Characterization of $(2m, m)$-paintable graphs

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Abstract

In this paper, we prove that for any graph G and any positive integer m, G is $(2m, m)$-paintable if and only if G is 2-paintable. It was asked by Zhu in 2009 whether k-paintable graphs are (km, m)-paintable for any positive integer m. Our result answers this question in the affirmative for $k = 2$.

1 Introduction

Graphs considered in this paper are finite. Suppose G is a graph, b is a positive integer. A *b-fold colouring* c of G assigns to each vertex v a set $c(v)$ of b colours, and colour sets assigned to adjacent vertices are disjoint. If, for $a \in \mathbb{Z}^+$, we have $c(v) \subseteq \{1, 2, \ldots, a\}$ for all vertices v of G, then c is called a *b-fold a-colouring* of G; we then say that G is *(a, b)-colourable.* A 1-fold a-colouring of G is also called an a-colouring of G. The *chromatic number* $\chi(G)$ of G is the minimum a such that G is a-colourable. The *fractional chromatic number* $\chi_f(G)$ of G is defined as

$$\chi_f(G) = \inf \left\{ \frac{a}{b} : G \text{ is } (a, b)\text{-colourable} \right\}.$$

It is well-known that "inf" in the definition of $\chi_f(g)$ can be replaced by "min."

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A *list assignment* of G is a mapping L which assigns to each vertex v of G a set $L(v)$ of permissible colours. If $|L(v)| = a$ for all vertices v of G, then L is called an *a-list assignment*. A *b-fold L-colouring* of G is a b-fold colouring c of G such that $c(v) \subseteq L(v)$ for each vertex v. Similarly, a 1-fold L-colouring of G is called an L-colouring of G. A graph G is called *a-choosable* if there is an L-colouring for any a-list assignment L of G; and is called *(a, b)-choosable* if there is a b-fold L-colouring for any a-list assignment L of G. The *choice number* $\mathrm{ch}(G)$ of G is the minimum a for which G is a-choosable, and the *fractional choice number* $\mathrm{ch}_f(G)$ is defined as

$$\mathrm{ch}_f(G) = \inf \left\{ \frac{a}{b} : G \text{ is } (a, b)\text{-choosable} \right\}.$$

List colouring of graphs was introduced independently in the 1970s by Vizing [8] and by Erdős, Rubin, and Taylor [3]; it has been studied extensively in the literature [7]. By definitions, we have $\chi(G) \leqslant \mathrm{ch}(G)$ and $\chi_f(G) \leqslant \mathrm{ch}_f(G)$ for any graph G. It is easy to see (cf. [3]) that the choice number of bipartite graphs can be arbitrarily large. In particular, $\mathrm{ch}(G) - \chi(G)$ (as well as $\mathrm{ch}(G)/\chi(G)$) is not bounded. On the other hand, it was proved by Alon, Tuza, and Voigt [1] that for any graph G, $\chi_f(G) = \mathrm{ch}_f(G)$. Moreover, the infimum in the definition of $\mathrm{ch}_f(G)$ is attained, and hence can be replaced by minimum.

In this paper, we consider the on-line version of list colouring of graphs. The on-line list colouring of a graph is defined through a two-person game. At the beginning of the game, instead of assigning to each vertex v a set of permissible colours, each vertex v is assigned a set of tokens. In the process of the game, each token is replaced by a permissible colour. The coloring algorithm, which we call *Painter*, needs to decide right away which independent set of vertices with this permissible colour will receive the colour. We now give a price definition. Let $V(G)$ denote the vertex set of a graph G.

Definition 1. Suppose $G = (V, E)$ is a graph and $f, g : V \to \mathbb{N}$ are functions that assign nonnegative integers to vertices of G. The *(f, g)-painting game* on G is played by two players: Lister and Painter. Initially, each vertex v has $f(v)$ tokens, and no colours. In the ith step, Lister chooses a nonempty subset M_i of the vertices v that have received less than $g(v)$ colours and takes away one token from each chosen vertex. Painter chooses an independent set X_i in G contained in M_i and assigns colour i to each vertex of X_i. If at the end of some step, there is a vertex v which has no tokens left, and received less than $g(v)$ colours, then Lister wins the game. Otherwise, each vertex v receives $g(v)$ colours at some step, and Painter wins the game.

The game was called the *on-line list colouring game* in [10], because the sets of permissible colours are given on-line, and the colouring of the graph is constructed on-line. The $(f, 1)$-painting game was originally described by Schauz [6] as a game between *Mr. Paint and Mrs. Correct*, and in [2], it was described as a game between Marker and Remover (as in the $(f, 1)$-painting game, once a vertex v is coloured, then it requires no further attention, and we can consider v to be removed from G).

Definition 2. A graph G is *(f, g)-paintable* when Painter has a winning strategy for the (f, g)-painting game on G. If $f(v) = a$ and $g(v) = b$ are constant functions, then an

(f,g)-paintable graph is called (a,b)-paintable. If G is $(a,1)$-paintable, then we say G is *a-paintable.* The *paint number* $\chi_p(G)$ of G is the minimum a such that G is a-paintable. The *fractional paint number* $\chi_{fp}(G)$ of G is defined as

$$\chi_{fp}(G) = \inf\left\{\frac{a}{b} : G \text{ is } (a,b)\text{-paintable}\right\}.$$

By definition, $\chi(G) \leqslant \text{ch}(G) \leqslant \chi_p(G)$ and $\chi_f(G) \leqslant \text{ch}_f(G) \leqslant \chi_{fp}(G)$ for every graph G. It is known (see [6],[10]) that there are graphs G for which $\chi_p(G) > \text{ch}(G)$. However, Gutowski [4] proved that for any graph G, $\chi_{fp}(G) = \text{ch}_f(G) = \chi_f(G)$. Nevertheless, example graphs are given in [4] to show that the infimum in the definition of $\chi_{fp}(G)$ may not be attained, and hence cannot be replaced by the minimum.

If G is a nonempty bipartite graph, then $\chi_{fp}(G) = \text{ch}_f(G) = \chi(G) = 2$. Since $\text{ch}_f(G) = \min\left\{\frac{a}{b} : G \text{ is } (a,b)\text{-choosable}\right\}$, we know that for some integer m, G is $(2m,m)$-choosable. Although the fractional paint number of G is also 2, in the definition of $\chi_{fp}(G)$, the infimum cannot be replaced by the minimum. Thus $\chi_{fp}(G) = 2$ does not imply that G is $(2m,m)$-paintable for some integer m. A natural question is which graphs are $(2m,m)$-paintable for some integer m.

It was conjectured in [3] that if a graph G is k-choosable, then for any positive integer m, G is (km,m)-choosable. More generally, it was conjectured that if G is (a,b)-choosable then for any positive integer m, G is (am,bm)-choosable. We propose the on-line version of the above conjecture.

Conjecture 3. If G is (a,b)-paintable, then G is (am,bm)-paintable for any $m \in \mathbb{Z}^+$.

It was asked in [10] (Question 24) whether k-paintable graphs are (km,m)-paintable for any positive integer m. The main result of this paper is the following theorem, which answers this question in affirmative for $k=2$, and surprisingly, in this case the converse is also true.

Theorem 4. *For every graph G and every positive integer m, G is $(2m,m)$-paintable if and only if G is 2-paintable.*

In the remaining paper, we prove Theorem 4. Section 2 contains some easy lemmas that we use to make later proofs more efficient. In Section 3, we prove the forward direction of the implication in Theorem 4. In Section 4, we prove the reverse direction of Theorem 4.

2 Lemmas

Denote by $N_G(v)$ (respectively, $N_G[v]$) the neighborhood of v (respectively, the closed neighborhood of v). We say v is an (x,y)-vertex when $f(v) = x$ and $g(v) = y$.

We begin with an observation about necessary conditions on f and g for a graph to be (f,g)-paintable. We call the pair (f,g) the *token-colour functions* of G. If $g(v) = 0$ for some v, then the game is equivalent to the game restricted to $G - v$. Using a term introduced in [2], we call the set M_i chosen by Lister the *set marked at step* i. When the step number is clear from the context or is irrelevant, we simply call M a marked set.

The following proposition is usually used as an equivalent, recursive definition of a graph to be (f,g)-paintable. Let δ_X be the characteristic function of a set X, defined as $\delta_X(v)=1$ for $v \in X$ and $\delta_X(v)=0$ otherwise.

Proposition 5. *Assume G is a graph and (f,g) are token-colour functions with $g(v)>0$ for all v. Then G is (f,g)-paintable if and only if any subset U of $V(G)$ contains an independent set X such that G is $(f-\delta_U, g-\delta_X)$-paintable.*

Proposition 6. *Assume G is a graph and (f,g) are token-colour functions with $g(v)>0$ for all v. If G is (f,g)-paintable, then*

- *$f(v) \geqslant g(v)$ for all $v \in V(G)$,*
- *$\max\{f(u), f(v)\} \geqslant g(u)+g(v)$ for all $uv \in E(G)$.*

Proof. If $f(v) < g(v)$, then Lister wins by marking $\{v\}$ until it has no more tokens, but still needs to be coloured. If $uv \in E(G)$ and $\max\{f(u), f(v)\} < g(u)+g(v)$, then Lister wins by marking $\{u,v\}$ as long as $f(u), f(v), g(u)$, and $g(v)$ are nonnegative. Each round, $g(u)+g(v)$ decreases by at most 1, so some vertex still needs to be coloured after losing all of its tokens. □

We say that a vertex v is *forced* when $f(v)=g(v)$, and an edge uv is *tight* when $\max\{f(u), f(v)\} = g(u)+g(v)$. For $uv \in E(G)$, we say that the ordered pair (u,v) is *strictly tight* when $f(u)=g(u)+g(v)$ and $f(v) < g(u)+g(v)$.

Corollary 7. *If the marked set contains a forced vertex v, then Painter must colour v; if the marked set contains u and v where uv is a tight edge, then Painter must colour one of u and v; if the marked set contains u and not v where (u,v) is a strictly tight pair, then Painter must colour u.*

To make studying the painting game more efficient, we make use of the following observations about Painter's responses on bipartite graphs:

Corollary 8. *Assume G is a bipartite graph and the set of tight edges induces a connected spanning subgraph of G. If Lister marks $V(G)$, then Painter must colour all vertices in one of the partite sets.*

We also use the following two results, which were first proven by Zhu [10].

Proposition 9. *If some $v \in V(G)$ is forced, then G is (f,g)-paintable if and only if $G-v$ is (f',g')-paintable where*

$$f'(w) = \begin{cases} f(w) - g(v) & \text{if } w \in N_G(v), \\ f(w) & \text{otherwise,} \end{cases}$$

and $g'(w) = g(w)$ for all $w \in V(G) - v$.

Proof. If G is (f, g)-paintable and Lister marks $N[v]$ for $g(v)$ consecutive rounds, then Corollary 7 implies Painter colours v each time. After these moves, each w of $G - v$ is an $(f'(w), g'(w))$-vertex. If G is (f, g)-paintable, then $G - v$ is (f', g')-paintable since Painter had no other possible responses against this Lister strategy. If $G - v$ is (f', g')-paintable, then in G, Painter "reserves" $g(v)$ tokens at each neighbor of v. Anytime v is marked, Painter colours v, and uses up at most one of the reserved tokens for each $u \in N(v)$. Since this happens at most $g(v)$ times and $G - v$ is (f', g')-paintable, Painter has a winning strategy in G. □

For a subset X of $V(G)$, let $f(X) = \sum_{v \in X} f(v)$ and $g(X) = \sum_{v \in X} g(v)$. We say a vertex v is *degenerate* if $f(v) \geqslant g(N[v])$.

Proposition 10. *If v is degenerate, then G is (f, g)-paintable if and only if $G - v$ is (f', g')-paintable where f' and g' are the restrictions of f and g to $G - v$.*

Proof. If G is (f, g)-paintable, then Painter has a winning strategy on every subgraph of G with restrictions of f and g. Suppose $G - v$ is (f', g')-paintable. Painter wins in G by following a winning strategy for $G - v$, colouring v when it is marked and none of its neighbors are coloured by that strategy. At most $g(N_G(v))$ tokens of v are used without colouring it, so v will receive enough colours. □

The following proposition is obvious. In our later proofs, we implicitly use it to restrict our attention to Painter responses that are maximal independent subsets of Lister's moves.

Proposition 11. *If G is (f, g)-paintable, then Painter has a winning strategy in which on each round, the vertices coloured form a maximal independent subset of the marked set.*

We now prove that Lister has a winning strategy in a particular position on C_4.

Lemma 12. *Let $G = C_4$ and $V(G) = \{v_0, \ldots, v_3\}$. Assume (f, g) are token-colour functions with $g(v) > 0$ for all v. If all the edges of G are tight and (v_1, v_0) and (v_3, v_0) are strictly tight pairs, then G is not (f, g)-paintable.*

Proof. We use induction on the total number of tokens $\sum f(v_i)$. First suppose that v_0 is forced and $f(v_0) = g(v_0) = r$. Proposition 9 implies that it suffices to show that $G - v_0$ is not (f', g')-paintable, where $f'(v_i) = f(v_i) - r = g(v_i)$ for $i \in \{1, 3\}$, $f'(v_2) = f(v_2)$, and g' is the restriction of g to $\{v_1, v_2, v_3\}$. However, with respect to (f', g'), both v_1 and v_3 are forced. Lister wins the game by applying Proposition 9 again.

We may now assume that v_0 is not forced. As (v_1, v_0) and (v_3, v_0) are strictly tight pairs, i.e, $1 \leqslant f(v_0) - g(v_0) < g(v_i)$ for $i = 1, 3$, we conclude that $g(v_1), g(v_3) \geqslant 2$.

Lister marks $\{v_2, v_3\}$. By Corollary 7, Painter colours v_3; now (v_1, v_2) is strictly tight. Then Lister marks $\{v_2, v_1\}$. By Corollary 7 again, Painter colours v_1; now (v_3, v_2) is strictly tight and (v_1, v_2) remains strictly tight. Since $g(v_i) > 0$ for all v_i and $\sum f(v_i)$ is smaller, the induction hypothesis implies that Lister wins. □

When a graph G is not $(2m,m)$-paintable, we wish to conclude that "similar" graphs are also not $(2m,m)$-paintable. We make this intuitive idea more precise with the following definition.

Definition 13. Assume H is a graph and U is a subset of vertices of H and $a \geqslant b$ are positive integers. We say (H,U) is an (a,b)-*gadget* if H is (a,b)-colourable, and in any (a,b)-colouring of H, all vertices in U are coloured by the same b-set.

Definition 14. Assume G and H are graphs, $v \in V(G)$ and $U \subseteq V(H)$. If G' is obtained from the disjoint union of G and H by splitting v into $|U|$ copies, arbitrarily partitioning the edges incident to v among those copies, and identifying the $|U|$ copies of v with the vertices of U in H, then we say G' is an (H,U)-*augmentation* of G.

We use the gadget from Definition 13 and the augmentation described in Definition 14 to build many non-(a,b)-paintable graphs G' from a single non-(a,b)-paintable graph G. The folowing lemma makes this idea precise by showing how "non-(a,b)-paintability" can be preserved when augmenting G to form G'.

Lemma 15. *If G is not (a,b)-paintable, (H,U) is an (a,b)-gadget, and G' is an (H,U)-augmentation of G, then G' is not (a,b)-paintable.*

Proof. Since G is not (a,b)-paintable, Lister has a winning strategy $\mathcal{S}$. Each round, Lister obtains a marked set $M \subseteq V(G)$ according to $\mathcal{S}$. If $v \in M$, then in G', Lister marks $V(H) \cup (M - v)$; otherwise Lister marks M as a subset of $V(G')$. Let D be the set that Painter colours. If $0 < |D \cap U| < |U|$, then Lister marks $V(H)$ in every remaining round. Since every (a,b)-colouring of H assigns vertices in U the same set of b colours, Painter will not be able to colour each vertex of H by a set of b colours. So Lister wins the game. If $D \cap U = \emptyset$, then Lister views Painter's response as $D - V(H)$ in the game on G. If $D \cap U = U$, then Lister views Painter's response as $D \cup \{v\} - V(H)$ in the game on G. Thus Lister can continue using strategy $\mathcal{S}$ and eventually wins the game. □

Note that every $(2m,m)$-colouring of a path assigns the same set of b colours to every other vertex along the path. We now apply Lemma 15 with $a = 2m$ and $b = m$ in the following corollary. In Section 3, we use it to reduce the number of cases that we must consider in Theorem 17.

Corollary 16. *Given a graph G and an edge uv of G, if G is not $(2m,m)$-paintable, then the graph obtained by replacing uv with a path of odd length is not $(2m,m)$-paintable.*

Proof. Assume G' is obtained from G by replacing uv with a path of length $2r+1$. Let H be a path of length $2r$ with vertices $\{v_1, v_2, \ldots, v_{2r+1}\}$ taken in order along the path. Then $(H, \{v_1, v_{2r+1}\})$ is a $(2m,m)$-gadget, and we obtain G' as in Definition 14 by taking the disjoint union of G and H, splitting u into two vertices u_1 and u_2, and let u_1 be adjacent to v and u_2 adjacent to the rest of $N_G(u)$, and finally, u_1 is identified with v_1 and u_2 with v_{2r+1}. Using this (H,U)-augmentation of G, Lemma 15 implies that G' is not $(2m,m)$-paintable. □

3 Non-$(2m, m)$-paintable graphs

Our goal in this section is to prove the following theorem:

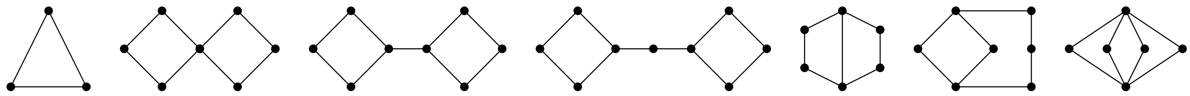
Theorem 17. *If G is not 2-paintable, then G is not $(2m, m)$-paintable for any m.*

A graph is *k-paint-critical* if $\chi_p(G) = k$ but $\chi_p(G - e) < k$ for all $e \in E(G)$. If G is not 2-paintable, then it must contain a 3-paint-critical subgraph. Thus to prove Theorem 17, it suffices to show that every 3-paint-critical graph is not $(2m, m)$-paintable. Riasat and Schauz [5], and independently Carraher et al. [2], characterized 3-paint-critical graphs. The characterization first requires the following definition: the *theta-graph* $\Theta_{r,s,t}$ consists of two vertices joined by internally disjoint paths of lengths r, s, t.

Theorem 18 ([5, 2])**.** *A graph is 3-paint-critical if and only if it is one of the following:*

- *An odd cycle.*
- *Two vertex-disjoint even cycles connected by a path.*
- *Two edge-disjoint even cycles having exactly one vertex in common.*
- $\Theta_{r,s,t}$ *where r, s, t have the same parity and* $\max\{r, s, t\} > 2$.
- $K_{2,4}$.

To prove Theorem 17, it suffices by Corollary 16 to show that the following seven graphs in Figure 1 are not $(2m, m)$-paintable for any positive integer m.



C_3 F_0 F_1 F_2 $\Theta_{1,3,3}$ $\Theta_{2,2,4}$ $K_{2,4}$

Figure 1: Family of graphs for Theorem 17

Since C_3 is not $(2m, m)$-colourable, it is not $(2m, m)$-paintable for any positive integer m. We now reduce this family further by applying Lemma 15 to F_0.

Proposition 19. *For $m \in \mathbb{Z}^+$, if F_0 is non-$(2m, m)$-paintable, then F_2 is non-$(2m, m)$-paintable.*

Proof. Let $G = F_0$, let u be the vertex of degree 4, and suppose that G is not $(2m, m)$-paintable. Let $H = P_3$ and $V(H) = \{v_1, v_2, v_3\}$ with v_1, v_3 as the endpoints. Note that $(H, \{v_1, v_3\})$ is a $(2m, m)$-gadget. We split u into two copies u_1, u_2 and partition the edges incident to u so that u_1 is incident to the two edges in the copy of C_4 on the left and u_2 is incident to the two edges in the copy of C_4 on the right. Identifying u_1 with v_1 and u_2 with v_3 yields the graph F_2. Therefore, F_2 is an $(H, \{v_1, v_3\})$-augmentation of F_0, and by Lemma 15, F_2 is not $(2m, m)$-paintable. □

It remains to show that each of $K_{2,4}, F_0, F_1, \Theta_{2,2,4}$, and $\Theta_{1,3,3}$ is not $(2m, m)$-paintable for any $m \in \mathbb{Z}^+$. All these graphs are bipartite graphs. We use A and B to denote the two partite sets. Vertices in A are named $a_1, \ldots, a_{|A|}$ and vertices in B are $b_1, \ldots, b_{|B|}$.

Theorem 20. *Let $G = K_{2,4}$ and f, g be positive on every vertex. If every edge of G is tight, then G is not (f, g)-paintable.*

Proof. We prove by induction on the total number of tokens. Let $A = \{a_1, a_2\}$ be the set of vertices of degree 4, and let $B = \{b_1, \ldots, b_4\}$ be the set of vertices of degree 2. Lister marks $L_1 = \{a_1, b_1, b_2\}$. Painter must colour a_1, for otherwise, (b_3, a_1) and (b_4, a_1) would be strictly tight pairs and Lister wins at the 4-cycle $C_4 = (a_1, b_3, a_2, b_4)$ by Lemma 12. If $g(a_1) = 1$, then after the first move, b_1, b_2 become forced vertices, and it is easy to check that Lister wins on the subgraph induced by $\{a_2, b_1, b_2\}$. Assume $g(a_1) \geqslant 2$. Next Lister marks $\{a_2, b_3, b_4\}$, and Painter must colour a_2 by Corollary 7 (as (a_2, b_1) is strictly tight). Similarly as above, it is easy handle the case when $g(a_2) = 1$ and we can assume $g(a_2) \geqslant 2$.

Now $g' = g - \delta_{\{a_1,a_2\}}$ is positive on every vertex v and every edge is tight, so Lister wins by induction hypothesis. □

Corollary 21. *$K_{2,4}$ is not $(2m, m)$-paintable for any positive integer m.*

Theorem 22. *Let $G = F_0$ and f, g be positive on every vertex. If every edge of G is tight, then G is not (f, g)-paintable.*

Proof. We prove by induction on the total number of tokens. Let F_0 be labelled as in Figure 2 and let $A = \{a_1, a_2, a_3, a_4\}$.

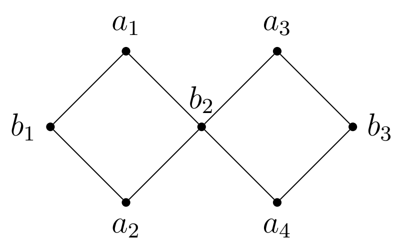


Figure 2: F_0 with vertices labeled

If there is a forced vertex x, then let $C = (x, y, z, w)$ be a 4-cycle in G containing x. We know that (w, x) and (y, x) are strictly tight pairs. It follows from Lemma 12 that Lister has a winning strategy. Thus we assume that there is no forced vertex.

Suppose there is a strictly tight pair (x, y). Note since y is not forced, $g(y) < f(y) < g(y) + g(x)$, and thus $g(x) \geqslant 2$. Also, since no vertex is forced, $f(v) \geqslant 2$ for all v. Lister marks $N_G[x] - \{y\}$, and by Corollary 7, Painter colours x. After this move, f, g are still positive on every vertex. By the induction hypothesis, Lister has a winning strategy. Thus we assume that there is no strictly tight pair. This implies that for each edge xy of G, $f(x) = f(y) = g(x) + g(y)$. Hence, f is constant on $V(G)$, and g is constant on each partite set of G.

If $g(v) \geqslant 2$ for every v, then Lister marks $V(G)$. By Corollary 8, Painter colours a whole partite set, and hence after this move, all the edges of G are still tight. Also after this move, f, g are still positive on every vertex. So Lister wins by induction hypothesis.

If $g(v) = 1$ for all v, then the conclusion follows from the fact that G is not 2-paintable. Assume $g(v) = 1$ for vertices v of one partite set, and $g(v) = r \geqslant 2$ for vertices v of the other partite set. We consider two cases.

Case 1: Vertices in A are $(r+1, 1)$-vertices.
Lister marks $\{a_3, b_3\}$. If Painter colours a_3, then b_3 becomes a forced vertex. Applying Corollary 7 to the forced vertex b_3 results in a_4 becoming a $(1, 1)$-vertex. Lister now marks $\{a_4, b_2\}$, Painter must colour a_4, and Lister wins by Lemma 12.

If Painter colours b_3, then Lister marks $V(G) \setminus \{a_3, b_3\}$. Since (b_2, a_3) is a strictly tight pair, by Corollary 7, Painter colours $\{b_1, b_2\}$. Now all the edges of G are tight and f, g are positive on every vertex, and Lister wins by induction hypothesis.

Case 2: Vertices in A are $(r+1, r)$-vertices.
Lister marks $\{a_3, b_3\}$. If Painter colours a_3, then Lister marks $V(G) \setminus \{a_3, b_3\}$. Since (a_4, b_3) is a strictly tight pair, by Corollary 7, Painter colours $\{a_1, a_2, a_4\}$. Now all the edges of G are tight and f, g are positive on every vertex, and Lister wins by induction hypothesis.

If Painter colours b_3, then a_3 is forced. Lister marks $\{a_3, b_2\}$, Painter must colour a_3. Now all the edges of the 4-cycle (a_1, b_2, a_2, b_1) are tight, and (a_1, b_2) and (a_2, b_2) are strictly tight pairs. Lister wins by Lemma 12. □

Corollary 23. *The graph F_0 is not $(2m, m)$-paintable for any positive integer m.*

Proposition 24. *Let $G = F_1$ and f, g be positive on every vertex. If every edge of G is tight, then G is not (f, g)-paintable.*

Proof. Lister marks $\{u, v\}$, and by symmetry we may assume Painter colours u. Lemma 12 implies that Lister wins by marking the copy of C_4 containing v. □

Theorem 25. *Let $G = \Theta_{2,2,4}$ and f, g be positive on every vertex. If every edge of G is tight, then G is not (f, g)-paintable.*

Proof. We prove by induction on the total number of tokens. Suppose (x, y) is a strictly tight pair. Lister marks $N_G[x] - \{y\}$, and by Corollary 7, Painter colours x. As in Theorem 22, after this move f, g are still positive on every vertex and every edge is tight. By the induction hypothesis, Lister has a winning strategy. Thus we assume that there is no strictly tight pair, implying that f is constant on $V(G)$, and g is constant on each partite set of G.

If $g(v) \geqslant 2$ for every v, then Lister marks $V(G)$. By Corollary 8, Painter colours a whole partite set, and hence after this move, all the edges of G are still tight. Also after this move, f, g are still positive on every vertex. So Lister wins by induction hypothesis.

If $g(v) = 1$ for all v, then the conclusion follows from the fact that G is not 2-paintable. Assume $g(v) = 1$ for vertices v of one partite set, $g(v) = r \geqslant 2$ for vertices v of the other partite set, and $f(v) = r + 1$ for all vertices v.

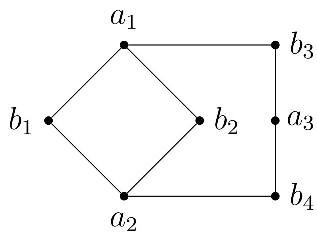


Figure 3: $\Theta_{2,2,4}$ with vertices labeled

Let vertices be labeled as shown in Figure 3, and let $A = \{a_1, \ldots, a_3\}$ and $B = \{b_1, \ldots, b_4\}$.

Case 1: Vertices in A are $(r+1, 1)$-vertices.
Lister marks $\{a_3, b_3\}$. If Painter colours a_3, then b_3 becomes a forced vertex. Lister marks $\{a_1, b_3\}$, Painter must colour b_3, and Lister now wins on $\{a_1, a_2, b_1, b_2\}$ by Lemma 12.

If Painter colours b_3, then Lister marks $\{a_3, b_4\}$ for r rounds, after which b_4 becomes a $(1, 1)$-vertex. Lister now marks $\{b_4, a_2\}$, Painter must colour b_4, and Lister wins by Lemma 12.

Case 2: Vertices in A are $(r+1, r)$-vertices.
Lister marks $\{a_3, b_3\}$. If Painter colours a_3, then Lister marks $\{a_1, b_1, b_2\}$. Painter colours a_1, otherwise Lister wins on $\{a_1, b_3\}$ by Proposition 6. Now Lister marks $\{a_2, b_4\}$, and Painter must colour a_2, otherwise Lister wins on $\{a_2, b_1\}$. Lister wins by the induction hypothesis.

If Painter colours b_3, then a_3 becomes forced. Applying Corollary 7 with the forced vertex a_3 results in b_4 becoming a $(1, 1)$-vertex. Lister marks $\{a_2, b_4\}$, and Painter must colour b_4. Lister now wins by Lemma 12. ☐

Corollary 26. *The graph $\Theta_{2,2,4}$ is not $(2m, m)$-paintable for any positive integer m.*

Theorem 27. *Let $G = \Theta_{1,3,3}$ and f, g be positive on every vertex. If every edge of G is tight, then G is not (f, g)-paintable.*

Proof. Let u, v be adjacent vertices of degree 2. Lister repeatedly marks $\{u, v\}$. Since uv is a tight edge, eventually, one vertex, say u, becomes a forced vertex. Lister marks u and its neighbor u' in $G - \{u, v\}$. Painter must colour u, as u is forced. Lemma 12 implies that Lister wins in the remaining graph. ☐

Corollary 28. *The graph $\Theta_{1,3,3}$ is not $(2m, m)$-paintable for any positive integer m.*

Since no 3-paint-critical graph is $(2m, m)$-paintable for any m, Theorem 17 follows.

4 $(2m, m)$-paintable graphs

The *core* of a connected graph G is the graph obtained from G by successively deleting vertices of degree 1; it is unique up to isomorphism. The following theorem characterizes 2-paintable graphs.

Theorem 29 ([10])**.** *A connected graph G is 2-paintable if and only if the core of G is K_1, an even cycle, or $K_{2,3}$.*

In the $(2m, m)$-painting game, vertices of degree 1 are degenerate. Thus for any positive integer m, a graph G is $(2m, m)$-paintable if and only if its core is $(2m, m)$-paintable. Thus to prove Theorem 4, it suffices to show that each of $K_1, C_{2n}, K_{2,3}$ is $(2m, m)$-paintable for all $m \in \mathbb{Z}^+$. Given $m \in \mathbb{Z}^+$, it is obvious that K_1 is $(2m, m)$-paintable, so we move on to considering even cycles. An oriented graph G is *kernel perfect* if every induced subgraph $G[M]$ has a kernel, i.e., M contains an independent set I such that every $v \in M - I$ has an out-neighbour in I. For an oriented graph D, $N^+_D(v)$ is the set of out-neighbours of v, and $N^+_D[v] = N^+_D(v) \cup \{v\}$.

Theorem 30. *Assume G has an orientation D which is kernel perfect and $g : V(G) \to N$ is a mapping which assigns to each vertex x a non-negative integer. If $f(v) \geqslant \sum_{x \in N^+_D[v]} g(x)$ for every $v \in V(G)$, then G is (f, g)-paintable.*

Proof. We prove this theorem by induction on the total number of tokens. For any subset M of $V(D)$ for which $g(v) > 0$ for $v \in M$, let I be the kernel of $D[M]$. Let $f' = f - \delta_M$ and $g' = g - \delta_I$. Since each vertex $x \in M - I$ has an out-neighbour in I, for every vertex v, $f'(v) \geqslant \sum_{x \in N^+_D[v]} g'(x)$. By induction, G is (f', g')-paintable. Therefore G is (f, g)-paintable. □

Corollary 31. *C_{2n} is $(2m, m)$-paintable for all $m \in \mathbb{Z}^+$.*

Proof. Orient C_{2n} as a directed cycle. This orientation is kernel perfect. By Theorem 30, C_{2n} is $(2m, m)$-paintable. □

Let $G = K_{2,3}$ be labeled as in Figure 4.

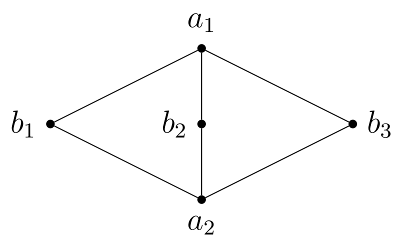


Figure 4: $K_{2,3}$ with vertices labeled

Let (f, g) be token-colour functions on G. For each edge a_ib_j of G, let

$$\begin{aligned} w_{A,f,g}(a_ib_j) &= f(a_i) - g(a_i) - g(b_j), \\ w_{B,f,g}(a_ib_j) &= f(b_j) - g(b_j) - g(a_i). \end{aligned}$$

For a set D of edges, we let $w_{A,f,g}(D) = \sum_{e \in D} w_{A,f,g}(e)$ and $w_{B,f,g}(D) = \sum_{e \in D} w_{B,f,g}(e)$. An edge set D is *special* if $|D| \geqslant 2$ and there is an edge $e \in D$ such that every other edge $e' \in D$ has no common endpoint with e. Observe that a special edge set D contains either two or three edges. Indeed, up to isomorphism, there are only two special edge sets $\{a_1b_1, a_2b_2\}$ and $\{a_1b_1, a_2b_2, a_2b_3\}$.

We say (f, g) has Property $(\star)$ if the following hold:

(1) For each edge uv, $\max\{f(u), f(v)\} \geqslant g(u) + g(v)$.

(2) For any special edge set D, $w_{A,f,g}(D) \geqslant 0$ and $w_{B,f,g}(D) \geqslant 0$.

Theorem 32. *If (f, g) has Property $(\star)$, then G is (f, g)-paintable.*

Proof. The proof is by induction on the total number of tokens.

Assume (f, g) has Property $(\star)$. First we consider the case that there exists a forced vertex.

Assume a_1 is forced, say a_1 is an (a, a)-vertex. Let a_2 be a $(c + d, d)$-vertex, and each b_i be a $(x_i + y_i, y_i)$-vertex. Then $w_{A,f,g}(\{a_2b_1, a_1b_2, a_1b_3\}) \geqslant 0$ implies that $c \geqslant y_1 + y_2 + y_3$, i.e., a_2 is degenerate. By Proposition 10, G is (f, g)-paintable if and only if $G - a_2$ is (f, g)-paintable. By (1) of Property $(\star)$, $x_i \geqslant a$ for $i \in \{1, 2, 3\}$. So in $G - a_2$, b_1, b_2, b_3 are all degenerate. So $G - a_2$ is (f, g)-paintable if and only if $G - \{a_2, b_1, b_2, b_3\}$ is (f, g)-paintable, which is obviously true.

Assume b_1 is forced, say b_1 is a (b, b)-vertex. Let b_j be a $(c_j + d_j, d_j)$-vertex for $j \in \{2, 3\}$, and let a_i be a $(x_i + y_i, y_i)$-vertex for $i \in \{1, 2\}$. Then $w_{B,f,g}(\{a_1b_2, a_2b_1\}) \geqslant 0$ implies that $c_2 \geqslant y_1 + y_2$. and $w_{B,f,g}(\{a_1b_3, a_2b_1\}) \geqslant 0$ implies that $c_3 \geqslant y_1 + y_2$. Thus both b_2 are b_3 are degenerate, and Proposition 10 implies that G is (f, g)-paintable if and only if $G - \{b_2, b_3\}$ is (f, g)-paintable. By (1) of Property $(\star)$, $x_i \geqslant b$ for $i \in \{1, 2\}$. Thus in $G - \{b_2, b_3\}$, a_1 and a_2 are degenerate. So $G - \{b_2, b_3\}$ is (f, g)-paintable if and only if $G - \{a_1, a_2, b_2, b_3\}$ is (f, g)-paintable, which is obviously true.

Assume there are no forced vertices. We shall prove that for any set M of vertices, there is an independent set X contained in M such that $(f - \delta_M, g - \delta_X)$ has Property $(\star)$.

(R1) If there exists $a_i \in M, b_j \notin M$ such that $w_{B,f,g}(a_ib_j) < 0$, then let $X = M \cap A$.

(R2) Else, if there exists $b_j \in M, a_i \notin M$ such that $w_{A,f,g}(a_ib_j) < 0$, then let $X = M \cap B$.

(R3) Else, if $|M \cap A| \geqslant |M \cap B|$, then let $X = M \cap A$.

(R4) Else, let $X = M \cap B$.

Let $f' = f - \delta_M$ and $g' = g - \delta_X$.

First we show that for any edge $e = uv$, $\max\{f'(u), f'(v)\} \geqslant g'(u) + g'(v)$.

Assume this is not true. Without loss of generality, assume $\max\{f'(a_1), f'(b_1)\} < g'(a_1) + g'(b_1)$. As (f, g) has Property $(\star)$, $\max\{f(u), f(v)\} \geqslant g(u) + g(v)$. If $f(a_1) \geqslant g(a_1) + g(b_1)$, then $f'(a_1) < g'(a_1) + g'(b_1)$ implies that $f'(a_1) = f(a_1) - 1$ and $g'(a_1) = g(a_1), g'(b_1) = g(b_1)$. Hence $a_1 \in M, b_1 \notin M$ and $X = M \cap B$. So (R1) is not applied, which implies that $w_{B,f,g}(a_1b_1) \geqslant 0$, i.e., $f(b_1) \geqslant g(b_1) + g(a_1)$, and hence $f'(b_1) = f(b_1) \geqslant g'(a_1) + g'(b_1)$.

Assume $f(a_1) < g(a_1) + g(b_1)$. Then $f(b_1) \geqslant g(a_1) + g(b_1)$, and $f'(b_1) < g'(a_1) + g'(b_1)$ implies that $b_1 \in M, a_1 \notin M$ and $X = M \cap A$. So $w_{A,f,g}(a_1b_1) < 0$ and yet (R2) is not applied. This implies that (R1) is applied. Without loss of generality, we may assume that $a_2 \in M, b_2 \notin M$ and $w_{B,f,g}(a_2b_2) < 0$. But $w_{B,f,g}(\{a_1b_1, a_2b_2\}) \geqslant 0$ implies that $f(b_1) \geqslant g(a_1) + g(b_1) + g(a_2) + g(b_2) - f(b_2) \geqslant g(a_1) + g(b_1) + 1$. This in turn implies that $f'(b_1) = f(b_1) - 1 \geqslant g(a_1) + g(b_1) \geqslant g'(a_1) + g'(b_1)$, a contradiction.

Next we show (2) of Property ($\star$) holds for (f', g'), i.e., for any special edge set D, $w_{A,f,g}(D) \geqslant 0$ and $w_{B,f,g}(D) \geqslant 0$.

Observe that if $X = M \cap A$, then for any edge $e = ab$, $w_{A,f,g}(e) = w_{A,f',g'}(e)$. Indeed, if $a \in M$, then $f'(a) = f(a) - 1$, $g'(a) = g(a) - 1$, and $g'(b) = g(b)$. So $w_{A,f,g}(e) = w_{A,f',g'}(e)$. If $a \notin M$, then $f'(a) = f(a)$, $g'(a) = g(a)$, and $g'(b) = g(b)$. Again $w_{A,f,g}(e) = w_{A,f',g'}(e)$. Similarly, if $X = M \cap B$, then for any edge e, we have $w_{B,f,g}(e) = w_{B,f',g'}(e)$.

Case 1 (R1) applies.

Since $X = M \cap A$, by the observation above, for any special edge set D, $w_{A,f',g'}(D) \geqslant w_{A,f,g}(D) \geqslant 0$. It remains to show that $w_{B,f',g'}(D) \geqslant 0$.

As (R1) applies, there is an edge, say $e = a_1b_1$, such that $w_{B,f,g}(a_1b_1) < 0$, $a_1 \in M$ and $b_1 \notin M$.

Straightforward calculation shows the following hold:

1. $w_{B,f',g'}(a_1b_1) = w_{B,f,g}(a_1b_1) + 1 \leqslant 0$.
2. If d is an edge incident to a_1 or incident to b_1, $w_{B,f',g'}(d) \geqslant w_{B,f,g}(d)$.
3. If an edge d is incident to neither a_1 nor b_1, i.e., $d \in \{a_2b_2, a_2b_3\}$, then $w_{B,f',g'}(d) \geqslant w_{B,f,g}(d) - 1$. However, for such an edge d, by (2) of Property ($\star$), we have $w_{B,f,g}(d) \geqslant -w_{B,f,g}(a_1b_1) \geqslant 1$, which implies that $w_{B,f',g'}(d) \geqslant 0$.

First we assume that $a_1b_1 \in D$. If D contains at most one of a_2b_2 and a_2b_3, then $w_{B,f',g'}(D) \geqslant w_{B,f,g}(D) \geqslant 0$ by the observations above. If D contains both a_2b_2, a_2b_3, then $D - \{a_2b_3\}$ is also special, and hence $w_{B,f',g'}(D - \{a_2b_3\}) \geqslant 0$. As $w_{B,f',g'}(a_2b_3) \geqslant 0$, we have $w_{B,f',g'}(D) = w_{B,f',g'}(D - \{a_2b_3\}) + w_{B,f',g'}(a_2b_3) \geqslant w_{B,f',g'}(D - \{a_2b_3\}) \geqslant 0$.

Next assume D does not contain a_1b_1. Then D contains at most one of the edges a_2b_2, a_2b_3 (for otherwise, D is not special). If D contains none of a_2b_2, a_2b_3, then by the observations above, $w_{B,f',g'}(D) \geqslant w_{B,f,g}(D) \geqslant 0$. Thus we may assume that $a_2b_2 \in D$ and $a_2b_3 \notin D$. If every other edge of D are non-adjacent to a_2b_2, then $D \cup \{a_1b_1\}$ is also special and $w_{B,f',g'}(D) = w_{B,f',g'}(D \cup \{a_1b_1\}) - w_{B,f',g'}(a_1b_1) \geqslant 0$ (as $w_{B,f',g'}(a_1b_1) \leqslant 0$). Assume D contains another edge adjacent to a_2b_2. The only possible special edge set is $D = \{a_2b_1, a_2b_2, a_1b_3\}$. In this case, $w_{B,f',g'}(D) = w_{B,f',g'}(D - \{a_2b_2\}) + w_{B,f',g'}(a_2b_2) \geqslant w_{B,f,g}(D - \{a_2b_2\}) + w_{B,f',g'}(a_2b_2)$. Since $D - \{a_2b_2\}$ is special, we have $w_{B,f,g}(D - \{a_2b_2\}) \geqslant 0$. As $w_{B,f',g'}(a_2b_2) \geqslant 0$, we conclude that $w_{B,f',g'}(D) \geqslant 0$. This completes the proof of Case 1.

Case 2 (R2) applies.

The proof of this case is the same as that of Case 1. One simply needs to interchange A and B in the subscripts and the roles of a_1 and b_1 in the marked set M.

Case 3 (R3) applies.

If $M \cap A = A$, then $w_{A,f',g'}(D) \geqslant w_{A,f,g}(D) \geqslant 0$ for any special edge set D. If $|M \cap A| = 2$, then $w_{B,f',g'}(e) \geqslant w_{B,f,g}(e)$ for every edge e. Assume that $|M \cap A| = 1$. Then $|M \cap B| \leqslant 1$. The case $M \cap B = \emptyset$ is trivial. Thus we may assume $M = \{a_1 b_1\}$. Then $w_{B,f',g'}(a_2 b_1) = w_{B,f,g}(a_2 b_1) - 1$, $w_{B,f',g'}(a_1 b_2) = w_{B,f,g}(a_1 b_2) + 1$, $w_{B,f',g'}(a_1 b_3) = w_{B,f,g}(a_1 b_3) + 1$, and for every other edge e, $w_{B,f',g'}(e) = w_{B,f,g}(e)$. If a special edge set D does not contain $a_2 b_1$, then $w_{B,f',g'}(D) \geqslant w_{B,f,g}(D) \geqslant 0$. If D contains $a_2 b_1$, then D contains at least one of $a_1 b_2$ and $a_1 b_3$. In this case, we also have $w_{B,f',g'}(D) \geqslant w_{B,f,g}(D) \geqslant 0$.

Case 4 (R4) applies.

As (R3) does not apply, $|M \cap B| \geqslant 2$. If $|M \cap B| = 3$, then $w_{A,f',g'}(e) \geqslant w_{A,f,g}(e)$ and $w_{B,f',g'}(e) \geqslant w_{B,f,g}(e)$ for every edge e. Thus we may assume $M = \{a_1, b_1, b_2\}$. If D is a special edge set not containing $a_1 b_3$, then $w_{A,f',g'}(e) \geqslant w_{A,f,g}(e)$ for each edge $e \in D$, and hence $w_{A,f',g'}(D) \geqslant w_{A,f,g}(D) \geqslant 0$. If $a_1 b_3 \in D$, then D contains at least one of $a_2 b_1, a_2 b_2$. As in Case 3, $w_{B,f',g'}(D) \geqslant w_{B,f,g}(D) \geqslant 0$. □

Corollary 33. *$K_{2,3}$ is $(2m, m)$-paintable for all positive integer m.*

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