On the chromatic number of the
Erdős-Rényi orthogonal polarity graph

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Abstract

For a prime power q, let ER_q denote the Erdős-Rényi orthogonal polarity graph. We prove that if q is an even power of an odd prime, then $\chi(ER_q) \leqslant 2\sqrt{q} + O(\sqrt{q}/\log q)$. This upper bound is best possible up to a constant factor of at most 2. If q is an odd power of an odd prime and satisfies some condition on irreducible polynomials, then we improve the best known upper bound for $\chi(ER_q)$ substantially. We also show that for sufficiently large q, every ER_q contains a subgraph that is not 3-chromatic and has at most 36 vertices.

Keywords: orthogonal polarity graphs; Turán number; forbidden subgraph; chromatic number

1   Introduction

Let q be a prime power and let V be a 3-dimensional vector space over $\mathbb{F}_q$. Let $PG(2, q)$ be the projective geometry whose points are the 1-dimensional subspaces of V and whose lines are the 2-dimensional subspaces of V. The *Erdős-Rényi orthogonal polarity graph*, denoted ER_q, is the graph whose vertices are the points of $PG(2, q)$, and distinct vertices (x_0, x_1, x_2) and (y_0, y_1, y_2) are adjacent if and only if $x_0 y_0 + x_1 y_1 + x_2 y_2 = 0$. One obtains an isomorphic graph if the equation for adjacency is $x_2 y_0 + x_0 y_2 = x_1 y_1$ (see [16]) and it is this definition of ER_q that we will use. Throughout the paper, if G is a graph and $S \subset V(G)$, we will use $G[S]$ to denote the subgraph induced by S.

These graphs were constructed independently by Brown [4], and Erdős, Rényi, and Sós [7] and have many applications to problems in extremal graph theory. The graph ER_q

has $q^2 + q + 1$ vertices, has $\frac{1}{2}q(q+1)^2$ edges, and has no 4-cycle as a subgraph. Füredi [9], [10] proved that for $q \geqslant 15$ a prime power, a graph with $q^2 + q + 1$ vertices and no 4-cycle has at most $\frac{1}{2}q(q+1)^2$ edges. The Erdős-Rényi graphs show that Füredi's upper bound is best possible. This is perhaps the most well-known application of ER_q to extremal graph theory, but there are many others. The interested reader is referred to [14], [13], and [3] for applications of ER_q to problems in hypergraph Turán theory, Ramsey theory, and structural graph theory.

Because of its important place in extremal graph theory, many researchers have studied the graph ER_q. Benny Sudakov posed the question of determining the independence number of ER_q (see [17]), and it has since been investigated in several papers. Mubayi and Williford [16] proved that if p is a prime, $n \geqslant 1$ is an integer, and $q = p^n$, then

$$\alpha(ER_q) \geqslant \begin{cases} \frac{1}{2}q^{3/2} + \frac{1}{2}q + 1 & \text{if } p \text{ is odd and } n \text{ is even,} \\ \frac{120q^{3/2}}{73\sqrt{73}} & \text{if } p \text{ is odd and } n \text{ is odd,} \\ \frac{q^{3/2}}{2\sqrt{2}} & \text{if } p = 2 \text{ and } n \text{ is odd,} \\ q^{3/2} - q + q^{1/2} & \text{if } p = 2 \text{ and } n \text{ is even.} \end{cases}$$

An upper bound $\alpha(ER_q) \leqslant q^{3/2} + q^{1/2} + 1$ can be obtained from Hoffman's bound. Therefore, the order of magnitude of $\alpha(ER_q)$ is $q^{3/2}$. Godsil and Newman refined the upper bound obtained from Hoffman's bound in [11]. Their result was then improved using the Lovász theta function in [6]. When q is even, Hobart and Williford [12] used coherent configurations to provide upper bounds for the independence number of general orthogonal polarity graphs. When q is an even square, the know upper bound and lower bound for $\alpha(ER_q)$ differ by at most 1. In the case when p is odd or when $p = 2$ and n is odd, it is still an open problem to determine an asymptotic formula for $\alpha(ER_q)$.

Since the independence number has been well-studied and its order of magnitude is known, it is natural to investigate the chromatic number of ER_q which is closely related to $\alpha(ER_q)$. As ER_q has $q^2 + q + 1$ vertices and $\alpha(ER_q) = \Theta(q^{3/2})$, a lower bound for $\chi(ER_q)$ is $\frac{q^2+q+1}{\alpha(ER_q)} \geqslant q^{1/2}$. One may ask whether this lower bound actually gives the right order of magnitude of $\chi(ER_q)$. We confirm this for q being an even power of an odd prime.

Theorem 1. *If $q = p^{2r}$ where p is an odd prime and $r \geqslant 1$ is an integer, then*

$$\chi(ER_q) \leqslant 2\sqrt{q} + O(\sqrt{q}/\log q).$$

This upper bound is within a factor of 2 of the lower bound $\chi(ER_q) \geqslant \frac{q^2+q+1}{\alpha(ER_q)} \geqslant q^{1/2}$. Any improvement in the coefficient of $q^{1/2}$ would give an improvement to the best known lower bound on the independence number of ER_q from [16]. The lower order term $O(q^{1/2}/\log q)$ is obtained using probabilistic methods [2] and while the implied constant is absolute, we have not made an effort to compute it. By using Brooks' Theorem instead

of the result of Alon et. al. [2], we obtain the upper bound $\chi(ER_q) \leqslant 4q^{1/2} + 1$ for all $q = p^{2r}$ where $r \geqslant 1$ is an integer and p is an odd prime.

When q is not an even power, we first prove the following general theorem.

Theorem 2. *Let q be an odd power of an odd prime and let $r \geqslant 1$ be an integer. If there is a $\mu \in \mathbb{F}_q$ such that $x^{2r+1} - \mu$ is irreducible in $\mathbb{F}_q[x]$, then*

$$\chi(ER_{q^{2r+1}}) \leqslant \frac{2r+5}{3} q^{\frac{4r}{3}+1} + (2r+1)q^{r+1} + 1.$$

Given an odd integer $2r + 1 \geqslant 3$, there are infinitely many primes p for which there is a $\mu \in \mathbb{F}_q$ such that $x^{2r+1} - \mu \in \mathbb{F}_q[x]$ is irreducible (see Section 5 for more details), where q is an arbitrary odd power of p. Our method can also be used to prove that if q is a power of any odd prime, then

$$\chi(ER_{q^3}) \leqslant 6q^2 + 1.$$

Here we do not need the existence of an irreducible polynomial $x^3 - \mu \in \mathbb{F}_q[x]$.

For q is an odd power of an odd prime, we have the following corollary.

Corollary 3. *Let $q = p^s$ for an odd prime p and an odd integer $s \geqslant 3$. If $t > 1$ is the smallest divisor of s such that $x^t - \mu$ is irreducible in $\mathbb{F}_{p^{s/t}}[x]$ for some $\mu \in \mathbb{F}_{p^{s/t}}$, then*

$$\chi(ER_q) \leqslant \frac{t+4}{3} p^{s(2t+1)/3t} + tp^{(t+1)/2} + 1.$$

We encountered difficulties in extending this upper bound to the general case. In particular, when p is a prime, we have not been able to improve the upper bound $\chi(ER_p) = O(p/\log p)$ which is obtained by applying the main result of [2].

Conjecture 4. *Let p be an odd prime. For any integer $r \geqslant 0$,*

$$\chi(ER_{p^{2r+1}}) = O(p^{r+1/2}).$$

Instead of working with ER_q, we work with a related graph that is a bit more suitable for our computations.

Definition 5. *Let q be a power of an odd prime and $A = \{(a, a^2) : a \in \mathbb{F}_q\}$. Let G_q be the graph with vertex set $\mathbb{F}_q \times \mathbb{F}_q$, and distinct vertices (x_1, x_2) and (y_1, y_2) are adjacent if and only if*

$$(x_1, x_2) + (y_1, y_2) \in A.$$

Let G_q° be the graph obtained from G_q by adding loops to all vertices (x_1, x_2) for which $(x_1, x_2) + (x_1, x_2) \in A$. Vinh [18] proved that the graph G_q° is a $(q^2, q, \sqrt{2q})$-graph. Recall an (n, d, λ) graph is an n-vertex d-regular graph whose second eigenvalue $\max\{|\lambda_2|, |\lambda_n|\}$ is at most λ. Vinh used the fact that G_q° is a $(q^2, q, \sqrt{2q})$-graph to count solutions to $x_1 + x_2 = (x_3 + x_4)^2$ where $(x_1, x_3) \subset B$, $(x_2, x_4) \subset C$, and $B, C \subset \mathbb{F}_q^2$. For similar results that are obtained using techniques from combinatorial number theory, see [5]. We prove that G_q is isomorphic to an induced subgraph of the Erdős-Rényi orthogonal polarity graph.

Theorem 6. *If q is a power of an odd prime, then the graph G_q is isomorphic to an induced subgraph of ER_q.*

In the course of proving Theorem 6 we will show how to obtain ER_q from G_q by adding vertices and edges to G_q. This will allow us to translate upper bounds on $\chi(G_q)$ to upper bounds on $\chi(ER_q)$.

In addition to finding a proper coloring of ER_q, we also investigate proper colorings of small subgraphs of ER_q. In particular, we obtain the following result concerning small subgraphs of ER_q that are not 3-colorable.

Theorem 7. *If q is a sufficiently large odd prime power, then ER_q contains a subgraph H with at most 36 vertices and $\chi(H) \geqslant 4$.*

If $\mathcal{F}$ is a family of graphs, we say that a graph G is $\mathcal{F}$-free if G does not contain a subgraph isomorphic to a graph in $\mathcal{F}$. One of the most well-studied problems in extremal graph theory is to determine the maximum number of edges in an $\mathcal{F}$-free graph with n vertices, and then describe the extremal $\mathcal{F}$-free graphs. When $\mathcal{F}$ contains bipartite graphs, the structure of extremal $\mathcal{F}$-free graphs is not very well understood and there are few general results. A notable exception is when $\mathcal{F} = \{C_4\}$. A result of Füredi [10] states that when $q \geqslant 15$ is a prime power, an extremal C_4-free graph with $q^2 + q + 1$ vertices is an orthogonal polarity graph of a projective plane (see Section 5 for the definition of an orthogonal polarity graph).

Let $\mathcal{C}^r$ be the family of graphs with chromatic number r, and $\mathcal{C}^r_k$ be the family of graphs with at most k vertices and chromatic number r. Theorem 7 is motivated by the following problem of Allen, Keevash, Sudakov, and Verstraëte [1].

Problem 8 (Allen, et al. [1])**.** Let $\mathcal{F}$ be a family of bipartite graphs. Determine if there is an integer k such that

$$\mathrm{ex}(n, \mathcal{F} \cup \mathcal{C}^r_k) \sim \mathrm{ex}(n, \mathcal{F} \cup \mathcal{C}^r).$$

When considering Problem 8, a question that arises is if every extremal $\mathcal{F}$-free n-vertex graph (here n is tending to infinity) must contain some member of $\mathcal{C}^r_k$? In other words, does forbidding $\mathcal{C}^r_k$ actually have an effect on extremal $\mathcal{F}$-free graphs. By Theorem 7, one cannot take ER_q to obtain a lower bound on the Turán number $\mathrm{ex}(n, \{C_4\} \cup \mathcal{C}^3_k)$ for $k \geqslant 36$ without modifying ER_q in some way. It seems likely that for any integer $r \geqslant 5$, there exists integers q_r and $f(r)$ such that for any $q \geqslant q_r$, the graph ER_q contains a subgraph with at most $f(r)$ vertices and chromatic number at least r.

In Section 2 we prove Theorem 6. In Section 3 we prove Theorems 1 and 2. In Section 4 we prove Theorem 7. Section 5 contains some concluding remarks.

2   Proof of Theorem 6

Let q be a power of an odd prime power and $A = \{(a, a^2) : a \in \mathbb{F}_q\}$. Let $\mathbb{F}_q = \{b_1, \ldots, b_q\}$ and assume that $b_q = 0$. Let $F = \{b_q\} \times \mathbb{F}_q$. Then F is a subgroup of $\mathbb{F}_q^2$ and we let

$$F_i = F + (b_i, 0)$$

be the cosets of F where $F_q = F$. Add new vertices $z_1, \ldots, z_q, y$ to G_q. Make z_i adjacent to all vertices in F_i, and make y adjacent to each z_i. Call this graph H_q. Observe that G_q is an induced subgraph of H_q. We define an isomorphism ϕ from H_q to ER_q as follows.

1. For any $b_j \in \mathbb{F}_q$, let $\phi((0, b_j)) = (1, 0, 2^{-1} b_j)$.

2. For any $b_i, b_j \in \mathbb{F}_q$ with $b_i \neq 0$, let $\phi((b_i, b_j)) = (1, b_i, 2^{-1}(b_j - b_i^2))$.

3. Let $\phi(y) = (0, 0, 1)$ and $\phi(z_i) = (0, 1, b_i)$ for $1 \leqslant i \leqslant q$.

We will show that ϕ is an isomorphism by considering the different types of vertices in H_q. Recall that the rule for adjacency in ER_q is that (x_0, x_1, x_2) is adjacent to (y_0, y_1, y_2) if and only if $x_0 y_2 + x_2 y_0 = x_1 y_1$.

Case 1: Vertices of type $(0, b_j)$.

Let $b_j \in \mathbb{F}_q$. In H_q, the neighborhood of $(0, b_j)$ is $\{z_q\} \cup \{(x, x^2 - b_j) : x \in \mathbb{F}_q\}$. In ER_q, the neighborhood of $(1, 0, 2^{-1} b_j)$ is

$$\{(0, 1, 0)\} \cup \{(1, x, -2^{-1} b_j) : x \in \mathbb{F}_q\}. \tag{1}$$

By definition, $\phi((0, -b_j)) = (1, 0, -2^{-1} b_j)$ and for $x \neq 0$,

$$\phi((x, x^2 - b_j)) = (1, x, 2^{-1}(x^2 - b_j - x^2)) = (1, x, -2^{-1} b_j).$$

This shows that (1) coincides with the set

$$\{\phi(z_q)\} \cup \{\phi((x, x^2 - b_j)) : x \in \mathbb{F}_q\}.$$

We conclude that for any $b_j \in \mathbb{F}_q$, $(0, b_j)$ is adjacent to u in H_q if and only if $\phi((0, b_j))$ is adjacent to $\phi(u)$ in ER_q.

Case 2: Vertices of type (b_i, b_j) with $b_i \neq 0$.

Let $b_i, b_j \in \mathbb{F}_q$ with $b_i \neq 0$. In H_q, the neighborhood of (b_i, b_j) is

$$\{z_i\} \cup \{(x - b_i, x^2 - b_j) : x \in \mathbb{F}_q\}.$$

In ER_q, the neighborhood of $(1, b_i, 2^{-1}(b_j - b_i^2))$ is

$$\{(0, 1, b_i)\} \cup \{(1, x, x b_i - 2^{-1}(b_j - b_i^2)) : x \in \mathbb{F}_q\}. \tag{2}$$

We have $\phi(z_i) = (0, 1, b_i)$ and

$$\phi((b_i - b_i, b_i^2 - b_j)) = (1, 0, -2^{-1}(b_j - b_i^2)).$$

For $y \neq b_i$,

$$\begin{aligned} \phi((y - b_i, y^2 - b_j)) &= (1, y - b_i, 2^{-1}(y^2 - b_j - (y - b_i)^2)) \\ &= (1, y - b_i, y b_i - 2^{-1}(b_j + b_i^2)). \end{aligned}$$

If we take $x = y - b_i$ in (2), we obtain

$$(1, y - b_i, yb_i - 2^{-1}(b_j + b_i^2))$$

using the fact that $2^{-1} - 1 = -2^{-1}$. We conclude that for any $b_i, b_j \in \mathbb{F}_q$ with $b_i \neq 0$, (b_i, b_j) is adjacent to u in H_q if and only if $\phi((b_i, b_j))$ is adjacent to $\phi(u)$ in ER_q.

Case 3: Vertices of type z_i.

Let $1 \leqslant i \leqslant q$ and consider z_i. The neighborhood of z_i is $\{y\} \cup \{(b_i, x) : x \in \mathbb{F}_q\}$. In ER_q, the neighborhood of $(0, 1, b_i)$ is

$$\{(0, 0, 1)\} \cup \{(1, b_i, x) : x \in \mathbb{F}_q\} = \{\phi(y)\} \cup \{(1, b_i, x) : x \in \mathbb{F}_q\}.$$

If $i = q$, then $\phi((0, y)) = (1, 0, 2^{-1}y)$. If $i \neq q$, then $\phi((b_i, y)) = (1, b_i, 2^{-1}(y - b_i^2))$. As y ranges over $\mathbb{F}_q$, we obtain $(1, b_i, x)$ for all $x \in \mathbb{F}_q$.

We have not checked the neighborhood condition for $y \in V(H_q)$ but since we have considered all other vertices, this is not necessary.

3 Proof of Theorems 1 and 2

Throughout this section p is an odd prime and q is a power of p. The set $\mathbb{F}_q^*$ consisting of the nonzero elements of $\mathbb{F}_q$ can be partitioned into two sets $\mathbb{F}_q^+$ and $\mathbb{F}_q^-$ where

$$a \in \mathbb{F}_q^+ \text{ if and only if } -a \in \mathbb{F}_q^-.$$

Observe that the vertices (x_1, x_2) and (y_1, y_2) are adjacent in G_q if and only if $x_1 + y_1 = a$ and $x_2 + y_2 = a^2$ for some $a \in \mathbb{F}_q$. This is equivalent to $(x_1 + y_1)^2 = x_2 + y_2$. It is often this relation that we will use in our calculations.

Lemma 9. *(i) If $\mathbb{F}_{q^2} = \{a\theta + b : a, b \in \mathbb{F}_q\}$ for some $\theta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, then both*

$$\{(x, y\theta + z) : x, z \in \mathbb{F}_q, y \in \mathbb{F}_q^+\} \text{ and } \{(x, y\theta + z) : x, z \in \mathbb{F}_q, y \in \mathbb{F}_q^-\}$$

are independent sets in G_q.
(ii) If $t \geqslant 3$ is odd and $\mathbb{F}_{q^t} = \{a_0 + \cdots + a_{t-1}\theta^{t-1} : a_i \in \mathbb{F}_q\}$ for some $\theta \in \mathbb{F}_{q^t}$, then both

$$\{(x_0 + \cdots + x_{(t-3)/2}\theta^{\frac{t-3}{2}}, y_0 + \cdots + y_{t-1}\theta^{t-1}) : x_i, y_j \in \mathbb{F}_q, y_{t-1} \in \mathbb{F}_q^+\}$$

and

$$\{(x_0 + \cdots + x_{(t-3)/2}\theta^{\frac{t-3}{2}}, y_0 + \cdots + y_{t-1}\theta^{t-1}) : x_i, y_j \in \mathbb{F}_q, y_{t-1} \in \mathbb{F}_q^-\}$$

are independent sets in G_q.

Proof. We prove the first case of (i) as the proofs of the remaining statements are very similar. Suppose $(x_1, y_1\theta + z_1)$ and $(x_2, y_2\theta + z_2)$ are vertices in G_q with $x_1, x_2, z_1, z_2 \in \mathbb{F}_q$ and $y_1, y_2 \in \mathbb{F}_q^+$. Then $(x_1 + x_2)^2 \in \mathbb{F}_q$ but $(y_1 + y_2)\theta + (z_1 + z_2) \notin \mathbb{F}_q$ since $y_1 + y_2 \neq 0$. Therefore, the vertices $(x_1, y_1\theta + z_1)$ and $(x_2, y_2\theta + z_2)$ are not adjacent. $\square$

Lemma 10. *For any $k \in \mathbb{F}_q^*$, the maps $\psi_k, \phi_k : V(G_q) \to V(G_q)$ given by*

$$\psi_k((x,y)) = (x+k, y+4kx+2k^2) \text{ and } \phi_k((x,y)) = (kx, k^2 y)$$

are automorphisms of G_q.

Proof. Let $k \in \mathbb{F}_q^*$. Suppose (x_1, x_2) is adjacent to (y_1, y_2) so that $(x_1 + y_1)^2 = x_2 + y_2$. In this case,

$$
\begin{aligned}
(x_1 + k + y_1 + k)^2 &= (x_1 + y_1)^2 + 4kx_1 + 4ky_1 + 4k^2 \\
&= (x_2 + 4kx_1 + 2k^2) + (y_2 + 4ky_1 + 2k^2).
\end{aligned}
$$

This shows that $(x_1+k, x_2+4kx_1+2k^2)$ is adjacent to $(y_1+k, y_2+4ky_1+2k^2)$. Conversely, if $(x_1 + k, x_2 + 4kx_1 + 2k^2)$ is adjacent to $(y_1 + k, y_2 + 4ky_1 + 2k^2)$, then it must be the case that $(x_1 + y_1)^2 = x_2 + y_2$ and so (x_1, y_1) is adjacent to (x_2, y_2).

To show that ϕ_k is an isomorphism it is enough to observe that $(x_1 + y_1)^2 = x_2 + y_2$ is equivalent to $(kx_1 + ky_1)^2 = k^2 x_2 + k^2 y_2$. $\qquad \square$

3.1 q a square

In this section we prove the following.

Theorem 11. *Let q be a power of an odd prime. The chromatic number of G_{q^2} satisfies*

$$\chi(G_{q^2}) \leqslant 2q + O(q/\log q).$$

Proof. Let θ be a root of an irreducible quadratic polynomial in $\mathbb{F}_q[x]$ so that $\mathbb{F}_{q^2} = \{a\theta + b : a, b, \in \mathbb{F}_q\}$. Assume that $\theta^2 = \mu_1 \theta + \mu_0$ where $\mu_0, \mu_1 \in \mathbb{F}_q$. Let $I^+ = \{(x, y\theta + z) : x, z \in \mathbb{F}_q, y \in \mathbb{F}_q^+\}$, $I^- = \{(x, y\theta + z) : x, z \in \mathbb{F}_q, y \in \mathbb{F}_q^-\}$, and $J = I^+ \cup I^-$. By Lemma 9, J is the union of two independent sets and so $\chi(G_{q^2}[J]) \leqslant 2$. Let

$$S = \bigcup_{k \in \mathbb{F}_q} \psi_{k\theta}(J).$$

By Lemma 10, each $\psi_{k\theta}$ is an isomorphism and so $\chi(G_{q^2}[S]) \leqslant 2q$. Let $X = V(G_{q^2}) \backslash S$. Since $\mathbb{F}_q^+ \cup \mathbb{F}_q^- = \mathbb{F}_q^*$, we can write

$$S = \{(x + k\theta, y\theta + z + 4k\theta x + 2k^2 \theta^2) : x, k, y, z \in \mathbb{F}_q, y \neq 0\}.$$

Given a vertex $(s, t) \in V(G_{q^2})$, say with $s = s_0 + s_1 \theta$ and $t = t_0 + t_1 \theta$, we can take $x = s_0$ and $k = s_1$ to obtain

$$\{(s_0 + s_1 \theta, y\theta + z + 4s_1 s_0 \theta + 2s_1^2(\mu_1 \theta + \mu_0)) : y, z \in \mathbb{F}_q, y \neq 0\} \subset S.$$

The second coordinate in the above subset of S simplifies to

$$(z + 2s_1^2 \mu_0) + (y + 4s_1 s_0 + 2s_1^2 \mu_1)\theta.$$

We can choose $z = t_0 - 2s_1^2\mu_0$ and as long as $t_1 \neq 4s_1s_0 + 2s_1^2\mu_1$, we can take $y = t_1 - 4s_1s_0 - 2s_1^2\mu_1$. Otherwise, $t_1 = 4s_1s_0 + 2s_1^2\mu_1$ and so

$$X = \{(s_0 + s_1\theta, t_0 + (4s_1s_0 + 2s_1^2\mu_1)\theta) : s_0, s_1, t_0 \in \mathbb{F}_q\}.$$

Partition X into q sets X_s where $s \in \mathbb{F}_q$ and

$$X_s = \{(s\theta + s_2, (2s^2\mu_1 + 4ss_2)\theta + t_2 : s_2, t_2 \in \mathbb{F}_q\}.$$

Claim 1: For any $s \in \mathbb{F}_q$, $\Delta(G_{q^2}[X_s]) \leqslant q$.

Let $s \in \mathbb{F}_q$. A pair of vertices

$$(s\theta + s_2, (2s^2\mu_1 + 4ss_2)\theta + t_2) \text{ and } (s\theta + u_2, (2s^2\mu_1 + 4su_2)\theta + v_2),$$

both in X_s, are adjacent if and only if $4s^2\mu_0 + (s_2 + u_2)^2 = t_2 + v_2$. If s_2 and t_2 are fixed, then there are q choices for u_2 and once u_2 is fixed, v_2 is determined. Therefore, the maximum degree of $G_{q^2}[X_s]$ is q.

Claim 2: $\Delta(G_{q^2}[X]) \leqslant 2q - 1$.

By Claim 1, a vertex in X_s has at most q other neighbors in X_s. Let $s, t \in \mathbb{F}_q$ where $s \neq t$. The vertex $(s\theta + s_2, (2s^2\mu_1 + 4ss_2)\theta + t_2) \in X_s$ is adjacent to the vertex $(t\theta + u_2, (2t^2\mu_1 + 4tu_2)\theta + v_2) \in X_t$ if and only if

$$\mu_1(s^2 + 2st + t^2) + 2(s + t)s_2 + 2(s + t)u_2 = \mu_1(2s^2 + 2t^2) + 4ss_2 + 4tu_2 \qquad (3)$$

and

$$(s + t)^2\mu_0 + (s_2 + u_2)^2 = t_2 + v_2. \qquad (4)$$

Equation (3) can be rewritten as

$$\mu_1(s - t)^2 = 2(t - s)s_2 + 2(s - t)u_2. \qquad (5)$$

Thus if s_2 and t_2 are fixed, then (5) and (4) determine u_2 and v_2 since $2(s - t) \neq 0$. This shows that a vertex in X_s has exactly one neighbor in X_t whenever $s \neq t$. Namely, given the vertex $(s\theta + s_2, (2s^2\mu_1 + 4ss_2)\theta + t_2) \in X_s$, its unique neighbor in X_t where $t \neq s$ is $(t\theta + u_2, (2t^2\mu_1 + 4tu_2)\theta + v_2)$ where

$$u_2 = 2^{-1}\mu_1(s - t) + s_2 \text{ and } v_2 = (s + t)^2\mu_0 + (2s_2 + 2^{-1}\mu_1(s - t))^2 - t_2.$$

We conclude that a vertex $x \in X$ has at most q neighbors in X_s when $x \in X_s$, and one neighbor in each X_t for $t \neq s$. Since $X = \cup_{s \in \mathbb{F}_q} X_s$, we have proved Claim 2.

Alon, Krivelevich, and Sudakov [2] proved that any graph with maximum degree d with the property that the neighborhood of every vertex contains at most d^2/f edges has chromatic number at most $c(d/\log f)$ where c is an absolute constant. A C_4-free graph with maximum degree d has the property that the neighborhood of every vertex contains at most $d/2$ edges. Applying the result of [2] to $G_{q^2}[X]$, we obtain $\chi(G_{q^2}[X]) = O(2q/\log q)$. Combining this coloring with our coloring of S, we obtain a proper coloring of G_{q^2} with $2q + O(q/\log q)$ colors. $\qquad\square$

To obtain a coloring of $ER_{q^2} \cong H_{q^2}$, we only need one additional color for the vertices $z_1, \ldots, z_{q^2}, y$. The vertices $z_1, \ldots, z_{q^2}$ form an independent set in H_{q^2} and so we use one new color on these vertices. The vertex y has no neighbors in G_{q^2} and so we may use any one of the $2q + O(q/\log q)$ colors used to G_{q^2} to color y. This proves Theorem 1.

3.2 q not a square

In this subsection we prove the following result.

Theorem 12. *Let q be a power of an odd prime. If $r \geqslant 1$ and for some $\mu \in \mathbb{F}_q$, the polynomial $x^{2r+1} - \mu \in \mathbb{F}_q[x]$ is irreducible, then*

$$\chi(G_{q^{2r+1}}) \leqslant \frac{2r+5}{3} q^{\frac{4r}{3}+1} + (2r+1)q^{r+1}.$$

Proof. Let $t = 2r + 1 \geqslant 3$ be an odd integer. Suppose there is a $\mu \in \mathbb{F}_q^*$ such that the polynomial $x^t - \mu \in \mathbb{F}_q[x]$ is irreducible. Let θ be a root of $x^t - \mu$ in an extension field of $\mathbb{F}_q$. We may view θ as an element of $\mathbb{F}_{q^t}$ and $\{1, \theta, \ldots, \theta^{2r}\}$ is a basis for $\mathbb{F}_{q^t}$ over $\mathbb{F}_q$. For $2r + 1 \leqslant l \leqslant 6r + 3$,

$$\theta^l = \begin{cases} \mu \theta^{l-2r-1} & \text{if } 2r+1 \leqslant l < 4r+2, \\ \mu^2 \theta^{l-4r-2} & \text{if } 4r+2 \leqslant l < 6r+3. \end{cases}$$

This identity will be used frequently throughout this subsection. Define

$$I^+ = \left\{ (x_0 + x_1\theta + \cdots + x_{r-1}\theta^{r-1}, y_0 + \cdots + y_{2r}\theta^{2r}) : x_i, y_j \in \mathbb{F}_q, y_{2r} \in \mathbb{F}_q^+ \right\}$$

and

$$I^- = \left\{ (x_0 + x_1\theta + \cdots + x_{r-1}\theta^{r-1}, y_0 + \cdots + y_{2r}\theta^{2r}) : x_i, y_j \in \mathbb{F}_q, y_{2r} \in \mathbb{F}_q^- \right\}.$$

By Lemma 9, both I^+ and I^- are independent sets. Let $J = I^+ \cup I^-$. Since J is the union of two independent sets, $\chi(G_{q^t}[J]) \leqslant 2$. For $k \in \mathbb{F}_{q^t}$, the map $\psi_k((x, y)) = (x + k, y + 4kx + 2k^2)$ is an isomorphism of G_{q^t} by Lemma 10. Let

$$S = \bigcup_{(x_r, \ldots, x_{2r}) \in \mathbb{F}_q^{r+1}} \psi_{x_r\theta^r + \cdots + x_{2r}\theta^{2r}}(J).$$

We properly color the vertices of S with at most $2q^{r+1}$ colors. Let $X = \mathbb{F}_{q^t}^2 \backslash S$. It remains to color the vertices in X. To do this, we will proceed as follows. By Lemma 10, for any $k \in \mathbb{F}_{q^t}^*$, the map $\phi_k((x, y)) = (kx, k^2y)$ is an isomorphism of G_{q^t}. Let $1 \leqslant l \leqslant 2r$ and consider $\phi_{\theta^l}(X)$. Let $Y_l = S \cap \phi_{\theta^l}(X)$. The graph $G_{q^t}[Y_l]$ is isomorphic to a subgraph of $G_{q^t}[S]$. We have shown that $\chi(G_{q^t}[S]) \leqslant 2q^{r+1}$ and so $\chi(G_{q^t}[Y_l]) \leqslant 2q^{r+1}$ for any $1 \leqslant l \leqslant 2r$. Therefore, we can properly color the vertices in

$$\phi_{\theta^l}^{-1}(Y_l) = \phi_{\theta^l}^{-1}(S) \cap X$$

with at most $2q^{r+1}$ colors. This gives a proper coloring that uses at most $(2r+1)2q^{r+1}$ colors. The only vertices that have not been colored are those that are in the set

$$Z := X \cap \phi_\theta(X) \cap \phi_{\theta^2}(X) \cap \cdots \cap \phi_{\theta^{2r}}(X).$$

We are now going to show that if $(s_0 + \cdots + s_{2r}\theta^{2r}, t_0 + \cdots + t_{2r}\theta^{2r}) \in Z$, then each t_i is determined by $s_0, \ldots, s_{2r}$. This will allow us to prove an upper bound on the maximum degree of $G_{q^t}[Z]$ and we can then color $G_{q^t}[Z]$ by applying Brooks' Theorem.

We will use the following notation for the rest of this section. If $s \in \mathbb{F}_{q^t}$, then $s_0, \ldots, s_{2r}$ will be the coefficients of s in the unique representation $s = s_0 + s_1\theta + \cdots + s_{2r}\theta^{2r}$ where $s_i \in \mathbb{F}_q$. Given a $2r+1$-tuple $(z_0, z_1, \ldots, z_{2r}) \in \mathbb{F}_q^{2r+1}$, define

$$\alpha(z_0, z_1, \ldots, z_{2r}) = 2z_r^2 + 4 \sum_{j=0}^{r-1} z_j z_{2r-j}.$$

Claim 1: If $(s_0 + \cdots + s_{2r}\theta^{2r}, t_0 + \cdots + t_{2r}\theta^{2r}) \in X$, then

$$t_{2r} = \alpha(s_0, s_1, \ldots, s_{2r}).$$

Proof of Claim 1. A vertex in S is of the form

$$(x_0 + \cdots + x_{r-1}\theta^{r-1} + x_r\theta^r + \cdots + x_{2r}\theta^{2r}, y_0 + \cdots + y_{2r}\theta^{2r}$$
$$+ 4(x_r\theta^r + \cdots + x_{2r}\theta^{2r})(x_0 + \cdots + x_{r-1}\theta^{r-1}) + 2(x_r\theta^r + \cdots + x_{2r}\theta^{2r})^2)$$

for some $x_i, y_j \in \mathbb{F}_q$, and $y_{2r} \in \mathbb{F}_q^+ \cup \mathbb{F}_q^- = \mathbb{F}_q^*$. The coefficient of θ^{2r} in the second coordinate is

$$y_{2r} + 2x_r^2 + 4 \sum_{j=0}^{r-1} x_j x_{2r-j}.$$

Thus, given any vertex $(s, t) \in \mathbb{F}_{q^t}^2$, we have that $(s, t) \in S$ unless

$$t_{2r} = 2s_r^2 + \sum_{j=0}^{r-1} s_j s_{2r-j}.$$

$\square$

Claim 2: If $1 \leqslant l \leqslant 2r$ and $(s, t) \in X \cap \phi_{\theta^l}(X)$, then

$$t_{2l-1} = \mu\alpha(s_l, s_{l+1}, \ldots, s_{2r}, \mu^{-1}s_0, \ldots, \mu^{-1}s_{l-1}) \quad \text{if } 1 \leqslant l \leqslant r,$$

and

$$t_{2l-2r-2} = \mu^2\alpha(s_l, s_{l+1}, \ldots, s_{2r}, \mu^{-1}s_0, \ldots, \mu^{-1}s_{l-1}) \quad \text{if } r+1 \leqslant l \leqslant 2r.$$

Proof of Claim 2. Suppose $(s, t) \in X \cap \phi_{\theta^l}(X)$. There is an $(x, y) \in X$ such that $(s, t) = \phi_{\theta^l}((x, y))$. From the equation $(s, t) = (\theta^l x, \theta^{2l} y)$ we obtain by equating coefficients of $\theta^0, \theta^1, \ldots, \theta^{2r}$ in the first component,

$$x_i = s_{l+i} \text{ for } 0 \leqslant i \leqslant 2r - l \quad \text{and} \quad \mu x_i = s_{i-2r+l-1} \text{ for } 2r - l + 1 \leqslant i \leqslant 2r. \qquad (6)$$

If $1 \leqslant l \leqslant r$, then we obtain $t_{2l-1} = \mu y_{2r}$ by considering the coefficient of θ^{2l-1} in the second component. Similarly, if $r + 1 \leqslant l \leqslant 2r$, we obtain $t_{2l-2r-2} = \mu^2 y_{2r}$ by considering the coefficient of $\theta^{2l-2r-2}$ in the second component. Since $(x, y) \in X$, we have by Claim 1 that

$$y_{2r} = \alpha(x_0, x_1, \ldots, x_{2r}). \qquad (7)$$

Using (6), we can solve for the x_i's in terms of the s_j's and then substitute into (7) to complete the proof of Claim 2. $\qquad \square$

For $0 \leqslant k \leqslant 2r$, let

$$U_k = \{\{i, j\} \subset \{0, 1, \ldots, 2r\} : i + j \equiv k \pmod{2r + 1}\}.$$

Given $\{i, j\} \subset \{0, 1, \ldots, 2r\}$, let

$$\mu_{\{i,j\}} = \begin{cases} 1 & \text{if } 1 \leqslant i + j \leqslant 2r, \\ \mu & \text{if } 2r + 1 \leqslant i + j \leqslant 4r - 1. \end{cases}$$

Claim 3: Suppose $(s, t) \in Z$. If $1 \leqslant l \leqslant r$, then

$$t_{2l-1} = 2\mu s_{l+r}^2 + 4 \sum_{\{i,j\} \in U_{2l-1}} \mu_{\{i,j\}} s_i s_j.$$

If $0 \leqslant l \leqslant r - 1$, then

$$t_{2l} = 2s_l^2 + 4 \sum_{\{i,j\} \in U_{2l}} \mu_{\{i,j\}} s_i s_j.$$

Proof of Claim 3. First suppose $1 \leqslant l \leqslant r$. By Claim 2,

$$t_{2l-1} = \mu \alpha(s_l, s_{l+1}, \ldots, s_{2r}, \mu^{-1} s_0, \ldots, \mu^{-1} s_{l-1}).$$

Using the definition of α, we get that

$$\begin{aligned} t_{2l-1} &= \mu(2s_{l+r}^2 + 4(s_l \mu^{-1} s_{l-1} + s_{l+1} \mu^{-1} s_{l-2} + \cdots + s_{2l-1} \mu^{-1} s_0 \\ &+ s_{2l} s_{2r} + \cdots + s_{l+r-1} s_{l+r+1})) \\ &= 2\mu s_{l+r}^2 + 4 \sum_{\{i,j\} \in U_{2l-1}} \mu_{\{i,j\}} s_i s_j. \end{aligned}$$

Assume now that $0 \leqslant l \leqslant r - 1$. By Claim 2,

$$t_{2l} = \mu^2 \alpha(s_{l+r+1}, s_{l+r+2}, \ldots, s_{2r}, \mu^{-1} s_0, \ldots, \mu^{-1} s_{l+r}).$$

We can now proceed as before using the definition of α. $\qquad \square$

Claim 4: Let $s, x \in \mathbb{F}_{q^t}$. If $0 \leqslant l \leqslant 2r$, then the coefficient of θ^l in $(s+x)^2$ is

$$(s_{l/2} + x_{l/2})^2 + 2 \sum_{\{i,j\} \in U_l} \mu_{\{i,j\}}(s_i + x_i)(s_j + x_j) \text{ if } l \text{ is even,}$$

and

$$\mu(s_{r+l/2+1/2} + x_{r+l/2+1/2})^2 + 2 \sum_{\{i,j\} \in U_l} \mu_{\{i,j\}}(s_i + x_i)(s_j + x_j) \text{ if } l \text{ is odd.}$$

Proof of Claim 4. Consider

$$(s+x)^2 = \sum_{i=0}^{2r}(s_i + x_i)^2\theta^{2i} + 2 \sum_{0 \leqslant i < j \leqslant 2r}(s_i + x_i)(s_j + x_j)\theta^{i+j}.$$

The claim follows from the definitions of $\mu_{\{i,j\}}$, U_l, and the identity $\theta^{2r+k} = \mu\theta^{k-1}$ for $1 \leqslant k \leqslant 2r$. $\qquad\square$

Claim 5: If $(s,t), (x,y) \in Z$ and $(s+x)^2 = t + y$, then

$$\mu(s_{l+r} - x_{l+r})^2 + 2 \sum_{\{i,j\} \in U_{2l-1}} \mu_{\{i,j\}}(s_i - x_i)(s_j - x_j) = 0 \text{ for } 1 \leqslant l \leqslant r,$$

and

$$(s_l - x_l)^2 + 2 \sum_{\{i,j\} \in U_{2l}} \mu_{\{i,j\}}(s_i - x_i)(s_j - x_j) = 0 \text{ for } 0 \leqslant l \leqslant r.$$

Proof of Claim 5. By Claim 4, equating coefficients of $1, \theta, \ldots, \theta^{2r}$ in the equation $(s+x)^2 = t + y$ gives

$$t_{2l-1} + y_{2l-1} = \mu(s_{l+r} + x_{l+r})^2 + 2 \sum_{\{i,j\} \in U_{2l-1}} \mu_{\{i,j\}}(s_i + x_i)(s_j + x_j) \text{ if } 1 \leqslant l \leqslant r,$$

and

$$t_{2l} + y_{2l} = (s_l + x_l)^2 + 2 \sum_{\{i,j\} \in U_{2l}} \mu_{\{i,j\}}(s_i + x_i)(s_j + x_j) \text{ if } 0 \leqslant l \leqslant r.$$

Now we apply Claim 3 to t_{2l-1} and y_{2l-1}. This gives

$$2\mu(s_{l+r}^2 + x_{l+r}^2) + 4 \sum_{\{i,j\} \in U_{2l-1}} \mu_{\{i,j\}}(s_i s_j + x_i x_j) =$$
$$\mu(s_{l+r} + x_{l+r})^2 + 2 \sum_{\{i,j\} \in U_{2l-1}} \mu_{\{i,j\}}(s_i + x_i)(s_j + x_j)$$

for $1 \leqslant l \leqslant r$. This can be rewritten as

$$\mu(s_{l+r} - x_{l+r})^2 + 2 \sum_{\{i,j\} \in U_{2l-1}} \mu_{\{i,j\}}(s_i - x_i)(s_j - x_j) = 0.$$

A similar application of Claim 3 (and Claim 1 in the case of t_{2r} and y_{2r}) gives

$$(s_l - x_l)^2 + 2 \sum_{\{i,j\} \in U_{2l}} \mu_{\{i,j\}}(s_i - x_i)(s_j - x_j) = 0$$

for $0 \leqslant l \leqslant r$. $\square$

We are now ready to find an upper bound on the maximum degree of the subgraph of G_{q^t} induced by Z. Fix a vertex $(s,t) \in Z$. Suppose (x,y) is a neighbor of (s,t) with $(x,y) \in Z$. By Claim 5, $(x_0, \ldots, x_{2r}) \in \mathbb{F}_q^{2r+1}$ is a solution to the system

$$\mu(s_{l+r} - x_{l+r})^2 + 2 \sum_{\{i,j\} \in U_{2l-1}} \mu_{\{i,j\}}(s_i - x_i)(s_j - x_j) = 0 \text{ for } 1 \leqslant l \leqslant r,$$

and

$$(s_l - x_l)^2 + 2 \sum_{\{i,j\} \in U_{2l}} \mu_{\{i,j\}}(s_i - x_i)(s_j - x_j) = 0 \text{ for } 0 \leqslant l \leqslant r.$$

If we set $z_i = s_i - x_i$ for $0 \leqslant i \leqslant 2r$, then we see that we have a solution to the following system of $2r + 1$ homogeneous quadratic equations in the $2r + 1$ unknowns $z_0, \ldots, z_{2r+1}$:

$$\mu z_{l+r}^2 + 2 \sum_{\{i,j\} \in U_{2l-1}} \mu_{\{i,j\}} z_i z_j = 0 \text{ for } 1 \leqslant l \leqslant r,$$

and

$$z_l^2 + 2 \sum_{\{i,j\} \in U_{2l}} \mu_{\{i,j\}} z_i z_j = 0 \text{ for } 0 \leqslant l \leqslant r.$$

Lemma 13. *The number of solutions $(z_0, \ldots, z_{2r}) \in \mathbb{F}_q^{2r+1}$ to the above system of $2r + 1$ homogeneous quadratic equations is at most*

$$\frac{2r + 5}{3} q^{\frac{4r}{3}+1}.$$

Proof. Let m be the largest integer such that $m \leqslant \frac{2(r+1)}{3}$. If there is set T of size m such that each z_i in the set $\{z_i : i \in T\}$ either is zero or is determined uniquely by the z_j's in the set $\{z_j : j \in \{0, 1, \ldots, 2r\} \backslash T\}$, then there are at most q^{2r+1-m} solutions. We will show there are at most $(m+1)$ choices for the index set T which implies that we have at most $(m+1)q^{2r+1-m}$ solutions in total.

Let A be the $2(r-m+1) \times m$ matrix with entries in $\mathbb{F}_q$, where for $1 \leqslant i \leqslant 2(r-m+1)$ and $1 \leqslant j \leqslant m$, the (i,j) entry of A is the coefficient of z_{j-1} in the equation

$$\mu z_{m+r+(i-1)/2}^2 + 2 \sum_{\{k,l\} \in U_{2(m+(i-1)/2)-1}} \mu_{\{k,l\}} z_k z_l = 0$$

if i is odd, and the coefficient of z_{j-1} in the equation

$$z_{m+(i-2)/2}^2 + 2 \sum_{\{k,l\} \in U_{2(m+(i-2)/2)}} \mu_{\{k,l\}} z_k z_l = 0$$

if i is even. If A is the matrix formed in this way, then one check that

$$A = \begin{pmatrix} z_{2m-1} & z_{2m-2} & z_{2m-3} & \cdots & z_m \\ z_{2m} & z_{2m-1} & z_{2m-2} & \cdots & z_{m+1} \\ z_{2m+1} & z_{2m} & z_{2m-1} & \cdots & z_{m+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{2r} & z_{2r-1} & z_{2r-2} & \cdots & z_{2r-m+1} \end{pmatrix}$$

As $m \leqslant \frac{2(r+1)}{3}$, we have $2(r - m + 1) \geqslant m$. Let $x = (z_0, z_1, \ldots, z_{m-1})^T$ and

$$b_i = \begin{cases} -2^{-1}\mu z_{m+r+(i-1)/2} - \sum\limits_{\substack{\{k,l\} \in U_{2(m+(i-1)/2)-1} \\ \{k,l\} \cap \{0,1,\ldots,m-1\} = \emptyset}} \mu_{\{k,l\}} z_k z_l & \text{if } i \text{ is odd} \\ -2^{-1} z_{m+(i-2)/2} - \sum\limits_{\substack{\{k,l\} \in U_{2(m+(i-2)/2)} \\ \{k,l\} \cap \{0,1,\ldots,m-1\} = \emptyset}} \mu_{\{k,l\}} z_k z_l & \text{if } i \text{ is even} \end{cases}$$

for $1 \leqslant i \leqslant 2(r - m + 1)$. Let $b = (b_1, b_2, \ldots, b_{2(r-m+1)})^T$.

We next show an upper bound $m + 1$ for the possible choices for the index set T. Let r_i be the i-th row of A so that

$$r_i = \left(z_{2m-1+(i-1)}, z_{2m-2+(i-1)}, z_{2m-3+(i-1)}, \ldots, z_{m+(i-1)}\right).$$

If $r_1 = \mathbf{0}$, then we take $T = \{m, m + 1, \ldots, 2m - 1\}$. We assume $r_1 \neq \mathbf{0}$ for the rest of the proof of the lemma.

Claim: If $i \geqslant 1$ and $r_{i+1} \in \mathrm{Span}_{\mathbb{F}_q}\{r_1, \ldots, r_i\}$, then

$$z_{m+i+j} \in \mathrm{Span}_{\mathbb{F}_q}\{z_m, z_{m+1}, \ldots, z_{m+i-1}\} \text{ for } j = 0, 1, \ldots, m - 1.$$

Proof of Claim. We prove the claim by induction on j. Suppose

$$r_{i+1} = \alpha_1 r_1 + \cdots + \alpha_i r_i \tag{8}$$

for some $\alpha_j \in \mathbb{F}_q$. By considering the last coordinate, we get that

$$z_{m+i} = \sum_{j=1}^{i} \alpha_j z_{m+(j-1)} \in \mathrm{Span}_{\mathbb{F}_q}\{z_m, \ldots, z_{m+i-1}\}$$

establishing the base case $j = 0$. If $z_{m+i+j_0} \in \mathrm{Span}_{\mathbb{F}_q}\{z_m, \ldots, z_{m+i-1}\}$ for $0 \leqslant j_0 \leqslant m - 2$, then by (8),

$$z_{m+i+j_0+1} = \sum_{j=1}^{i} \alpha_j z_{m+j_0+1+(j-1)} = \alpha_i z_{m+i+j_0} + \sum_{j=1}^{i-1} \alpha_j z_{m+j+j_0}.$$

By the inductive hypothesis, this is in $\mathrm{Span}_{\mathbb{F}_q}\{z_m, \ldots, z_{m+i-1}\}$. $\qquad\square$

By the Claim, if there is an $i \in \{1, 2, \ldots, m-1\}$ such that $r_{i+1} \in \operatorname{Span}_{\mathbb{F}_q}\{r_1, \ldots, r_i\}$, then there exist m z_i's that are uniquely determined by the other z_j's and we can take $T = \{z_{m+i}, z_{m+i+1}, \ldots, z_{2m+i-1}\}$. Otherwise, $r_1, \ldots, r_m$ are linearly independent which implies that the rank of A is at least m. It is at this step where we need $m \leqslant \frac{2(r+1)}{3}$ as we require the number of rows of A, which is $2(r-m+1)$, to be at least m. Since the rank of A is at least m, there is at most one solution x to $Ax = b$. In this case we take $T = \{0, 1, \ldots, m-1\}$ and each of $\{z_0, z_1, \ldots, z_{m-1}\}$ is determined by $\{z_m, z_{m+1}, \ldots, z_{2r}\}$.

Altogether, we have at most

$$(m+1)q^{\frac{4r}{3}-\frac{1}{3}} \leqslant \frac{2r+5}{3}q^{\frac{4r}{3}+1}$$

solutions which proves the lemma.

$\square$

By Lemma 13, the maximum degree of $G_{q^t}[Z]$ is at most $\frac{2r+5}{3}q^{\frac{4r}{3}+1}$. Therefore,

$$\chi(G_{q^t}) \leqslant \frac{2r+5}{3}q^{\frac{4r}{3}+1} + (2r+1)q^{r+1}$$

$\square$

We obtain a coloring of ER_{q^t} from a coloring of G_{q^t} as before. We use one new color on the vertices $z_1, \ldots, z_t$, and then give y any color that is used on G_{q^t}. This gives a coloring of ER_{q^t} that uses at most $\frac{2r+5}{3}q^{\frac{4r}{3}+1} + (2r+1)q^{r+1} + 1$ colors which proves Theorem 2.

4 Proof of Theorem 7

The following lemma is easily proved using the definition of adjacency in G_q.

Lemma 14. *Suppose $\alpha_i, \alpha_j, \alpha_k$ are distinct elements of $\mathbb{F}_q$ such that*

$$\alpha_i + \alpha_j = a^2, \alpha_j + \alpha_k = b^2, \text{ and } \alpha_k + \alpha_i = c^2$$

for some $a, b, c \in \mathbb{F}_q$. If $x + y = a$, $y + z = b$, and $z + x = c$, then $\{(x, \alpha_i), (y, \alpha_j), (z, \alpha_k)\}$ induces a triangle in G_q.

Given an odd prime power q, let $\chi : \mathbb{F}_q \to \{0, \pm 1\}$ be the quadratic character on $\mathbb{F}_q$. That is, $\chi(0) = 0$, $\chi(a) = 1$ if a is a nonzero square in $\mathbb{F}_q$, and $\chi(a) = -1$ otherwise. For the next lemma, we require some results on finite fields (see Chapter 5 of [15]).

Proposition 15. *Let q be an odd prime and $f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{F}_q[x]$ where $a_2 \neq 0$. If $a_1^2 - 4a_0a_2 \neq 0$, then*

$$\sum_{c \in \mathbb{F}_q} \chi(f(c)) = -\chi(a_2).$$

Proposition 16 (Weil)**.** *If $f(x) \in \mathbb{F}_q[x]$ is a degree $d \geqslant 1$ polynomial that is not the square of another polynomial, then*

$$\left| \sum_{x \in \mathbb{F}_q} \chi(f(x)) \right| \leqslant (d-1)\sqrt{q}.$$

Lemma 17. *If $q > 487$ is a power of an odd prime, then there are elements $\alpha_1, \ldots, \alpha_5 \in \mathbb{F}_q^*$ such that $\alpha_1, \ldots, \alpha_5$ are all distinct, and*

$$\chi(\alpha_i + \alpha_j) = 1$$

for $1 \leqslant i < j \leqslant 5$.

Proof. Choose $\alpha_1 \in \mathbb{F}_q^*$ arbitrarily. There are $\frac{q-1}{2}$ nonzero squares in $\mathbb{F}_q^*$ so we can easily find an $\alpha_2 \in \mathbb{F}_q \backslash \{0, \alpha_1\}$ such that $\chi(\alpha_1 + \alpha_2) = 1$. Observe that this implies $\alpha_2 \neq -\alpha_1$ otherwise $\chi(\alpha_1 + \alpha_2) = 0$. Assume that we have chosen $\alpha_1, \ldots, \alpha_k \in \mathbb{F}_q^*$ so that $\alpha_1, \ldots, \alpha_k$ are all distinct and

$$\chi(\alpha_i + \alpha_j) = 1$$

for $1 \leqslant i < j \leqslant k$. Let

$$f(x) = \prod_{i=1}^{k} (1 + \chi(\alpha_i + x))$$

and $X = \{\beta \in \mathbb{F}_q : f(\beta) = 2^k\}$. If $\beta \in \mathbb{F}_q$ and $f(\beta) \neq 0$, then $\chi(\alpha_i + \beta) \in \{0, 1\}$ for $1 \leqslant i \leqslant k$. We have $\chi(\alpha_i + \beta) = 0$ if and only if $\beta = -\alpha_i$. Therefore, there are at most k distinct β's in $\mathbb{F}_q$ such that $0 < f(\beta) < 2^k$ which implies

$$2^k |X| + k2^{k-1} \geqslant \sum_{x \in \mathbb{F}_q} f(x) = q + \sum_{\emptyset \neq S \subset [k]} \sum_{x \in \mathbb{F}_q} \chi\left(\prod_{\alpha \in S} (x + \alpha) \right).$$

For any $i \in [k]$, $\sum_{x \in \mathbb{F}_q} \chi(\alpha_i + x) = 0$. For any $1 \leqslant i < j \leqslant k$,

$$\sum_{x \in \mathbb{F}_q} \chi(x^2 + (\alpha_i + \alpha_j)x + \alpha_i \alpha_j) = -1$$

by Proposition 15. When $k = 2$, if $\frac{q - 3 - 3 \cdot 2^2}{2^3} \geqslant 3 + 1$, then $|X| \geqslant 4$ and we can choose an $\alpha_3 \in \mathbb{F}_q \backslash \{0, \alpha_1, \alpha_2\}$ that has the desired properties. When $k \in \{3, 4\}$, we use Weyl's inequality to obtain a lower bound on $|X|$. Observe that since $\alpha_1, \ldots, \alpha_k$ are distinct, no product $(x + \alpha_{i_1}) \cdots (x + \alpha_{i_l})$ for $1 \leqslant i_1 < \cdots < i_l \leqslant k$ is the square of a polynomial in $\mathbb{F}_q[x]$. By Weyl's inequality, for any $1 \leqslant i_1 < \cdots < i_l \leqslant k$,

$$\left| \sum_{x \in \mathbb{F}_q} \chi((x + \alpha_{i_1}) \cdots (x + \alpha_{i_l})) \right| \leqslant l\sqrt{q}.$$

The hypothesis on q implies that q is large enough so that $|X| \geqslant 5$ when $k = 3$, and $|X| \geqslant 6$ when $k = 4$. Therefore, we can inductively choose α_4 and α_5 so that $\alpha_1, \ldots, \alpha_5$ satisfy all of the required properties. $\qquad\square$

Choose elements $\alpha_1, \ldots, \alpha_5 \in \mathbb{F}_q^*$ satisfying the properties of Lemma 17. Let

$$\alpha_i + \alpha_j = a_{i,j}^2$$

for $1 \leqslant i < j \leqslant 5$. Since $\alpha_i \neq -\alpha_j$, no $a_{i,j}$'s are zero. For $1 \leqslant i < j < k \leqslant 5$, let $x_{i,j,k}$, $y_{i,j,k}$, and $z_{i,j,k}$ be any elements of $\mathbb{F}_q$ that satisfy

$$x_{i,j,k} + y_{i,j,k} = a_{i,j}, \quad y_{i,j,k} + z_{i,j,k} = a_{j,k}, \quad \text{and} \quad z_{i,j,k} + x_{i,j,k} = a_{i,k}.$$

Then the vertices $(x_{i,j,k}, \alpha_i)$, $(y_{i,j,k}, \alpha_j)$, and $(z_{i,j,k}, \alpha_k)$ form a triangle in G_q and this holds for any $1 \leqslant i < j < k \leqslant 5$.

Now we use these triangles, which are in G_q, together with the new vertices $z_1, \ldots, z_q, y$ that are added to G_q to from $H_q \cong ER_q$ to obtain a subgraph with chromatic number at least four. For $1 \leqslant i \leqslant 5$, the vertex z_{α_i} is adjacent to all vertices of the form (x, α_i). The vertex y is adjacent to each z_i. Consider the subgraph H_q whose vertices are $y, z_{\alpha_1}, \ldots, z_{\alpha_5}$ together with all $(x_{i,j,k}, \alpha_i)$, $(y_{i,j,k}, \alpha_j)$, and $(z_{i,j,k}, \alpha_k)$ for $1 \leqslant i < j < k \leqslant 5$. Suppose we have a proper 3-coloring of this subgraph, say with colors 1, 2, and 3. If the color 1 is given to three distinct vertices z_{α_i}'s, say $z_{\alpha_i}, z_{\alpha_j}$, and z_{α_k}, then only colors 2 and 3 may be used on the triangle whose vertices are $(x_{i,j,k}, \alpha_i)$, $(y_{i,j,k}, \alpha_j)$, and $(z_{i,j,k}, \alpha_k)$. Therefore, all three colors must be used to color the vertices in the set $\{z_{\alpha_1}, \ldots, z_{\alpha_5}\}$ but then no color may be used on y. The number of vertices in this subgraph is at most $1 + 5 + \binom{5}{3}3 = 36$.

5   Concluding Remarks

The upper bounds of Theorems 12 and 2 can be improved for large q by applying the result of Alon et. al. [2] to the graph $G_{q^t}[Z]$ instead of using Brooks' Theorem. We have chosen to use Brooks' Theorem as then there is no issue of how large the implicit constant is, and because we believe that the upper bound should be closer to $O(q^{t/2})$.

Using a similar argument as the one used to prove Theorem 2, we can prove the following.

Theorem 18. *If q is a power of an odd prime, then*

$$\chi(G_{q^3}) \leqslant 6q^2.$$

A consequence is that $\chi(ER_{q^3}) \leqslant 6q^2 + 1$ whenever q is a power of an odd prime. Unfortunately, we were not able to extend this bound to the general case. In the q^3 case, one can explicitly compute the relations satisfied by vertices in X (see Section 3.2) and then use these equations to bound the maximum degree of $G_{q^3}[X]$. Dealing with the set X is one of the main obstacles in our approach.

We remark that if one could improve the bound in Lemma 13, it would improve our result in Theorem 2. It seems likely that Lemma 13 could be strengthened enough to improve the bound in Theorem 2 all the way down to

$$\chi(ER_{q^{2r+1}}) \leqslant (2r + 1 + o(1))q^{r+1}.$$

Perhaps this can be done using techniques from algebraic geometry.

The conditions on q for which there exists an irreducible polynomial $x^t - \mu \in \mathbb{F}_q[x]$ are known (see Theorem 3.75 of [15]). Let q be a power of an odd prime and let $t \geqslant 3$ be an odd integer. Let $\mathrm{ord}(\mu, q)$ be the order of μ in group $\mathbb{F}_q^*$. Then $x^t - \mu \in \mathbb{F}_q[x]$ is irreducible if and only if each prime factor of t divides $\mathrm{ord}(\mu, q)$ but does not divide $\frac{q-1}{\mathrm{ord}(\mu,q)}$. Since $\mathbb{F}_q^*$ is cyclic, for any divisor d of $q - 1$, there is an element $a \in \mathbb{F}_q^*$ with $\mathrm{ord}(a, q) = d$. As long as t divides $q - 1$, we can choose an element $\mu \in \mathbb{F}_q^*$ so that t divides $\mathrm{ord}(\mu, q)$ but t does not divide $\frac{q-1}{\mathrm{ord}(\mu,q)}$. Therefore, if $q \equiv 1 (\mathrm{mod}\ t)$, then Theorem 2 applies and we obtain the upper bound $\chi(ER_{q^t}) \leqslant \frac{2r+5}{3} q^{\frac{4r}{3}+1} + (2r+1)q^{r+1} + 1$ in this case. For a fixed $t \geqslant 3$, Dirichlet's Theorem on primes in arithmetic progressions implies that there are infinitely many primes p such that $p \equiv 1 (\mathrm{mod}\ t)$. Then for any q which is an odd power of such p, we have $q \equiv 1 (\mathrm{mod}\ t)$.

The graph ER_q is an example of an orthogonal polarity graph. Let Π be a finite projective plane of order q with point set $\mathcal{P}$ and line set $\mathcal{L}$. A *polarity* of Π is a bijection $\pi : \mathcal{P} \cup \mathcal{L} \to \mathcal{P} \cup \mathcal{L}$ such that $\phi(\mathcal{P}) = \mathcal{L}$, $\phi(\mathcal{L}) = \mathcal{P}$, ϕ^2 is the identity, and point p is on line l if and only if point $\phi(l)$ is on line $\phi(p)$. A point p for which $p \in \phi(p)$ is called an *absolute point*. The polarity ϕ is called *orthogonal* if it has exactly $q + 1$ absolute points. The corresponding orthogonal polarity graph is the graph with vertex set $\mathcal{P}$, and vertices p_1 and p_2 are adjacent if and only if $p_1 \in \phi(p_2)$. The graph ER_q is an orthogonal polarity graph which can be obtained from the projective plane $PG(2, q)$ and the polarity that sends the point (x, y, z) to the line $[x, y, z]$, and sends the line $[x, y, z]$ to the point (x, y, z) (loops in this graph must be removed to obtain ER_q). There are non-desaurgesian projective planes that have orthogonal polarities and this leads us to the following problem.

Problem 19. Determine if there is an absolute constant C such that the following holds. If G is an orthogonal polarity graph of a projective plane of order q, then

$$\chi(G) \leqslant Cq^{1/2}.$$

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