text{ even} \\
p_n^0 - (4,4) & \text{if } n = m_k^3,\ k \text{ odd} \\
p_n^0 - (4,4) & \text{if } n = m_k^4,\ k \text{ even} \\
p_n^0 - (2,2) & \text{if } n = m_k^4,\ k \text{ odd} \\
p_n^0 - (3,3) & \text{otherwise.}
\end{cases}$$

We first show that the above formula is well-defined because no n can fall into multiple cases.

Proposition 1. *The* (m_k^i) *are disjoint.*

Proof. We can inductively show that $m_k^1 < m_k^2 < m_k^3 < m_k^4 < m_{k+1}^1$. This is true for $k = 0$ and $k = 1$ by definition of the sequences (note that $m_2^1 = 27 > m_1^4$).

Then if

$$\begin{aligned}
m_k^1 < m_k^2 < m_k^3 < m_k^4 < m_{k+1}^1, \\
m_{k+1}^1 < m_{k+1}^2 < m_{k+1}^3 < m_{k+1}^4 < m_{k+2}^1,
\end{aligned}$$

adding these inequality chains together yields $m_{k+2}^1 < m_{k+2}^2 < m_{k+2}^3 < m_{k+2}^4 < m_{k+3}^1$. □

Before proving Theorem 7, we need the following technical propositions.

Proposition 2. *Consider a positive integer sequence* $a_0, a_1, a_2, \ldots$ *satisfying* $a_{n+2} = a_{n+1} + a_n$ *for* $n \geqslant 0$*. Then we can find constants* c_1 *and* c_2 *such that* $a_n = c_1\phi^n + c_2\psi^n$*, where* $\phi = \frac{1+\sqrt{5}}{2}$ *and* $\psi = -\frac{1}{\phi}$*. Furthermore, if* $|c_2| < -\frac{1}{\psi^{-1}+\psi^1} = 1/\sqrt{5} \approx 0.447$*, then*

$$(\lfloor \phi a_n \rfloor, \lfloor \phi^2 a_n \rfloor) = \begin{cases} (a_{n+1}, a_{n+2}) & \textit{if } c_2(-1)^n \geqslant 0 \\ (a_{n+1} - 1, a_{n+2} - 1) & \textit{if } c_2(-1)^n < 0 \end{cases}$$

Proof. The fact that we can write $a_n = c_1\phi^n + c_2\psi^n$ is a standard result of linear recurrences because ϕ, ψ are the roots of the characteristic equation $x^2 - x - 1 = 0$.

Therefore $\phi a_n = \phi(c_1\phi^n + c_2\psi^n) = c_1\phi^{n+1} - c_2\psi^{n-1} = c_1\phi^{n+1} + c_2\psi^{n+1} - (c_2\psi^{n+1} + c_2\psi^{n-1}) = a_{n+1} + c_2\psi^n(-\psi^{-1} - \psi^1)$. Then the expression for $\lfloor \phi a_n \rfloor$ follows immediately, and $\lfloor \phi^2 a_n \rfloor = \lfloor \phi a_n \rfloor + a_n$ which simplifies using $a_n + a_{n+1} = a_{n+2}$. □

Let T_2' denote the sequence of positions described in Theorem 7; our final goal is to show that $T_2' = T_2$, the sequence of 2-positions.

We use Proposition 2 to show the following proposition, which tells us about the distribution of the elements of T_2'.

Proposition 3. *Define* $p_n' = (a_n', b_n')$ *to be the nth element of* T_2'*. Consider the sequence*

$$S' := (a_{10}', b_{10}', a_{11}', b_{11}', a_{12}', \ldots).$$

Then this sequence contains no duplicates and covers the set

$$\mathbb{N}_0 \setminus \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 21\}.$$

Proof. We will compare the elements of S' to the sequence

$$S := (\lfloor 10\phi \rfloor - 3, \lfloor 10\phi^2 \rfloor - 3, \lfloor 11\phi \rfloor - 3, \lfloor 11\phi^2 \rfloor - 3, \ldots)$$

. Because $(\lfloor n\phi \rfloor)_{n \geqslant 1}$ and $(\lfloor n\phi^2 \rfloor)_{n \geqslant 1}$ are complementary Beatty sequences [2], the elements of S are distinct and consist of the integers

$$\mathbb{N} \setminus \{\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor \,:\, 1 \leqslant n \leqslant 9\} = \mathbb{N}_0 \setminus \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 17, 20\}$$

By definition of T_2', S' and S agree everywhere except along (m^1), (m^2), (m^3), (m^4). It remains to compare these exceptions.

More precisely, let

$$\begin{aligned} M^1 &:= (\lfloor \phi m_0^1 \rfloor - 3, \lfloor \phi^2 m_0^1 \rfloor - 3, \lfloor \phi m_1^1 \rfloor - 3, \lfloor \phi^2 m_1^1 \rfloor - 3, , \lfloor \phi m_2^1 \rfloor - 3, \ldots) \\ M'^1 &:= (\lfloor \phi m_0^1 \rfloor - 2, \lfloor \phi^2 m_0^1 \rfloor - 2, \lfloor \phi m_1^1 \rfloor - 4, \lfloor \phi^2 m_1^1 \rfloor - 4, \lfloor \phi m_2^1 \rfloor - 2, \ldots) \end{aligned}$$

be the subsequences of S and S', respectively, along the indices given by (m^1). Define M^2, M'^2, M^3, M'^3, M^4, M'^4 similarly.

Note that we can write $m^1_k = (5+\frac{12}{\sqrt{5}})\phi^k + (5-\frac{12}{\sqrt{5}})\psi^k$ because it satisfies the Fibonacci recurrence, and $5 - \frac{12}{\sqrt{5}} \approx -0.367$. Therefore by Proposition 2, we can rewrite these sequences as

$$\begin{aligned} M'^1 &= (m^1_1 - 3, m^1_2 - 3, m^1_2 - 4, m^1_3 - 4, m^1_3 - 3, \dots) \\ M^1 &= (m^1_1 - 4, m^1_2 - 4, m^1_2 - 3, m^1_3 - 3, m^1_3 - 4, \dots) \end{aligned}$$

Thus $\{M'^1\} = \{M^1\} \cup (m^1_1 - 3) \setminus (m^1_1 - 4) = \{M^1\} \cup 14 \setminus 13$.

Similarly,

$$\begin{aligned} m^2_k &= \left(\frac{11}{2} + \frac{25}{2\sqrt{5}}\right)\phi^k + \left(\frac{11}{2} - \frac{25}{2\sqrt{5}}\right)\psi^k \\ m^3_k &= \left(6 + \frac{14}{\sqrt{5}}\right)\phi^k + \left(6 - \frac{14}{\sqrt{5}}\right)\psi^k \\ m^4_k &= \left(\frac{15}{2} + \frac{33}{2\sqrt{5}}\right)\phi^k + \left(\frac{15}{2} - \frac{33}{2\sqrt{5}}\right)\psi^k \end{aligned}$$

and $\frac{11}{2} - \frac{25}{2\sqrt{5}} \approx -0.090, 6 - \frac{14}{\sqrt{5}} \approx -0.261, \frac{15}{2} - \frac{33}{2\sqrt{5}} \approx 0.121$. Applying Proposition 2 in the same way as on (m^1) gives $\{M'^2\} = \{M^2\} \cup 15 \setminus 14$, $\{M'^3\} = \{M^3\} \cup 17 \setminus 16$, and $\{M'^4\} = \{M^4\} \cup 20 \setminus 21$.

Therefore

$$\begin{aligned} \{S'\} &= \{S\} \setminus \big(\{M^1\} \cup \{M^2\} \cup \{M^3\} \cup \{M^4\}\big) \cup \{M'^1\} \cup \{M'^2\} \cup \{M'^3\} \cup \{M'^4\} \\ &= \mathbb{N}_0 \setminus \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 21\}. \end{aligned}$$

□

Corollary 1. *The elements of T'_2 (and their reflections) cover every row.*

Proof. It suffices to show that $\{a'_n, b'_n : n \geqslant 0\}$ covers $\mathbb{N}_0$. Using the notation of Proposition 3, this set can be written as $\{a'_n, b'_n : 0 \leqslant n < 10\} \cup \{S'\}$. The result follows immediately from the values defined in Theorem 7 and Proposition 3. □

As a tool for proving Theorem 7, we will use the following characterization of the sequence T_g due to Jiao [4].

Proposition 4 (Lemma 8 in [4])**.** *Every T_g consists exactly of the positions having Sprague-Grundy value g if and only if every T_g satisfies*

1. *$T_g \cap T_h = \emptyset$ for $h < g$.*
2. *If $p \in T_g$, then p has no follower in T_g.*
3. *If $p \notin T_0 \cup \dots \cup T_g$, then p has a follower in T_g.*

Now we can prove Theorem 7.

21													2	2								
20													0									
19												1										
18												0										
17											1											
16										2	2											
15										0												
14																						
13									0													2
12								1													0	2
11							2	2											0	1		
10							0										2	1				
9						1										0	2					
8						2								0	1							
7					0							2	1									
6				1	1						0	2										
5				0					2	1												
4				2			1	0														
3					2	0	1															
2	2	0	1																			
1	1	2	0																			
0	0	1	2																			
	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**

Figure 2: Positions of Sprague-Grundy value 0, 1, 2

Proof of Theorem 7. It suffices to show that T_2' satisfies the properties of Proposition 4. First, direct computation using Algorithm $\mathcal{R}$WSG gives $p_0^2, \ldots, p_9^2$, and we can manually check that these values satisfy the appropriate properties (see Figure 2).

It suffices to check the properties for the rest of T_2'.

1. Note that the nth element of T_2', $n \geqslant 10$ lies on diagonal n. By Theorem 4, every diagonal contains exactly one 0- and one 1-position. By Theorem 5 and Theorem 6 these positions are given by $p_n^0 - (0,0)$ and $p_n^0 - (1,1)$, respectively. In other words, they are offset from p_n^0 along the diagonal by $0, 1$ respectively. However, the corresponding position in T_2' is offset by 2 or 4. Therefore the positions in T_2' do not collide with positions in T_0 and T_1.

2. Note that $\{b_n' - a_n' \,:\, 0 \leqslant n \leqslant 9\} = \{0, 1, 2, \ldots, 9\}$. Furthermore $b_n' - a_n' = n$ for $n \geqslant 10$. Therefore no two positions in T_2' are diagonal followers. Finally, Proposition 3 and noting that $\{S'\}$ and $\{a_n', b_n' \,:\, 0 \leqslant n \leqslant 9\}$ are disjoint imply that no two positions are row followers.

3. Let $q \notin T_0 \cup T_1 \cup T_2'$. We will find a $p \in T_2'$ that is a follower of q. Without loss of generality let q be below the main diagonal and write q as $(r, r+d)$ with $d \geqslant 0$. Let

$p \in T_2'$ be the unique element on diagonal d. If p is below q we are done, so assume otherwise. If $d \leqslant 9$ it is clear from inspecting Figure 2 that the claim holds. Suppose $d \geqslant 10$.

Then $p = p_d'$. By Corollary 1, there exists k such that $a_k' = r$ or $b_k' = r$. In the latter case (b_k', a_k') is a 2-follower of q. In the former case, $a_k' = r < a_d'$. Since $d \geqslant 10$ and $(a_n' : n \geqslant 10)$ is increasing, $k < d$. Then the diagonal p_k' lies on is less than the diagonal p_d' lies on, so $b_k' < r + d$, and p_k' is a row follower of q.

□

Similarly to the positions of Sprague-Grundy value 2, we may determine positions of Sprague-Grundy value 3. To save space we let $F^{(a,b)}$ denote the sequence $(m_k)_{k \geqslant 0} : m_0 = a, m_1 = b, m_{k+2} = m_{k+1} + m_k$.

Theorem 8. *The first* 36 *values of* T_3 *are* $(0,3)$, $(1,3)$, $(2,3)$, $(4,4)$, $(5,10)$, $(6,12)$, $(7,14)$, $(8,12)$, $(9,17)$, $(11,20)$, $(13,23)$, $(15,27)$, $(16,29)$, $(18,29)$, $(19,34)$, $(21,37)$, $(22,39)$, $(24,38)$, $(25,44)$, $(26,44)$, $(28,49)$, $(30,50)$, $(31,53)$, $(32,55)$, $(33,57)$, $(35,60)$, $(36,62)$, $(40,67)$, $(41,69)$, $(42,71)$, $(43,73)$, $(45,76)$, $(46,78)$, $(47,80)$, $(48,82)$, $(51,86)$.

For $n \geqslant 36$,

$$p_n^3 = p_n^0 - (5,5) + \begin{cases} \big((-1)^{k+1}, (-1)^{k+1}\big) & \text{if } n = F_k^{(36,58)} \\ \big((-1)^{k}, (-1)^{k}\big) & \text{if } n = F_k^{(42,68)} \\ \big((-1)^{k+1}, (-1)^{k+1}\big) & \text{if } n = F_k^{(44,71)} \\ \big((-1)^{k+1}, (-1)^{k+1}\big) & \text{if } n = F_k^{(45,72)} \\ \big((-1)^{k+1}, (-1)^{k+1}\big) & \text{if } n = F_k^{(46,74)} \\ \big((-1)^{k+1}, (-1)^{k+1}\big) & \text{if } n = F_k^{(47,76)} \\ \big((-1)^{k}, (-1)^{k}\big) & \text{if } n = F_k^{(53,86)} \\ \big((-1)^{k}, (-1)^{k}\big) & \text{if } n = F_k^{(54,88)} \\ \big((-1)^{k}, (-1)^{k}\big) & \text{if } n = F_k^{(55,89)} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Follows similarly to the proof of Theorem 7. □

3.2 Positions of Sprague-Grundy Value g

Next we turn out attention to greater g-positions. For the remainder of this section, we assume g is a fixed integer greater than 3. When talking about positions $p_n^g = (a_n^g, b_n^g)$ we will drop the superscripts when it is clear we are using g.

We conjecture that in general, the positions of Sprague-Grundy value g follow a similar pattern to that of the 2- and 3-positions.

Conjecture 1. *There exists constants N, C, k and disjoint sequences $F^{(m_1,n_1)}, F^{(m_2,n_2)}, \ldots, F^{(m_k,n_k)}$ and bits $x_i \in \{0,1\}$ for $1 \leqslant i \leqslant k$ such that for all $n \geqslant N$,*

$$(a_n, b_n) = \begin{cases} p_n^0 - \left(C + (-1)^{k+x_i}, C + (-1)^{k+x_i}\right) & \textit{if } n = F_k^{(m_i,n_i)} \\ p_n^0 - (C, C) & \textit{otherwise} \end{cases}$$

Essentially, there is a offset C such that outside of a finite number of positions in T_g, the rest lie at $p_n^0 - (C, C)$ except along a finite number of Fibonacci-recurrence sequences, where they alternate above and below.

We reduce this to a weaker conjecture about the g-positions.

Conjecture 2. *There exists a constant M such that for all $n \geqslant M$, $d_n = n$.*

We will show that Conjecture 2 implies Conjecture 1, and express C in a computable way.

Definition. *In a sequence $(c_n)_{n\geqslant 0}$, let an element c_j be called a repeat if there exists some $i < j$ such that $c_i = c_j$.*

Proposition 5. *Suppose Conjecture 2 is true. Then Conjecture 1 is true and C is equal to the total number of repeats in $(b_n)_{n\geqslant 0}$.*

Proof. We start off with the following useful fact.

Claim 1. *The sequence $b_M, b_{M+1}, b_{M+2}, \ldots$ is increasing and does not contain two consecutive integers.*

Proof. For $n \geqslant M$, $b_{n+1} = a_{n+1} + n + 1 \geqslant a_n + 1 + n + 1 = b_n + 2$. □

Claim 1 implies the sequence $(b_k)_{k\geqslant 0}$ has a finite number of repeats. Let R denote the total number of repeats. We will show Conjecture 1 holds and C is this quantity R.

Without loss of generality, we also make the assumption that none of b_M, $b_{M+1}, \ldots$ are repeats of earlier elements; if this is not true, increase M until it is true, and Conjecture 2 and Claim 1 still hold for this higher value of M.

For $n > M$, let $f(n)$ be defined such that $a_{f(n)} = b_n - 1$ and $a_{f(n)+1} = b_n + 1$. This is well-defined because Claim 1 implies that $b_n - 1$ and $b_n + 1$ are not in the sequence (b_k), so they must be in the sequence (a_k). Furthermore note that

$$f(n) = |\{a_0, a_1, \ldots\} \cap \{0, 1, \ldots, b_n - 2\}|$$

because $a_0, \ldots, a_{f(n)-1}$ are the only terms of (a_k) that lie in the set $\{0, 1, \ldots, b_n - 2\}$.

The sequence $b_0, b_1, \ldots, b_{n-1}$ contains $n - R$ distinct values, so of the values $\{0, 1, \ldots, b_n - 2\}$ exactly $n - R$ of them appear in the sequence (b_n) and the rest appear in (a_n). Finally, there is exactly one k such that $a_k = b_k$ by Theorem 4. Combining these facts gives

$$f(n) = b_n - n + R = a_n + R.$$

Now consider the "error" quantity $x_n = a_n - (\phi n - R)$, similar to a technique used by Nivasch [5]. Using the above equations we can determine that

$$\begin{aligned} x_{f(n)+1} &= a_{f(n)+1} - \phi\,(f(n)+1) + R = b_n + 1 - \phi(a_n + R + 1) \\ &= a_n(1-\phi) + n + R(1-\phi) + 1 - \phi \\ &= -\frac{1}{\phi}x_n - \frac{1}{\phi} \end{aligned}$$

and similarly

$$x_{f(n)} = -\frac{1}{\phi}x_n - 1.$$

Define the functions $g_1(x) = -\frac{1}{\phi}x - \frac{1}{\phi}$ and $g_2(x) = -\frac{1}{\phi}x - 1$.

Recall that we are trying to show that the a_n tend to $\phi n + R$, so it suffices to show that x_n is close to 0. To do this we will show that for large n, x_n can be written in terms of many compositions of g_1 or g_2.

Claim 2. *For every integer $k \geqslant 1$, there exists a constant M_k such that for all $n \geqslant M_k$, there is a sequence $n = n_1 > n_2 > \cdots > n_\ell$ such that $\ell \geqslant k$, for each i either $n_i = f(n_{i+1})$ or $n_i = f(n_{i+1}) + 1$, and $M \leqslant n_\ell \leqslant b_M + R$.*

Proof. Consider some $m > b_M + R$. Then there exists n such that $m = f(n)$ or $m = f(n) + 1$: otherwise, neither $m - R$ nor $m - R - 1$ are in (a_n), so they are in (b_n), contradicting Claim 1.

Now we induct on k. For $k = 1$, we claim the constant $M_1 = M$ works. Consider any $n > M$. If $n \leqslant b_M + R$ the singleton sequence consisting of itself works. Otherwise, choose n_2 such that $n = f(n_2)$ or $f(n_2) + 1$, and continue until we reach $n_\ell \leqslant b_M + R$.

Finally, suppose the statement is true for $1, \ldots, k-1$. We claim $M_k = f(M_{k-1}) + 1$ works. Consider any $n > M_k$. Let $n_1 = n$ and let n_2 be such that $n_1 = f(n_2)$ or $f(n_2)+1$. Noting that $n_1 > M_k$ implies $n_2 > M_{k-1}$, and applying the inductive hypothesis finishes the claim. □

Now consider the function $h(x) = \min\{|x - \alpha| \,:\, \alpha \in [-1, 0]\} = \max(x - 0, -1 - x, 0)$. We establish a few properties of h in relation to g_1 and g_2.

Claim 3 (Properties of $\boldsymbol{h}$)**.**

(a) *$h(g_1(x)) \leqslant \frac{1}{\phi}h(x)$ and $h(g_2(x)) \leqslant \frac{1}{\phi}h(x)$.*

(b) *If $h(x) = 0$, then $h(g_1(x)) = h(g_2(x)) = 0$.*

(c) *If $0 < h(x) < \frac{1}{\phi}$, then*

 i. *If $x > 0$, then $h(g_1(x)) = 0$ and $g_2(x) < -1$.*

 ii. *If $x < -1$, then $h(g_2(x)) = 0$ and $g_1(x) > 0$.*

Proof. (a) We case on x. First suppose $x < 0$. Note that $g_i(x) > -1$. If $g_i(x) \leqslant 0$ then $h(g_i(x)) = 0$ and the inequality is true. In particular, if $x \geqslant -1$ then both $g_1(x), g_2(x) \leqslant 0$. So assume $x < -1$, and at least one $g_i(x) > 0$. Since $h(x) = -1 - x$, we have $h(g_2(x)) \leqslant h(g_1(x)) = \frac{-1}{\phi}(x+1) = \frac{1}{\phi}h(x)$.

Next assume $x > 0$. Clearly $g_i(x) < 0$. If both $g_i(x) \geqslant -1$, then $h(g_i(x)) = 0$ and we are done. Otherwise, $h(g_2(x)) \leqslant h(g_1(x)) = -1 - (-\frac{1}{\phi}x - 1) = \frac{1}{\phi}h(x)$.

(b) If $h(x) = 0$, then $-1 \leqslant x \leqslant 0$. Then $0 \leqslant -\frac{1}{\phi}x \leqslant \frac{1}{\phi}$. This implies $-1 \leqslant g_i(x) \leqslant 0$, so $h(g_i(x)) = 0$.

(c) i. Assume $0 < x < \frac{1}{\phi}$. Then $-1 < g_1(x) < -\frac{1}{\phi}$ so $h(g_1(x)) = 0$, and $x > 0 \implies g_2(x) = -\frac{1}{\phi}x - 1 < 0$.

ii. Assume $-1 - \frac{1}{\phi} < x < -1$. Then $-1 + \frac{1}{\phi} < g_2(x) < 0$ so $h(g_2(x)) = 0$, and $-\frac{1}{\phi}x > \frac{1}{\phi} \implies g_1(x) > 0$.

□

Let $P = \max\{h(x_n) : M \leqslant n \leqslant b_M + R\}$. For every $n \geqslant M_k$, apply Claim 2 to find a sequence $n_1 > \cdots > n_\ell$. For each $1 \leqslant i < \ell$, $n_i = f(n_{i+1})$ or $f(n_{i+1}) + 1$, so $x_{n_i} = g_1(x_{n_{i+1}})$ or $x_{n_i} = g_2(x_{n_{i+1}})$. By Claim 3(a), $h(x_{n_i}) \leqslant \frac{1}{\phi}h(x_{n_{i+1}})$. Therefore $h(x_{n_1}) \leqslant \frac{1}{\phi^{\ell-1}}h(x_{n_\ell}) \leqslant \frac{1}{\phi^{k-1}}P$.

In particular, there exists a large M' such that for all $n \geqslant M'$, $h(x_n) < \frac{1}{\phi}$.

Claim 4. *For every $n \geqslant M'$, $h(x_n) < \frac{1}{\phi}$, and there is a sequence $n = n_1 > n_2 > \cdots > n_\ell$ such that for each i either $n_i = f(n_{i+1})$ or $n_i = f(n_{i+1}) + 1$, and $M' \leqslant n_\ell \leqslant b_{M'} + R$.*

Proof. Analogous to the proof of Claim 2. □

Thus for every $n \geqslant M'$, by working backwards we can find a sequence $n = n_1 > n_2 > \cdots > n_\ell$ such that $x_{n_i} = g_1(x_{n_{i+1}})$ or $x_{n_i} = g_2(x_{n_{i+1}})$. Furthermore we can ensure that $n_k \leqslant b_{M'} + R$ and $n_{k-1} \leqslant f(b_{M'} + R) + 1$.

Finally, suppose that $h(x_n) > 0$. By Claim 3(b), none of the n_i satisfy $h(x_{n_i}) = 0$. By Claim 3(c), the sequence (n_i) satisfies either

$$x_{n_1} = g_1(x_{n_2}) = g_1(g_2(x_{n_3})) = g_1(g_2(g_1(x_{n_4}))) = \cdots$$
$$x_{n_1} = g_2(x_{n_2}) = g_2(g_1(x_{n_3})) = g_2(g_1(g_2(x_{n_4}))) = \cdots$$

This implies that any three consecutive terms of (n_i) has the form $m, f(m), f(f(m)) + 1$ or $m, f(m) + 1, f(f(m) + 1)$. But by the definition and formula for f, we have

$$f(f(m)) + 1 = a_{f(m)} + R + 1 = b_m + R = m + f(m)$$

and similarly $f(f(m) + 1) = a_{f(m)+1} + R = b_n + 1 + R = n + (f(n) + 1)$. Therefore the sequence (n_i) satisfies the Fibonacci recurrence.

To relate everything back to the g-positions (a_n, b_n), knowing x_n tells us a_n since $x_n = a_n - (\phi n - R)$. For example, if $x_n \in (0, 1]$, then $a_n = x_n + \phi n - R = 1 + \lfloor \phi n \rfloor - R$. The three relevant cases are

$$\begin{aligned} 0 < x_n \leqslant 1 &\iff a_n = (\lfloor \phi n \rfloor - R) + 1 \\ -1 < x_n \leqslant 0 &\iff a_n = \lfloor \phi n \rfloor - R \\ -2 < x_n \leqslant -1 &\iff a_n = \lfloor \phi n \rfloor - R - 1. \end{aligned}$$

In summary we have shown that for all $n > M'$, $h(x_n) < 1/\phi \implies -2 < x_n < 1$, which implies that (a_n, b_n) has the form $p_n^0 - (R, R)$ or $p_n^0 - (R \pm 1, R \pm 1)$ according to the above cases. If it has the latter form, then we can find a sequence $n = n_1 > n_2 > \cdots > n_\ell$ satisfying the Fibonacci recurrence where the terms alternate between the form $a_{n_i} = p_{n_i}^0 - (R+1, R+1)$ and $a_{n_i} = p_{n_i}^0 - (R-1, R-1)$. Since we put bounds on n_ℓ and $n_{\ell-1}$ there are a finite number of choices for this pair, but the sequence is uniquely determined by the two smallest values, so there are a finite number of these Fibonacci-recurrence "exception" sequences. We conclude that Conjecture 1 follows Conjecture 2.

□

4 Additional Comments

The best result we have shown toward Conjecture 2 is a bound on how far away $b_n^g - a_n^g$ can be from n as a constant depending on g.

Proposition 6. *For all g and $n \geqslant 0$, $|d_n^g - n| \leqslant g$.*

Proof. According to step 2 of Algorithm $\mathcal{R}$WSG, a diagonal is only skipped when a previous g-position already occupies it, or any h-position for $h < g$. When calculating p_n^g, the former can occur n times, and the latter g times since there is at most one h-position per row. Thus $d_n^g \leqslant n + g$.

Now suppose that $d_n^g < n - g$. Then out of the n previous g-positions, more than g of them skipped diagonal d_n^g. Since it had no g-positions up to this point, it could only have been skipped by step 2(ii) of $\mathcal{R}$WSG because an h-position was already there. This can happen at most g times since there can be at most one h-position per diagonal, which is a contradiction.

Therefore $n - g \leqslant d_n^g \leqslant n + g$ as desired. □

4.1 Empirical Data and Further Directions

We have verified Conjecture 2, and hence Conjecture 1, up to $g = 20$ using Algorithm $\mathcal{R}$WSG with computer assistance. Figure 3 give values for constants in the conjectures, where C_g denotes the value of C in Conjecture 1 for the g-positions and M_g denotes the value of M in Conjecture 2.

It would be helpful to determine more properties of these values. The following conjectures were attempted as progress towards Conjecture 1 and Conjecture 2.

g	C_g	M_g
1	1	4
2	3	10
3	5	22
4	12	81
5	9	242
6	22	151
7	53	638
8	33	734
9	86	1456
10	143	3712

g	C_g	M_g
11	108	6783
12	160	37083
13	198	16207
14	257	35947
15	277	74332
16	1890	116760
17	496	159488
18	1009	339201
19	1174	6150670
20	546	778123

Figure 3: Conjecture constants

Conjecture 3. *For every g, the sequence $B^g = b_0^g, b_1^g, \ldots$ has a finite number of repeats.*

Conjecture 3 would imply that the values C_g would exist independently of whether Conjecture 1 is true.

Intuitively, our work shows that having R repeats in B^g means the g-positions lie approximately at positions $p_n^0 - (R, R)$. If R grows much larger than any of the values $C_0, C_1, \ldots, C_{g-1}$, this position will not collide with any positions of Sprague-Grundy value $0, 1, \ldots, g-1$. Then Algorithm $\mathcal{R}$WSG will greedily select the first available diagonal, which leads to the property in Conjecture 2, which bounds the total number of repeats.

Other partial results that may help are showing that no two C_g are consecutive, aside from C_0 and C_1 (this ensures that their positions are consistent with Conjecture 1), or showing some type of growth behavior of the C_g. Indeed, any further information about (C_g) would be of interest.

Another interesting statistic is the number of n such that $b_n^g - a_n^g \neq n$, or the number of g-positions which lie on the wrong diagonals. We can call p_n^g a wrong diagonal position if it does not lie on diagonal n.

In fact, from viewing the empirical data, most of the repeats come from "single skipped diagonals," which occur when p_n^g skips a diagonal and p_{n+1}^g fills it in; more precisely, when $d_n^g = n + 1$ and $d_{n+1}^g = n$. Notice that when a_n^g and a_{n+1}^g are consecutive, $b_n^g = b_{n+1}^g$. Furthermore this phenomenon occurs at a roughly fixed ratio: it is known that the sequence A^0 of P-positions Wythoff's game has a $1/\phi$ proportion of consecutive a_n values. Looking at the empirical data of number of repeats C_g versus number of wrong diagonals positions M_g, their ratio is a relatively stable constant that seems to hover between 4 and 7 (Figure 4). This supports the close relationship between repeats and wrong diagonals positions.

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g	C_g	M_g	g	C_g	M_g
1	1	4	11	108	602
2	3	6	12	160	1080
3	5	19	13	198	1004
4	12	58	14	257	1635
5	9	51	15	277	1547
6	22	103	16	1890	13385
7	53	185	17	496	2353
8	33	177	18	1009	4142
9	86	456	19	1174	6287
10	143	623	20	546	3713

Figure 4: Repeats vs. wrong diagonals

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