When Does the Set of (a, b, c)-Core Partitions Have a Unique Maximal Element?

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Abstract

In 2007, Olsson and Stanton gave an explicit form for the largest (a, b)-core partition, for any relatively prime positive integers a and b, and asked whether there exists an (a, b)-core that contains all other (a, b)-cores as subpartitions; this question was answered in the affirmative first by Vandehey and later by Fayers independently. In this paper we investigate a generalization of this question, which was originally posed by Fayers: for what triples of positive integers (a, b, c) does there exist an (a, b, c)-core that contains all other (a, b, c)-cores as subpartitions? We completely answer this question when a, b, and c are pairwise relatively prime; we then use this to generalize the result of Olsson and Stanton.

Keywords: Young diagram; hook length; core partition; numerical semigroup; UM-set; poset-UM

1 Introduction

A *partition* is a finite, nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of positive integers. The sum $\sum_{i=1}^{r} \lambda_i$ is the *size* of λ and is denoted by $|\lambda|$; the integer r is the *length* of λ. A partition $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ is a *subpartition* of λ if $s \leqslant r$ and $\mu_i \leqslant \lambda_i$ for each integer $i \in [1, s]$; in this case, we say that $\mu \subseteq \lambda$.

We may represent λ by a *Young diagram*, which is a collection of r left-justified rows of cells with λ_i cells in row i. The *hook length* of any cell C in the Young diagram is defined to be the number of cells to the right of, below, or equal to C. For instance, Figure 1 shows the Young diagrams and hook lengths of the partitions $(6, 4, 2, 2, 1, 1)$ and $(5, 3, 1, 1)$.

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Let $\beta(\lambda)$ denote the set of hook lengths in the leftmost column of the Young tableaux associated with λ; equivalently, $\beta(\lambda) = (\lambda_1 + r - 1, \lambda_2 + r - 2, \dots, \lambda_r)$. For instance, Figure 1 shows that $\beta(6,4,2,2,1,1) = \{11,8,5,4,2,1\}$ and $\beta(5,3,1,1) = (8,5,2,1)$.

For any set of positive integers $A = \{a_1, a_2, \dots, a_k\}$, a partition is an A-*core* if no cell of its Young diagram has hook length in A. Let the set of A-cores be $C(A)$; Figure 1 shows that $(6,4,2,2,1,1) \in C(3,7)$ and $(5,3,1,1) \in C(3,7,11)$.

Core partitions are known to be related to representations of the symmetric group; for instance, Olsson and Stanton use simultaneous core partitions in [12] to prove the Navarro-Willems conjecture for symmetric groups. Core partitions are also known to be related to the alcove geometry for certain types of Coxeter groups (see [4, 8, 9]). Recently, there has been a growing interest in simultaneous core partitions because of their relationship with numerical semigroups (see [1, 2, 4, 15, 17]).

11	8	5	4	2	1
8	5	2	1		
5	2				
4	1				
2					
1					

8	5	4	2	1
5	2	1		
2				
1				

Figure 1: To the left is the Young diagram of $\kappa_{3,7} = (6,4,2,2,1,1)$ and to the right is the Young diagram of $(5,3,1,1)$; each cell contains its hook length.

During the past decade, combinatorialists have studied properties of $C(A)$ when $|A| = 2$ (see [1, 3, 4, 5, 6, 8, 9, 10, 11, 12, 15, 16]). For instance, Anderson showed that $|C(a,b)| = \binom{a+b}{a}/(a+b)$ if a and b are relatively prime; in particular, there are finitely many (a,b)-cores [3].

This implies that there is an (a,b)-core of maximum size. Auckerman, Kane, and Sze conjectured in [5] that this size is $(a^2-1)(b^2-1)/24$. This was verified in 2007 by Olsson and Stanton, who also found the core of this size explicitly in terms of a and b [12]. Specifically, they established the following result.

Theorem 1.1. *For any relatively prime positive integers a and b, there is a unique (a,b)-core $\kappa_{a,b}$ of maximum size; a positive integer is in $\beta(\kappa_{a,b})$ if and only if it is of the form $ab - ia - jb$ for some positive integers i and j.*

Figure 1 depicts the Young diagram of $\kappa_{3,7} = (6,4,2,2,1,1)$. In their proof of Theorem 1.1, Stanton and Olsson showed that $\kappa_{n,n+1}$ contains every other $(n,n+1)$-core as a subpartition, for each integer $n \geqslant 2$ [12]. They then asked whether $\kappa_{a,b}$ contains all other (a,b)-cores as subpartitions, for every pair of relatively prime positive integers (a,b). Vandehey answered this question in the affirmative in 2009 through the use of abacus diagrams [16]. Recently, Fayers obtained the same result by analyzing actions of the affine symmetric group on the set of a-cores (and on the set of b-cores) [8].

To see an example of Vandehey's theorem, let $A = (3, 5)$. The nonempty partitions in $C(A)$ are $\{(1), (2), (1, 1), (3, 1), (2, 1, 1), (4, 2, 1, 1)\}$, and every element of $C(A)$ is contained in the $(3, 5)$-core $(4, 2, 1, 1)$. However, this containment phenomenon does not necessarily hold when $|A| \geqslant 3$ and $\gcd A = 1$. For instance, if $A = \{3, 4, 5\}$, then the nonempty elements of $C(A)$ are $(1, 1)$ and (2); neither of these is contained in the other.

For any set of positive integers A, we say that $C(A)$ has a *unique maximal element* if there is an A-core κ_A that contains every other A-core as a subpartition; in this case, the set A is said to be *UM*. In [7], Fayers asked the following question.

Question 1.2. *What triples of positive integers (a, b, c) are UM?*

Vandehey's result implies a partial result in this direction. For any set of positive integers A, let $S(A)$ be the numerical semigroup generated by A; equivalently, A consists of all linear combinations of elements in A with nonnegative integer coefficients. Due to the known fact that an (a, b)-core is an $(a + b)$-core (see [2], for instance), Vandehey's result implies that (a, b, c) is UM if a and b are relatively prime and $c \in S(a, b)$. Recently, Yang, Zhong, and Zhou showed that $(2k + 1, 2k + 2, 2k + 3)$ is not UM for any positive integer k [17].

In this paper we give a partial answer to Question 1.2. We call a triple of positive integers (a, b, c) *aprimitive* if either $a \in S(b, c)$, $b \in S(a, c)$, or $c \in S(a, b)$. The following theorem gives a restriction on triples that can be UM.

Theorem 1.3. *Suppose that (a, b, c) is a triple of positive integers such that $\gcd(a, b, c) = 1$; let $p = \gcd(a, b)$, $q = \gcd(a, c)$, $r = \gcd(b, c)$, and d, e, f be integers such that $(a, b, c) = (dpq, epr, fqr)$ as ordered triples. If (a, b, c) is UM, then (d, e, f) is aprimitive.*

Not all triples of the form given by the above theorem are UM; for instance, we will see in Section 2 that $(4, 5, 6)$ is not UM. However, we may use Theorem 1.3 to answer Question 1.2 completely when a, b, and c are pairwise relatively prime.

Corollary 1.4. *If $a < b < c$ are pairwise relatively prime positive integers, then (a, b, c) is UM if and only if $c \in S(a, b)$.*

Proof. As noted previously, Vandehey's theorem implies that (a, b, c) is UM if $c \in S(a, b)$. Setting $p = q = r = 1$ in Theorem 1.3 yields the converse. $\square$

Corollary 1.4 can be viewed as a converse to Vandehey's theorem; it also generalizes the previously mentioned result of Yang, Zhong, and Zhou.

Using Theorem 1.3, we will also be able to express the unique maximal (a, b, c)-core $\kappa_{a,b,c}$ in terms of a, b, and c if the triple (a, b, c) is UM.

Theorem 1.5. *Suppose that $A = (a, b, c)$ is a triple of positive integers that is UM; let $p = \gcd(a, b)$, $q = \gcd(a, c)$, $r = \gcd(b, c)$, and d, e, f be integers such that $(a, b, c) = (dpq, epr, fqr)$ as ordered triples. If $f \in S(d, e)$, then a positive integer is in $\beta(\kappa_A)$ if and only if it is of the form $(de + f)pqr - ia - jb - kc$ for some positive integers i, j, and k.*

Observe that letting $p = q = r = 1$ in Theorem 1.5 yields Theorem 1.1 of Olsson and Stanton, due to Vandehey's theorem.

The proofs of Theorem 1.3 and Theorem 1.5 use a recently developed characterization of simultaneous cores through numerical semigroups, which we will explain further in the next section.

2 Proofs of Theorems 1.3 and 1.5

In this section, we first explain a bijection (originally due to Stanley and Zanello in [15] when $|A| = 2$ and later generalized to arbitrary sets A by Amdeberhan and Leven in [2]) between A-cores and order ideals of some poset $P(A)$. We will then use this bijection to obtain a preliminary necessary condition for a set A to be UM. Using this condition, we will establish Theorem 1.3 and Theorem 1.5.

Now let us define the poset $P(A)$. The elements of $P(A)$ are those of $\mathbb{Z}_{\geqslant 0} \backslash S(A)$, the set of positive integers not contained in the numerical semigroup generated by A. Notice that if $\gcd A = 1$, then $|P(A)| < \infty$; we will suppose that this is the case for the remainder of the section. The order on $P(A)$ is fixed by requiring $p \in P(A)$ to be greater than $q \in P(A)$ if $p - q \in S(A)$. Under this partial order, $P(A)$ is a poset; we will follow the poset terminology given in Chapter 3 of Stanley's text [13, 14]. Figure 2 depicts the Hasse diagrams of the posets $P(3, 7)$ and $P(3, 7, 11)$.

Figure 2: The Hasse diagrams of $P(3, 7)$ and $P(3, 7, 11)$ are shown on the left and right, respectively.

The following lemma is due to Amdeberhan and Leven in [2].

Lemma 2.1. *There is a bijection between $C(A)$ and the set of order ideals of $P(A)$. Specifically, for each partition λ, the set $\beta(\lambda)$ is an order ideal of $P(A)$ if and only if λ is an A-core.*

For instance, suppose that $A \subseteq \{3, 7, 11\}$; then, $(5, 3, 1, 1)$ is an A-core and its beta set $\{8, 5, 2, 1\}$ is an order ideal of $P(A)$. Furthermore, $\kappa_{3,7} = (6, 4, 2, 2, 1, 1)$ is a $(3, 7)$-core and its beta set $\{1, 2, 4, 5, 8, 11\}$ is an order ideal of $P(3, 7)$; however, $\kappa_{3,7}$ is not a $(3, 7, 11)$-core and its beta set is not an order ideal of $P(3, 7, 11)$.

From Lemma 2.1, there is an A-core κ'_A such that $\beta(\kappa'_A) = P(A)$. The following result states that κ'_A is the unique maximal element of $C(A)$ if A is UM.

Corollary 2.2. *If a set of positive integers A is UM, then $\kappa'_A = \kappa_A$.*

Proof. The bijection in Lemma 2.1 is length preserving; since the longest order ideal of $P(A)$ is $P(A)$, the longest A-core is κ'_A. Therefore, κ'_A is not contained in any other A-core, which implies that κ_A is the unique maximal element of $C(A)$ because A is UM. Thus, $\kappa'_A = \kappa_A$. $\square$

Now, we call a poset P *poset-UM* if P contains a unique maximal element. For instance, Figure 2 shows that $P(3, 7)$ is poset-UM with unique maximal element 11 and that $P(3, 7, 11)$ is not poset-UM since it has both 4 and 8 as maximal elements. The following known lemma gives examples of posets that are poset-UM.

Lemma 2.3. *Suppose that a and b are relatively prime positive integers; then, $P(a, b)$ is poset-UM with maximal element $ab - a - b$. Equivalently, a positive integer is in $P(a, b)$ if and only if it is of the form $ab - ja - hb$ for some integers $j \in [1, b - 1]$ and $h \in [1, a - 1]$.*

The following proposition yields a preliminary necessary condition for a set of positive integers to be UM.

Proposition 2.4. *If a set of positive integers A is UM, then $P(A)$ is poset-UM.*

Proof. Suppose that $P(A)$ is not poset-UM but that A is UM. Consider the element $m \in P(A)$ of maximum magnitude; for instance, if $A = (3, 7, 11)$, then $m = 8$. Since $P(A)$ is not poset-UM, there is an order ideal $I \subseteq P(A)$ containing m but not equal to $P(A)$. By Lemma 2.1, there are A-cores $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ and $\kappa = \kappa_A = (\kappa_1, \kappa_2, \ldots, \kappa_s)$ such that $\beta(\lambda) = I$ and $\beta(\kappa) = P(A)$. By Corollary 2.2, κ is the unique maximal element of $C(A)$; therefore, $\lambda \subset \kappa$. Hence, $m-r+1 = \lambda_1 \leqslant \kappa_1 = m-s+1$, which is a contradiction since $s = |P(A)| > |I| = r$. $\square$

As an application, the above proposition implies that $(3, 7, 11)$ is not UM. Observe that the converse of Proposition 2.4 does not always hold. For instance, suppose that $A = \{4, 5, 6\}$; then, $P(A) = \{1, 2, 3, 7\}$ is poset-UM with unique maximal element 7. Therefore the longest A-core is $\kappa_A = (3, 1, 1, 1)$, which does not contain the A-core $(2, 2)$ as a subpartition.

We will now classify all triples of positive integers (a, b, c) whose associated posets $P(a, b, c)$ are poset-UM. The following proposition forms a bijection between the maximal elements of $P(a, b, c)$ and the maximal elements of $P(d, e, f)$ for particular triples (a, b, c) and (d, e, f).

Proposition 2.5. *Suppose that (a, b, c) is a triple of positive integers with $\gcd(a, b, c) = 1$; let $\gcd(a, b) = p$, $\gcd(a, c) = q$, $\gcd(b, c) = r$, and d, e, f be integers such that $(a, b, c) = (dpq, epr, fqr)$ as ordered triples. For m and n positive integers, $(m - 1)d + (n - 1)e - f$ is a maximal element of $P(d, e, f)$ if and only if $(mr - 1)dpq + (nq - 1)epr - fqr$ is a maximal element of $P(a, b, c)$.*

Proof. Let $s = (m-1)d+(n-1)e-f$ and $t = (mr-1)dpq+(nq-1)epr-fqr$. Suppose that s is a maximal element of $P(d, e, f)$; we will show that t is a maximal element of $P(a, b, c)$. Let us first verify that $t \in P(a, b, c)$. Suppose to the contrary that $t \in S(a, b, c)$. Then, there are nonnegative integers h, i, j such that $(mr-1)dpq + (nq-1)epr - fqr = hdpq + iepr + jfqr$. Since $\gcd(a,b,c) = 1$, we have that $\gcd(r,p) = \gcd(r,q) = 1 = \gcd(r,d)$. The previous equality implies that $(mr-1-h)dpq$ is a multiple of r, which yields $h = rh' - 1$ for some positive integer h'. Similarly, $i = qi' - 1$ for some positive integer i'; therefore, $(m-h')dp + (n-i')ep = (j+1)f$. Thus, $j = j'p - 1$ for some positive integer j'; hence $s = (m-1)d + (n-1)e - f = (h'-1)d + (i'-1)e + (j'-1)f \in S(d,e,f)$, which is a contradiction.

Now, to see that t is maximal, it suffices to verify that $t+a, t+b, t+c \in S(a,b,c)$. Observe that $t + a = mdpqr + (nq-1)epr - fqr = pqr(s+d) + epr(q-1) + fqr(p-1) \in S(dpq, epr, fqr)$ because $s + d \in S(d,e,f)$ by the maximality of s. By similar reasoning, $t + b \in S(a,b,c)$; since $t + c = (mr-1)dpq + (nq-1)epr \in S(a,b,c)$, it follows that t is a maximal element of $P(a,b,c)$.

This implies that t is a maximal element of $P(a,b,c)$ if s is a maximal element of $P(d,e,f)$. Through similar reasoning, one may show that s is a maximal element of $P(d,e,f)$ if t is a maximal element of $P(a,b,c)$. □

The following corollary reduces the classification of triples (a,b,c) whose associated posets $P(a,b,c)$ are poset-UM to the case when a, b, and c are pairwise relatively prime.

Corollary 2.6. *Suppose that (a,b,c) is a triple of positive integers such that $\gcd(a,b,c) = 1$; let $\gcd(a,b) = p$, $\gcd(a,c) = q$, $\gcd(b,c) = r$, and d, e, f be integers such that $(a,b,c) = (dpq, epr, fqr)$ as ordered triples. Then, $P(a,b,c)$ is poset-UM if and only if $P(d,e,f)$ is poset-UM.*

Proof. Suppose that $P(d,e,f)$ is not poset-UM and let s_1 and s_2 be two distinct maximal elements of $P(d,e,f)$. Since s_1 and s_2 are maximal, we have that $s_1 + f, s_2 + f \in S(d,e)$; thus, there are positive integers m_1, n_1, m_2, n_2 such that $s_1 = (m_1-1)d + (n_1-1)e - f$ and $s_2 = (m_2-1)d + (n_2-1)e - f$. Let $t_1 = (rm_1-1)dpq + (qn_1-1)epr - fqr$ and $t_2 = (rm_2-1)dpq + (qn_2-1)epr - fqr$; by Proposition 2.5, t_1 and t_2 are maximal elements of $P(a,b,c)$. Since s_1 and s_2 are distinct, $t_1 \neq t_2$; thus $P(a,b,c)$ has two distinct maximal elements and is therefore not poset-UM.

Now, suppose that $P(a,b,c)$ is not poset-UM and let t_1 and t_2 be two distinct maximal elements of $P(a,b,c)$. As above, there are positive integers m_1', n_1', m_2', n_2' such that $t_1 = (m_1'-1)dpq + (n_1'-1)epr - fqr$ and $t_2 = (m_2'-1)dpq + (n_2'-1)epr - fqr$. We claim that m_1' is a multiple of r. Indeed, since t_1 is maximal, $m_1'dpq + (n_1'-1)epr - fqr = t_1 + a \in S(a,b,c)$; therefore, there are nonnegative integers h, i, j such that $m_1'dpq + (n_1'-1)epr - fqr = hdpq + iepr + jfqr$. Since $t_1 \notin S(a,b,c)$, we have that $h = 0$. Therefore, r divides $m_1'dpq$; the fact that $\gcd(d,r) = \gcd(p,r) = \gcd(q,r) = 1$ thus yields r divides m_1'. Hence, there is an integer m_1 such that $m_1' = m_1 r$; by similar reasoning, there are integers n_1, m_2, n_2 such that $n_1' = n_1 q$, $m_2' = m_2 r$, and $n_2' = n_2 q$. By Proposition 2.5, $s_1 = (m_1-1)d + (n_1-1)e - f$ and $s_2 = (m_2-1)d + (n_2-1)e - f$ are unique maximal elements of $P(d,e,f)$. Since t_1 and t_2 are distinct, $s_1 \neq s_2$; this implies that $P(d,e,f)$ has two maximal elements and is therefore not poset-UM. □

The following proposition, which we will prove in the next section, classifies all triples of pairwise relatively prime positive integers (a,b,c) such that $P(a,b,c)$ is poset-UM.

Proposition 2.7. *If $a < b < c$ are pairwise relatively prime positive integers, then $P(a,b,c)$ is poset-UM if and only if $c \in S(a,b)$.*

Assuming Proposition 2.7, we may classify all triples of positive integers (a,b,c) for which $P(a,b,c)$ is poset-UM.

Corollary 2.8. *Suppose that (a,b,c) is a triple of positive integers such that $\gcd(a,b,c) = 1$; let $p = \gcd(a,b)$, $q = \gcd(a,c)$, $r = \gcd(b,c)$, and d,e,f be integers such that $(a,b,c) = (dpq, epr, fqr)$ as ordered triples. Then, $P(a,b,c)$ is poset-UM if and only if (d,e,f) is aprimitive.*

Proof. This follows from Corollary 2.6 and Proposition 2.7. □

We may now establish Theorem 1.3.

Proof of Theorem 1.3. This follows from Proposition 2.4 and Corollary 2.8. □

Using Theorem 1.3, we may now establish Theorem 1.5.

Proof of Theorem 1.5. Let $f = md + ne$ for some nonnegative integers m and n. By Lemma 2.3, the unique maximal element of $P(d,e,f)$ is $de - d - e = (d+n-1)e + (m-1)d - f$. By Proposition 2.4, $P(A)$ has a unique maximal element; by Proposition 2.5, the unique maximal element of $P(A)$ is $(de+md+ne)pqr - dpq - epr - fqr = (de+f)pqr - a - b - c$. Hence, a positive integer is in $P(A)$ if and only if it is of the form $(de+f)pqr - ia - jb - kc$ for some positive integers i, j, and k. This implies the corollary since $\beta(\kappa_A) = P(A)$. □

3 Proof of Proposition 2.7

Throughout this section, we will adopt the notation given in Proposition 2.7. If $c \in S(a,b)$, then $P(a,b,c) = P(a,b)$ because $S(a,b,c) = S(a,b)$; therefore, the proposition follows from Lemma 2.3.

So, let us suppose that $c \notin S(a,b)$. By Lemma 2.3, there are positive integers $s_1 \in [1, b-1]$ and $t_1 \in [1, a-1]$ such that $c = ab - s_1 a - t_1 b$. Let k be the largest positive integer such that $ic \notin S(a,b)$ for each integer $i \in [1,k]$; observe that $k < b$.

By Lemma 2.3, there are integers $s_1, s_2, \ldots, s_k \in [1, b-1]$ and $t_1, t_2, \ldots, t_k \in [1, a-1]$ such that $ic = ab - s_i a - t_i b$ for each integer $i \in [1,k]$; moreover, let $(s_0, t_0) = (0, a)$ and $(s_{k+1}, t_{k+1}) = (b, 0)$. Observe that the s_i are distinct, since we would otherwise have that $(j-i)c = (t_i - t_j)b$ for some integers $1 \leqslant i < j \leqslant k$, which contradicts the facts that $k < b$ and $\gcd(b,c) = 1$; similarly, we see that the t_i are distinct.

By establishing properties about the s_i and t_i, we will show that $P(a,b,c)$ has at least two distinct maximal elements, which would imply that $P(a,b,c)$ is not UM. Let us begin with the following property.

Lemma 3.1. *For all integers $i, j \in [0, k+1]$, we have that $t_i < t_j$ if and only if $s_i > s_j$.*

Proof. Suppose to the contrary that there exist integers $i, j \in [0, k+1]$ such that $t_i < t_j$ and $s_i < s_j$. Then, $ic = ab - s_i a - t_i b > ab - s_j a - t_j b = jc$, which implies that $i > j$. Furthermore, $i \neq k+1$ since $s_{k+1} = b \geqslant s_h$ for any integer $h \in [0, k+1]$. Therefore, $i - j \in [1, k]$, so $(i-j)c \notin S(a,b)$ by the definition of k. However, $(i-j)c = (s_j - s_i)a + (t_j - t_i)b \in S(a,b)$, which is a contradiction. Therefore, the proposition holds. ☐

Now, let $m, n \in [1, k]$ be integers such that $s_m = \min_{i \in [1,k]} s_i$ and $t_n = \min_{i \in [1,k]} t_i$. By Lemma 3.1, we have that $s_n = \max_{i \in [1,k]} s_i$ and $t_m = \max_{i \in [1,k]} t_i$.

Lemma 3.2. *We have that $k = m + n - 1$.*

Proof. We will first show that $k < m + n$. Suppose otherwise, so in particular $2ab - (s_m + s_n)a - (t_m + t_n)b = (m+n)c \notin S(a,b)$. Thus $2ab - (s_m + s_n)a - (t_m + t_n)b = ab - s_{m+n}a - t_{m+n}b$, so $ab = (s_m + s_n - s_{m+n})a + (t_m + t_n - t_{m+n})b$. Since $|s_m + s_n - s_{m+n}| < 2b$, $|t_m + t_n - t_{m+n}| < 2a$, and $\gcd(a,b) = 1$, this implies that either $s_m + s_n - s_{m+n} = b$ and $t_m + t_n = t_{m+n}$ or $s_m + s_n = s_{m+n}$ and $t_m + t_n - t_{m+n} = a$. Without loss of generality, suppose that the former holds; then $s_{m+n} = s_m + s_n - b < s_m$, which contradicts the minimality of s_m. Therefore, $k \leqslant m + n - 1$.

To see that $k \geqslant m+n-1$, observe that $(m+n-i)c = ab - (s_m + s_n - s_i)a - (t_m + t_n - t_i)b$ for each integer $i \in [1, k]$. Since $0 < s_m \leqslant s_i \leqslant s_n$ and $0 < t_n \leqslant t_i \leqslant t_m$, we obtain that $(m+n-i)c$ is of the form $ab - ah - bj$ for some positive integers h and j. Thus Lemma 2.3 implies that $ic \notin S(a,b)$ for each integer $i \in [1, m+n-1]$, which yields $k \geqslant m+n-1$. Hence, $k = m + n - 1$. ☐

Now let $\{r_0, r_1, \ldots, r_{k+1}\}$ be a permutation of $\{0, 1, 2, \ldots, k, k+1\}$ such that $s_{r_0} < s_{r_1} < \cdots < s_{r_{k+1}}$. By Lemma 3.1, we have that $t_{r_k} < t_{r_{k-1}} < \cdots < t_{r_1}$. Furthermore, observe that $r_0 = 0$; $r_{k+1} = k+1$; $r_1 = m$; and $r_k = n$.

Let $p_i = ab - (s_{r_i} + 1)a - (t_{r_{i+1}} + 1)b$ for each integer $i \in [0, k]$. Observe that the p_i are positive since $p_i \geqslant ab - s_{r_i+1}a - t_{r_{i+1}}b - b = r_{i+1}c - b$, which is positive since $b < c$. We will show that p_j and p_{j+1} are both maximal elements of $P(a,b,c)$ for some integer $j \in [0, k]$. However, let us first show the following properties about p_i for any integer $i \in [0, k]$.

Lemma 3.3. *For any integer $i \in [0, k]$, we have that $p_i \in P(a,b,c)$ but $p_i + a, p_i + b \notin P(a,b,c)$.*

Proof. Let us first show that the former claim holds; suppose to the contrary that $p_i \in S(a,b,c)$ for some integer $i \in [0,k]$. Then, there exist integers $f \in [0,b]$, $h \in [0,a]$, and $j \in [0,k]$ such that $ab - (s_{r_i} + 1)a - (t_{r_{i+1}} + 1)b = p_i = fa + hb + jc = ab - (s_j - f)a - (t_j - h)b$. Thus $s_j > s_{r_i}$ and $t_j > t_{r_{i+1}}$. Letting $j = r_h$ for some integer $h \in [0, k+1]$, the former inequality implies that $h > i$ and the latter inequality implies that $h < i + 1$; this is a contradiction, which yields $p_i \in P(a,b,c)$.

Now, for the second claim, observe that $p_i + a = ab - s_{r_i}a - (t_{r_{i+1}} + 1)b = r_i c + (t_{r_i} - t_{r_{i+1}} - 1)b$, which is in $S(a,b,c)$ since $t_{r_i} > t_{r_{i+1}}$. Therefore, $p_i + a \notin P(a,b,c)$. By similar reasoning, $p_i + b \notin P(a,b,c)$. ☐

Let k' be such that $r_{k'} = k$. We will show that p_j and p_{j+1} are distinct maximal elements of $P(a,b,c)$ with $j = k'-1$. First, let us observe the following property about k'.

Lemma 3.4. *We have that $r_{k'-1} = k-m$ and $r_{k'+1} = k-n$.*

Proof. We will only show the former statement since the proof of the latter is similar. Observe that, for any integer $i \in [0,k']$, we have that $ab - s_{k-r_i}a - t_{k-r_i}b = (k-r_i)c = ab - (s_{r_k'} - s_{r_i})a - (a + t_{r_{k'}} - t_{r_i})b$. Since $s_{r_i} < s_{r_{k'}}$, we have that $s_{k-r_i} = s_{r_{k'}} - s_{r_i}$; this is minimal when $i = k'-1$. Therefore, the minimality of s_m implies that $r_{k'-1} = k-m$. □

Now we can establish the following lemma.

Lemma 3.5. *We have that $p_{k'-1}$ and $p_{k'}$ are maximal elements of $P(a,b,c)$.*

Proof. By Lemma 3.3, it suffices to show that $p_{k'-1} + c \notin P(a,b,c)$ and that $p_{k'} + c \notin P(a,b,c)$. We will only show the first statement, since the proof of the second is similar.

Suppose to the contrary that $p_{k'-1} + c \in P(a,b,c) \subset P(a,b)$. Then, Lemma 2.3 implies that there exist positive integers j and h such that $2ab - (s_{r_{k'-1}} + s_1 + 1)a - (t_{r_{k'}} + t_1 + 1)b = p_{k'-1} + c = ab - ja - hb$. Using Lemma 3.4 and the fact that $r_{k'} = k$, we find that $(s_{k-m} + s_1 + 1 - j)a + (t_k + t_1 + 1 - h)b = ab$. Therefore, we have that either $s_{k-m} + s_1 \geqslant b$ or $t_k + t_1 \geqslant a$.

By Lemma 3.2, we have that $k = m+n-1$. Thus, $ab - s_{k-m}a - t_{k-m}b = (k-m)c = (n-1)c = ab - (s_n - s_1)a - (a + t_n - t_1)b$, which implies that $s_{k-m} + s_1 = s_n < b$. Hence, $t_k + t_1 \geqslant a$.

We claim that $t_k + t_1 \leqslant a$ holds as well. Indeed, applying Lemma 3.2 again, we find that $ab - s_k a - t_k b = kc = (m+n-1)c = ab - (s_m + s_n - s_1)a - (t_m + t_n - t_1)b$. Thus, $s_k = s_m + s_n - s_1$ and $t_k = t_m + t_n - t_1$. Moreover, since $ab - (s_m + s_n)a - (t_m + t_n - a)b = (m+n)c \in S(a,b)$, we have that $t_k + t_1 = t_m + t_n \leqslant a$ by Lemma 2.3.

Therefore, $t_m + t_n = t_k + t_1 = a$ and hence $a(b - s_m - s_n) = (m+n)c$. Since a and c are relatively prime, $b - s_m - s_n$ is a positive multiple of c; this contradicts the fact that $b < c$. Thus, $p_{k'-1} + c \notin P(a,b,c)$. □

Using Lemma 3.5, we may establish Proposition 2.7.

Proof of Proposition 2.7. If $c \in S(a,b)$, then $P(a,b,c) = P(a,b)$ because $S(a,b,c) = S(a,b)$; therefore, the proposition follows from Lemma 2.3. If $c \notin S(a,b)$, then $P(a,b,c)$ has two distinct maximal elements by Lemma 3.5, which implies that $P(a,b,c)$ cannot be UM. Therefore, $P(a,b,c)$ is UM if and only if $c \in S(a,b)$. □

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