Intersections of shifted sets

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Abstract

We consider shifts of a set $A \subseteq \mathbb{N}$ by elements from another set $B \subseteq \mathbb{N}$, and prove intersection properties according to the relative asymptotic size of A and B. A consequence of our main theorem is the following: If $A = \{a_n\}$ is such that $a_n = o(n^{k/k-1})$, then the k-recurrence set $R_k(A) = \{x \mid |A \cap (A+x)| \geqslant k\}$ contains the distance sets of some arbitrarily large finite sets.

Keywords: Asymptotic density, Delta-sets, k-Recurrence sets.

1 Introduction

It is a well-know fact that if a set of natural numbers A has positive upper asymptotic density, then its *set of distances*

$$\Delta(A) \;=\; \{a' - a \mid a', a \in A, a' > a\}$$

meets the set of distances $\Delta(X)$ of any infinite set X (see, *e.g.*, [1]). In consequence, $\Delta(A)$ is *syndetic*, that is there exists k such that $\Delta(A) \cap I \neq \emptyset$ for every interval I of length k. It is a relevant theme of research in combinatorial number theory to investigate properties of distance sets according to their "asymptotic size" (see, *e.g.*, [7, 8, 4, 2].)

The sets of distances are generalized by the *k-recurrence sets*, namely the sets of those numbers that are the common distance of at least k-many pairs:

$$R_k(A) \;=\; \{x \mid |A \cap (A+x)| \geqslant k\}\,.$$

Trivially, $R_{k+1}(A) \subseteq R_k(A)$; notice also that $R_1(A) = \Delta(A)$. We now further generalize this notion.

Let $[A]^h = \{Z \subseteq A \mid |Z| = h\}$ denote the family of all finite subsets of A of cardinality h, namely the *h-tuples* of A.

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Definition 1. For $k, h \in \mathbb{N}$ with $h > 1$, the (h, k)-*recurrence set* of A is the following set of h-tuples:

$$R_k^h(A) = \left\{ \{t_1 < \ldots < t_h\} \in [\mathbb{N}]^h \,\middle|\, |(A + t_1) \cap \ldots \cap (A + t_h)| \geqslant k \right\}.$$

Also in this case, we trivially have the inclusions $R_{k+1}^h(A) \subseteq R_k^h(A)$. Notice that for $j < h$, any j-tuple included in some h-tuple of $R_k^h(A)$, belongs to $R_k^j(A)$. Notice also that a pair $\{t < t'\} \in R_k^2(A)$ if and only if the distance $t' - t \in R_k(A)$, because trivially $|(A + t) \cap (A + t')| = |A \cap (A + (t' - t))|$. More generally, one has the property:

Proposition 2. *If $Z \in R_k^h(A)$ then its set of distances $\Delta(Z) \subseteq R_k(A)$.*

Proof. Let $z < z'$ be elements of Z. Then $\{z < z'\} \subseteq Z \in R_k^h(A)$, and hence $\{z < z'\} \in R_k^2(A)$, which is equivalent to $z' - z \in R_k(A)$. $\square$

We remark that the implication in the above proposition cannot be reversed when $h > 2$. *E.g.*, if $A = \{1, 2, 3, 5, 8\}$ and $F = \{1, 2, 4\}$ then $|A \cap (A + 1)| = |A \cap (A + 2)| = |A \cap (A + 3)| = 2$, and so $\Delta(F) = \{1, 2, 3\} \subseteq R_2(A)$. However $F \notin R_2^3(A)$ because $(A + 1) \cap (A + 2) \cap (A + 4) = \emptyset$.

For sets of natural numbers, we write $A = \{a_n\}$ to mean that elements a_n of A are arranged in increasing order. We adopt the usual "little-O" notation, and for functions $f : \mathbb{N} \to \mathbb{R}$, we write $a_n = o(f(n))$ to mean that $\lim_{n\to\infty} a_n/f(n) = 0$.

Our main result is the following.

- **Theorem 4.** *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers, and let:*[1]
$$\liminf_{n,m\to\infty} \frac{a_n + b_m}{n\sqrt[k]{m}} \;=\; l\,.$$
If $l < \frac{1}{\sqrt[k]{h-1}}$ then $R_k^h(A) \cap [B]^h \neq \emptyset$; and if $l = 0$ then $R_k^h(A) \cap [B]^h$ is infinite for all h.

(Notice that when $k = 1$, for every infinite set A one has $R_1^h(A) \neq \emptyset$ for all h). As a consequence of the theorem above, the following intersection property is obtained.

- **Theorem 9.** *Let $k \geqslant 2$. If the infinite set $A = \{a_n\}$ is such that $a_n = o(n^{k/k-1})$ then $R_k(A)$ is a "finitely Delta-set", that is there exist arbitrarily large finite sets Z such that $\Delta(Z) \subseteq R_k(A)$.*

[1] By *limit inferior* of a double sequence $\langle c_{nm} \mid (n, m) \in \mathbb{N} \times \mathbb{N}\rangle$ we mean

$$\liminf_{n,m\to\infty} c_{nm} = \lim_{k\to\infty}\left(\inf_{n,m\geqslant k} c_{nm}\right).$$

(When $k = 1$, $R_1(A) = \Delta(A)$ is trivially a "finitely Delta-set".)

All proofs contained in this paper have been first obtained by working with the *hyperintegers* of nonstandard analysis. (Nonstandard integers seem to provide a convenient framework to investigate combinatorial properties of numbers which depend on density; see, *e.g.*, [5, 6, 3].) However, all arguments used in our original proof could be translated in terms of limits of subsequences in an (almost) straightforward manner, with the only inconvenience being a heavier notation. So, we eventually decided to keep to the usual language of elementary combinatorics.

2 The main theorem

The following finite combinatorial property will be instrumental for the proof of our main result.

Lemma 3. *Let $A = \{a_1 < \ldots < a_n\}$ and $B = \{b_1 < \ldots < b_m\}$ be finite sets of natural numbers. For every $k \leqslant n$ there exists a subset $Z \subseteq B$ such that*

1. $|\bigcap_{z\in Z}(A+z)| \geqslant k$.
2. $|Z| \geqslant L \cdot \left(\frac{n\sqrt[k]{m}}{a_n+b_m}\right)^k$ *where* $L = \prod_{i=1}^{k-1} \frac{1-\frac{i}{n}}{1-\frac{i}{a_n+b_m}}$.

Proof. For every $i \leqslant m$, let $A_i = A + b_i$ be the shift of A by b_i. Notice that $|A_i| = |A| = n$ and $A_i \subseteq I = [1, a_n + b_m]$ for all i. For $H \in [\mathbb{N}]^k$, denote by $f(H) = |\{i \mid H \subseteq [A_i]^k\}|$. Then:

$$\sum_{H\in[I]^k} f(H) \;=\; \sum_{i=1}^{m} |[A_i]^k| \;=\; \sum_{i=1}^{m} \binom{n}{k} \;=\; m \cdot \binom{n}{k}.$$

Since $|[I]^k| = \binom{a_n+b_m}{k}$, by the *pigeonhole principle* there exists $H_0 \in [I]^k$ such that

$$\begin{aligned} f(H_0) \;&\geqslant\; \frac{m\cdot\binom{n}{k}}{\binom{a_n+b_m}{k}} \;=\; m \cdot \frac{n(n-1)(n-2)\cdots(n-(k-1))}{(a_n+b_m)(a_n+b_m-1)\cdots(a_n+b_m-(k-1))} \\ &=\; m \cdot L \cdot \left(\frac{n}{a_n+b_m}\right)^k \;=\; L \cdot \left(\frac{n\sqrt[k]{m}}{a_n+b_m}\right)^k, \end{aligned}$$

where L is the number defined in the statement of this lemma. Now consider the set $\Gamma = \{i \in [1, m] \mid H_0 \in [A_i]^k\}$. We have that

$$|\Gamma| \;=\; f(H_0) \;\geqslant\; L \cdot \left(\frac{n\sqrt[k]{m}}{a_n+b_m}\right)^k.$$

Now, $H_0 = \{h_1 < \ldots < h_k\} \in \bigcap_{i\in\Gamma}[A_i]^k \Rightarrow |\bigcap_{i\in\Gamma} A_i| \geqslant k$, and the set $Z = \{b_i \mid i \in \Gamma\}$ satisfies the thesis. $\square$

We are finally ready to prove our main theorem.

Theorem 4. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers, and let*

$$\liminf_{n,m\to\infty} \frac{a_n + b_m}{n\sqrt[k]{m}} \;=\; l\,.$$

If $l < \frac{1}{\sqrt[k]{h-1}}$ then $R^h_k(A) \cap [B]^h \neq \emptyset$; and if $l = 0$ then $R^h_k(A) \cap [B]^h$ is infinite for all h.

Proof. Pick increasing functions $\sigma, \tau : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{a_{\sigma(n)} + b_{\tau(n)}}{\sigma(n)\sqrt[k]{\tau(n)}} \;=\; l\,.$$

For every $n \geqslant k$, apply Lemma 3 to the finite sets $A_n = \{a_1 < \ldots < a_{\sigma(n)}\}$ and $B_n = \{b_1 < \ldots < b_{\tau(n)}\}$, and get the existence of a subset $Z_n \subseteq B_n$ such that

1. $\left|\bigcap_{z\in Z_n}(A_n + z)\right| \geqslant k$.
2. $|Z_n| \geqslant L_n \cdot \left(\frac{\sigma(n)\sqrt[k]{\tau(n)}}{a_{\sigma(n)}+b_{\tau(n)}}\right)^k$ where $L_n = \prod_{i=1}^{k-1} \frac{1-\frac{i}{\sigma(n)}}{1-\frac{i}{a_{\sigma(n)}+b_{\tau(n)}}}$.

Since $\lim_{n\to\infty} L_n = 1$, we have that

$$\liminf_{n\geqslant k} |Z_n| \;\geqslant\; \lim_{n\to\infty} L_n \cdot \left(\frac{\sigma(n)\sqrt[k]{\tau(n)}}{a_{\sigma(n)} + b_{\tau(n)}}\right)^k \;=\; 1 \cdot \left(\frac{1}{l}\right)^k \;>\; h-1.$$

Pick an index $t \geqslant k$ with $|Z_t| > h-1$, and pick $\{z_1 < \ldots < z_h\} \subseteq Z_t$. Then:

$$\left|\bigcap_{i=1}^{h}(A+z_i)\right| \;\geqslant\; \left|\bigcap_{i=1}^{h}(A_t+z_i)\right| \;\geqslant\; \left|\bigcap_{z\in Z_t}(A_t+z)\right| \;\geqslant\; k.$$

As $Z_t \subset B$, we conclude that $\{z_1 < \ldots < z_h\} \in R^h_k(A) \cap [B]^h$.

Now let us turn to the case $l = 0$. Given $s > 1$, pick $j \leqslant s$ such that the set $T_j = \{\tau(n) \mid \tau(n) \equiv j \mod s\}$ is infinite, let $\xi, \zeta : \mathbb{N} \to \mathbb{N}$ be the increasing functions such that $T_j = \{\tau(\xi(n))\} = \{s \cdot \zeta(n) + j\}$, and let $B = \{b'_n\}$ be the set where $b'_n = b_{sn+j}$. Then for every $h > 1$:

$$\liminf_{n,m\to\infty} \frac{a_n + b'_m}{n \cdot \sqrt[k]{m}} \;\leqslant\; \lim_{n\to\infty} \frac{a_{\sigma(\xi(n))} + b'_{\zeta(n)}}{\sigma(\xi(n)) \cdot \sqrt[k]{\zeta(n)}} \;=\;$$

$$\lim_{n\to\infty} \frac{a_{\sigma(\xi(n))} + b_{\tau(\xi(n))}}{\sigma(\xi(n)) \cdot \sqrt[k]{\tau(\xi(n))}} \cdot \sqrt[k]{\frac{s\cdot\zeta(n)+j}{\zeta(n)}} \;=\; l \cdot \sqrt[k]{s} \;=\; 0 \;<\; \frac{1}{\sqrt[k]{h-1}}\,.$$

By what already proved above, we get the existence of an h-tuple

$$Z \;=\; \{z_1 < z_2 < \ldots < z_h\} \;\subseteq\; B'$$

such that $|\bigcap_{i=1}^h (A+z_i)| \geqslant k$. It is clear from the definition of B' that $\max Z \geqslant b'_h \geqslant sh + j > s$. Since s can be taken arbitrarily large, we conclude that $R^h_k(A) \cap [B]^h$ is infinite, as desired. □

Corollary 5. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers. If there exists a function $f : \mathbb{N} \to \mathbb{R}^+$ such that*

$$\limsup_{n\to\infty} \frac{a_n}{n \cdot f(n)} < \infty \quad \textit{and} \quad \lim_{n\to\infty} \frac{f(b_n)}{\sqrt[k]{n}} = 0\,,$$

then $R_k^h(A) \cap [B]^h$ is infinite for all h.

Proof. It directly follows from Theorem 4, since

$$\begin{aligned}\liminf_{n,m\to\infty} \frac{a_n + b_m}{n\sqrt[k]{m}} &\leqslant \liminf_{m\to\infty} \frac{a_{b_m} + b_m}{b_m \cdot \sqrt[k]{m}} = \liminf_{m\to\infty} \frac{a_{b_m}}{b_m \cdot \sqrt[k]{m}} \\ &\leqslant \limsup_{m\to\infty} \frac{a_{b_m}}{b_m \cdot f(b_m)} \cdot \liminf_{m\to\infty} \frac{f(b_m)}{\sqrt[k]{m}} = 0\,.\end{aligned}$$

□

An an example, we now see a property that also applies to all zero density sets having at least the same "asymptotic size" as the prime numbers.

Corollary 6. *Assume that the sets $A = \{a_n\}$ and $B = \{b_n\}$ satisfy the conditions $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$ and $\log b_n = o(n^\varepsilon)$ for all $\varepsilon > 0$. Then for every h and k, there exist infinitely many h-tuples $\{\beta_1 < \ldots < \beta_h\} \subset B$ such that each distance $\beta_j - \beta_i$ equals the distance of k-many pairs of elements of A.*

Proof. By the hypothesis $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$ it follows that $a_n = o(n \log^2 n)$, and so the previous corollary applies with $f(n) = \log^2 n$. Clearly, every h-tuple $\{\beta_1 < \ldots < \beta_h\} \in R_k^h(A) \cap [B]^h$ satisfies the desired property. □

3 Finitely Δ-sets

Recall that a set $A \subseteq \mathbb{N}$ is called a *Delta-set* (or Δ-*set* for short) if $\Delta(X) \subseteq A$ for some infinite X. A basic result is the following: "If A has positive upper asymptotic density, then $\Delta(A) \cap \Delta(X) \neq \emptyset$ for all infinite sets X." (See, *e.g.*, [1].) Another relevant property is that Δ-sets are *partition regular*, *i.e.* the family $\mathcal{F}$ of Δ-sets satisfies the following property:

- If a set $A = A_1 \cup \ldots \cup A_r$ of $\mathcal{F}$ is partitioned into finitely many pieces, then at least one of the pieces A_i belongs to $\mathcal{F}$.

To see this, let an infinite set of distances $\Delta(X) = C_1 \cup \ldots \cup C_r$ be finitely partitioned, and consider the partition of the pairs $[X]^2 = D_1 \cup \ldots \cup D_r$ where $\{x < x'\} \in D_i \Leftrightarrow x' - x \in C_i$. By the infinite Ramsey Theorem, there exists an infinite $Y \subseteq X$ and an index i such that $[Y]^2 \subseteq D_i$, which means $\Delta(Y) \subseteq C_i$.

A convenient generalization of Δ-sets is the following.

Definition 7. A is a *finitely* Δ*-set* (or Δ_f*-set* for short) if it contains the distances of finite sets of arbitrarily large size, *i.e.*, if for every k there exists $|X| = k$ such that $\Delta(X) \subseteq A$.

Trivially every Δ-set is a Δ_f-set, but not conversely. For example, take any sequence $\{a_n\}$ such that $a_{n+1} > a_n \cdot n$, let $A_n = \{a_n \cdot i \mid i = 1, \dots, n\}$, and consider the set $A = \bigcup_{n\in\mathbb{N}} A_n$. Notice that for every n, one has $\Delta(A_n) \subseteq A_n$, and hence A is a Δ_f-set. However A is not a Δ-set. Indeed, assume by contradiction that $\Delta(X) \subseteq A$ for some infinite $X = \{x_1 < x_2 < \dots\}$; then $x_2 - x_1 = a_k \cdot i$ for some k and some $1 \leqslant i \leqslant k$. Pick a large enough m so that $x_m > x_2 + a_k \cdot k$. Then $x_m - x_1, x_m - x_2 \in \bigcup_{n>k} A_n$, and so $x_2 - x_1 = (x_m - x_1) - (x_m - x_2) \geqslant a_{k+1} > a_k \cdot k \geqslant x_2 - x_1$, a contradiction. We remark that there exist "large" sets that are not Δ_f-sets. For instance, consider the set O of odd numbers; it is readily seen that $\Delta(Z) \not\subseteq O$ whenever $|Z| \geqslant 3$.

The following property suggests the notion of Δ_f-set as combinatorially suitable.

Proposition 8. *The family of* Δ_f*-sets is partition regular.*

Proof. Let A be a Δ_f-set, and let $A = C_1 \cup \ldots \cup C_r$ be a finite partition. Given k, by the finite Ramsey theorem we can pick n large enough so that every r-partition of the pairs $[\{1, \dots, n\}]^2$ admits a homogeneous set of size k. Now pick a set $X = \{x_1 < \ldots < x_n\}$ with n-many elements such that $\Delta(X) \subseteq A$, and consider the partition $[\{1, \dots, n\}]^2 = D_1 \cup \ldots \cup D_r$ where $\{i < j\} \in D_t \Leftrightarrow x_j - x_i \in C_t$. Then there exists an index t_k and a set $H = \{h_1 < \ldots < h_k\}$ of cardinality k such that $[H]^2 \subseteq D_{t_k}$. This means that the set $Y = \{x_{h_1} < \ldots < x_{h_k}\}$ is such that $\Delta(Y) \subseteq C_{t_k}$. Since there are only finitely many pieces $C_1, \dots, C_r$, there exists t such that $t_k = t$ for infinitely many k. In consequence, C_t is a Δ_f-set. $\square$

As a straight consequence of Theorem 4, we can give a simple sufficient condition on the "asymptotic size" of a set A that guarantees the corresponding k-recurrence sets be finitely Δ-sets.

Theorem 9. *Let* $k \geqslant 2$ *and let the infinite set* $A = \{a_n\}$ *be such that* $a_n = o(n^{k/k-1})$. *Then* $R_k(A)$ *is a* Δ_f*-set.*

Proof. Let $B = \mathbb{N}$, so $b_m = m$. By taking $m = a_n$, we obtain that

$$\liminf_{n,m\to\infty} \frac{a_n + m}{n\sqrt[k]{m}} \leqslant \lim_{n\to\infty} \frac{a_n + a_n}{n\sqrt[k]{a_n}} = \lim_{n\to\infty} \left(2^{\frac{k}{k-1}} \cdot \frac{a_n}{n^{\frac{k}{k-1}}}\right)^{\frac{k-1}{k}} = 0\,.$$

Then Theorem 4 applies, and for every h we obtain the existence of a finite set Z of cardinality h such that $Z \in R_k^h(A) \cap [B]^h = R_k^h(A)$. But then, by Proposition 2, $\Delta(Z) \subseteq R_k(A)$. $\square$

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