# Nonexistence of a Class of Distance-regular Graphs

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**Abstract**

Let $\Gamma$ denote a distance-regular graph with diameter $D \geqslant 3$ and intersection numbers $a_1 = 0, a_2 \neq 0$, and $c_2 = 1$. We show a connection between the $d$-bounded property and the nonexistence of parallelograms of any length up to $d+1$. Assume further that $\Gamma$ is with classical parameters $(D, b, \alpha, \beta)$, Pan and Weng (2009) showed that $(b, \alpha, \beta) = (-2, -2, ((-2)^{D+1}-1)/3)$. Under the assumption $D \geqslant 4$, we exclude this class of graphs by an application of the above connection.

**Keywords:** Distance-regular graph; classical parameters; parallelogram; strongly closed subgraph; $D$-bounded

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# 1 Introduction

Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geqslant 3$. A sequence $x$, $z$, $y$ of vertices of $\Gamma$ is *geodetic* whenever

$$\partial(x, z) + \partial(z, y) = \partial(x, y),$$

where $\partial$ is the distance function of $\Gamma$. A sequence $x$, $z$, $y$ of vertices of $\Gamma$ is *weak-geodetic* whenever

$$\partial(x, z) + \partial(z, y) \leqslant \partial(x, y) + 1.$$

We consider subsets of the vertex set of $\Gamma$ that are closed under the sense of weak-geodetic sequences as the following definition.

**Definition 1.** A subset $\Delta \subseteq X$ is *strongly closed* if for any weak-geodetic sequence $x$, $z$, $y$ of $\Gamma$,

$$x, y \in \Delta \Longrightarrow z \in \Delta.$$

A subgraph of $\Gamma$ which is induced by a strongly closed subset of $X$ is called a *strongly closed subgraph* of $\Gamma$. Strongly closed subgraphs are also called *weak-geodetically closed subgraphs* in [14]. If a strongly closed subgraph $\Delta$ of diameter $d$ is regular then it has valency $a_d + c_d = b_0 - b_d$, where $a_d, c_d, b_0, b_d$ are intersection numbers of $\Gamma$. Furthermore $\Delta$ is distance-regular with intersection numbers $a_i(\Delta) = a_i(\Gamma)$ and $c_i(\Delta) = c_i(\Gamma)$ for $1 \leqslant i \leqslant d$ [14, Theorem 4.6].

The following property is considered for a distance-regular graph.

**Definition 2.** $\Gamma$ is said to be *d-bounded* whenever for all $x, y \in X$ with $\partial(x, y) \leqslant d$, there is a regular strongly closed subgraph of diameter $\partial(x, y)$ which contains $x$ and $y$.

Note that a $(D - 1)$-bounded distance-regular graph is clear to be $D$-bounded. The properties of $D$-bounded distance-regular graphs were studied in [13], and these properties were used in the classification of classical distance-regular graphs of negative type [15]. Other applications of $D$-bounded distance-regular graphs are given in [3, 12, 13, 15]. Before stating our main results, we show one more definition and some known results.

**Definition 3.** A 4-tuple $xyzw$ consisting of vertices of $\Gamma$ is called a *parallelogram of length d* if $\partial(x, y) = \partial(z, w) = 1$, $\partial(x, w) = d$, and $\partial(x, z) = \partial(y, w) = \partial(y, z) = d - 1$.

The following theorem is a combination of three previous results.

**Theorem 4.** *Let $\Gamma$ denote a distance-regular graph with diameter $D \geqslant 3$. Suppose that the intersection numbers $a_1, a_2, c_2$ satisfy one of the following.*

*(i) [4, Theorem 2] $a_2 > a_1 = 0$, $c_2 > 1$;*

*(ii) [14, Theorem 1] $a_1 \neq 0$, $c_2 > 1$; or*

*(iii) [9, Theorem 1.1]* $a_2 > a_1 \geqslant c_2 = 1$.

*Fix an integer* $1 \leqslant d \leqslant D-1$ *and suppose that* $\Gamma$ *contains no parallelograms of any length up to* $d+1$. *Then* $\Gamma$ *is* $d$*-bounded.*

We deal with the case "$a_1 = 0, a_2 \neq 0$, and $c_2 = 1$" in the following, which is the key point among our main results.

**Theorem 5.** *Let* $\Gamma = (X, R)$ *denote a distance-regular graph with diameter* $D \geqslant 3$, *and intersection numbers* $a_1 = 0$, $a_2 \neq 0$ *and* $c_2 = 1$. *Fix an integer* $1 \leqslant d \leqslant D-1$ *and suppose that* $\Gamma$ *contains no parallelograms of any length up to* $d+1$. *Then* $\Gamma$ *is* $d$*-bounded.*

The proof of Theorem 5 is given in Section 4. Theorem 5 is a generalization of [2, Lemma 4.3.13] and [7]. Combining Theorem 4 and Theorem 5, we have the (ii) $\Rightarrow$ (i) part of the following theorem.

**Theorem 6.** *Suppose* $\Gamma$ *is a distance-regular graph with diameter* $D \geqslant 3$ *and the intersection number* $a_2 \neq 0$. *Fix an integer* $2 \leqslant d \leqslant D-1$. *Then the following two conditions (i), (ii) are equivalent:*

*(i)* $\Gamma$ *is* $d$*-bounded.*

*(ii)* $\Gamma$ *contains no parallelograms of any length up to* $d+1$ *and* $b_1 > b_2$.

The complete proof of Theorem 6 is given in Section 4. Theorem 6 answers the problem proposed in [14, p. 299]. The following is an application of Theorem 6, which excludes a class of distance-regular graphs mentioned in [8, Theorem 2.2].

**Theorem 7.** *There is no distance-regular graph with classical parameters* $(D, b, \alpha, \beta) = (D, -2, -2, ((-2)^{D+1} - 1)/3)$, *where* $D \geqslant 4$.

We prove Theorem 7 in Section 5. Since Witt graph $M_{23}$ [2, Table 6.1] is a distance-regular graph with classical parameters $(D, b, \alpha, \beta)$ with $D = 3$, $b = -2$, $\alpha = -2$, and $\beta = 5$, the condition $D \geqslant 4$ in Theorem 7 can not be loosened to $D \geqslant 3$.

## 2 Preliminaries

In this section we review some definitions, basic concepts and some previous results concerning distance-regular graphs. See Bannai and Ito [1] or Terwilliger [10] for more background information.

Let $\Gamma$=($X$, $R$) denote a finite undirected, connected graph without loops or multiple edges with vertex set $X$, edge set $R$, distance function $\partial$, and diameter $D$:=max$\{\, \partial(x, y) \mid x, y \in X\}$. By a *pentagon*, we mean a 5-tuple $u_1u_2u_3u_4u_5$ consisting of distinct vertices in $\Gamma$ such that $\partial(u_i, u_{i+1}) = 1$ for $1 \leqslant i \leqslant 4$ and $\partial(u_5, u_1) = 1$.

For a vertex $x \in X$ and an integer $0 \leqslant i \leqslant D$, set $\Gamma_i(x) := \{\, z \in X \mid \partial(x, z) = i\}$. The *valency* $k(x)$ of a vertex $x \in X$ is the cardinality of $\Gamma_1(x)$. The graph $\Gamma$ is called *regular* (with *valency* $k$) if each vertex in $X$ has valency $k$. The graph $\Gamma$ is said to be *distance-regular* whenever for all integers $0 \leqslant h, i, j \leqslant D$, and all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of $x, y$. The constants $p^h_{ij}$ are known as the *intersection numbers* of $\Gamma$.

From now on let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geqslant 3$. For two vertices $x, y \in X$ with $\partial(x, y) = i$, set

$$\begin{aligned} B(x,y) &:= \Gamma_1(x) \cap \Gamma_{i+1}(y), \\ C(x,y) &:= \Gamma_1(x) \cap \Gamma_{i-1}(y), \\ A(x,y) &:= \Gamma_1(x) \cap \Gamma_i(y). \end{aligned}$$

Note that

$$\begin{aligned} |B(x,y)| &= p^i_{1\ i+1}, \\ |C(x,y)| &= p^i_{1\ i-1}, \\ |A(x,y)| &= p^i_{1\ i} \end{aligned}$$

are independent of $x, y$. For convenience, set $c_i := p^i_{1\ i-1}$ for $1 \leqslant i \leqslant D$, $a_i := p^i_{1\ i}$ for $0 \leqslant i \leqslant D$, $b_i := p^i_{1\ i+1}$ for $0 \leqslant i \leqslant D-1$ and put $b_D := 0$, $c_0 := 0$, $k := b_0$. Note that $k$ is the valency of each vertex in $\Gamma$. It is immediate from the definition of $p^h_{ij}$ that $b_i \neq 0$ for $0 \leqslant i \leqslant D-1$ and $c_i \neq 0$ for $1 \leqslant i \leqslant D$. Moreover $c_1 = 1$ and

$$k = a_i + b_i + c_i \quad \text{for} \ \ 0 \leqslant i \leqslant D. \tag{1}$$

A subset $\Omega$ of $X$ is *strongly closed with respect to a vertex* $x \in \Omega$ if for any $z \in X$ with $x, z, y$ being a weak-geodetic sequence for some $y \in \Omega$, we have $z \in \Omega$. Note that $\Omega$ is strongly closed if and only if for any vertex $x \in \Omega$, $\Omega$ is strongly closed with respect to $x$. A subset $\Omega$ of $X$ is strongly closed with respect to a vertex $x \in \Omega$ if and only if [14, Lemma 2.3]

$$C(y, x) \subseteq \Omega \ \ \text{and} \ \ A(y, x) \subseteq \Omega \qquad \text{for all} \ \ y \in \Omega. \tag{2}$$

We quote two more theorems from [14] that will be used later in this paper to end this section.

**Theorem 8.** *([14, Theorem 4.6]) Let $\Gamma$ be a distance-regular graph with diameter $D \geqslant 3$. Let $\Omega$ be a regular subgraph of $\Gamma$ with valency $\gamma$ and set $d := \min\{i \mid \gamma \leqslant c_i + a_i\}$. Then the following (i),(ii) are equivalent.*

*(i) $\Omega$ is strongly closed with respect to at least one vertex $x \in \Omega$.*

*(ii) $\Omega$ is strongly closed with diameter $d$.*

*Suppose (i) or (ii) holds. Then $\Omega$ is a distance-regular subgraph of $\Gamma$ with diameter $d$ and $\gamma = c_d + a_d$.*

**Theorem 9.** *([14, Lemma 6.5]) Let $\Gamma$ be a distance-regular graph with diameter $D \geqslant 2$. Suppose $\Gamma$ is $d$-bounded for some $1 \leqslant d \leqslant D-1$, then $\Gamma$ contains no parallelograms of any length up to $d+1$.*

## 3 The Shape of Pentagons

Throughout this section, let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geqslant 3$, and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Such graphs are also studied in [4, 5, 6, 7, 8].

Fix a vertex $x \in X$, a pentagon $u_1u_2u_3u_4u_5$ has *shape* $i_1, i_2, i_3, i_4, i_5$ *with respect to* $x$ if $i_j = \partial(x, u_j)$ for $1 \leqslant j \leqslant 5$. Note that under the assumption $a_1 = 0$ and $a_2 \neq 0$, any two vertices at distance 2 in $\Gamma$ are always contained in a pentagon, and two nonconsecutive vertices in a pentagon of $\Gamma$ have distance 2. In this section we give a few lemmas which will be used in the next section.

**Lemma 10.** *Fix an integer* $1 \leqslant d \leqslant D - 1$*, and suppose* $\Gamma$ *contains no parallelograms of any length up to* $d + 1$ *for some integer* $d \geqslant 2$*. Let* $x$ *be a vertex in* $\Gamma$*, and let* $u_1u_2u_3u_4u_5$ *be a pentagon of* $\Gamma$ *such that* $\partial(x, u_1) = i - 1$ *and* $\partial(x, u_3) = i + 1$ *for* $1 \leqslant i \leqslant d$*. Then the pentagon* $u_1u_2u_3u_4u_5$ *has shape* $i - 1, i, i + 1, i + 1, i$ *with respect to* $x$*.*

*Proof.* Since $\partial(u_3, u_4) = 1$ and $\partial(u_3, x) = i + 1$, $\partial(x, u_4) = i + 2, i + 1$, or $i$. Since $\partial(u_1, u_4) = 2$ and $\partial(u_1, x) = i - 1$, $\partial(x, u_4) \leqslant i - 1 + 2 = i + 1$. Consequently we we have $\partial(x, u_4) = i + 1$ or $i$. It suffices to prove $\partial(x, u_4) = i + 1$. We prove this lemma by induction on $i$.

The case $i = 1$ holds otherwise $\partial(x, u_4) = i = 1$ and $\partial(x, u_5) = 1$, which contradicts the assumption $a_1 = 0$.

Suppose the assertion holds for any $i < \ell \leqslant d$. For the case $i = \ell$, suppose to the contrary that $u_1u_2u_3u_4u_5$ is a pentagon with $\partial(x, u_1) = \ell - 1$ and $\partial(x, u_3) = \ell + 1$, but $\partial(x, u_4) = \ell$. We can choose $y \in C(x, u_1)$ and hence $\partial(y, u_1) = \ell - 2$. Since $\partial(x, u_3) = \ell + 1$ and $\partial(x, y) = 1$, we have $\partial(y, u_3) = \ell + 2, \ell + 1$ or $\ell$. Since $\partial(y, u_1) = \ell - 2$ and $\partial(u_1, u_3) = 2$, we have $\partial(y, u_3) \leqslant \ell - 2 + 2 = \ell$. Consequently we have $\partial(y, u_3) = \ell$. By the induction hypothesis, the pentagon $u_1u_2u_3u_4u_5$ has shape $\ell - 2, \ell - 1, \ell, \ell, \ell - 1$ with respect to $y$. In particular, $\partial(y, u_3) = \partial(y, u_4) = \ell$. Then $xyu_4u_3$ is a parallelogram of length $\ell + 1$, a contradiction.

□

Other versions of Lemma 10 can be seen in [14, Lemma 6.9] and [9, Lemma 4.1] under various assumptions on intersection numbers.

The following three lemmas were formulated by A. Hiraki in [4] under an additional assumption $c_2 > 1$, but this assumption is essentially not used in his proofs. For the sake of completeness, we still provide the proofs.

**Lemma 11.** *Fix an integer* $1 \leqslant d \leqslant D - 1$*, and suppose* $\Gamma$ *contains no parallelograms of any length up to* $d + 1$*. Then for any two vertices* $z, z' \in X$ *such that* $\partial(x, z) \leqslant d$ *and* $z' \in A(z, x)$*, we have* $B(x, z) = B(x, z')$*.*

*Proof.* Note that $z' \in A(z, x)$ implies $\partial(x, z) = \partial(x, z')$, hence it suffices to show $B(x, z) \subseteq B(x, z')$ since $|B(x, z)| = |B(x, z')| = b_{\partial(x,z)}$. Suppose to the contrary that there exists $w \in B(x, z) - B(x, z')$. Then $\partial(w, z) = \partial(x, z) + 1$ and $\partial(w, z') \neq \partial(x, z) + 1$. Note that

$\partial(w, z') \leqslant \partial(w, x) + \partial(x, z') = 1 + \partial(x, z)$ and $\partial(w, z') \geqslant \partial(w, z) - \partial(z, z') = \partial(x, z)$. Consequently $\partial(w, z') = \partial(x, z)$ and $wxz'z$ forms a parallelogram of length $\partial(x, z) + 1$, a contradiction. □

**Lemma 12.** *Fix integers $1 \leqslant i \leqslant d \leqslant D - 1$, and suppose $\Gamma$ contains no parallelograms of any length up to $d + 1$. Let $x$ be a vertex in $\Gamma$. Then there is no pentagon of shape $i, i, i, i, i + 1$ with respect to $x$ in $\Gamma$.*

*Proof.* We prove this lemma by induction on $i$.

The case $i = 1$ holds otherwise we have a pentagon having shape $1, 1, 1, 1, 2$ with respect to $x$. In particular we have three vertices $x, u_1, u_2$ with $\partial(x, u_1) = \partial(x, u_2) = \partial(u_1, u_2) = 1$, which is a contradiction to the initial assumption $a_1 = 0$.

Suppose the assertion holds for any $i < \ell \leqslant d$. For the case $i = \ell$, suppose to the contrary that $u_1u_2u_3u_4u_5$ is a pentagon of shape $\ell, \ell, \ell, \ell, \ell + 1$ with respect to $x$. This implies $u_2 \in A(u_1, x), u_3 \in A(u_2, x)$, and $u_4 \in A(u_3, x)$. Hence we have $B(x, u_1) = B(x, u_2) = B(x, u_3) = B(x, u_4)$ by Lemma 11. We shall prove $C(x, u_1) = C(x, u_2) = C(x, u_3) = C(x, u_4)$ in the following.

First we prove $C(x, u_1) = C(x, u_2)$. It suffices to show $C(x, u_2) \subseteq C(x, u_1)$ since $|C(x, u_1)| = |C(x, u_2)| = c_\ell$. Suppose to the contrary that there exists $v \in C(x, u_2) - C(x, u_1)$. By our choice of $v$, we have $v \notin C(x, u_1)$. We also have $v \notin B(x, u_1)$, since $B(x, u_1) = B(x, u_2)$ and $v \notin B(x, u_2)$. Consequently we have $v \in A(x, u_1)$ since $v$ is a neighbor of $x$. Then $B(u_1, x) = B(u_1, v)$ by Lemma 11. Note that $v \in A(x, u_1)$ implies $\partial(v, u_1) = \partial(x, u_1) = \ell$, and hence $\partial(v, u_5) = \ell + 1$ since $u_5 \in B(u_1, x) = B(u_1, v)$. Applying Lemma 10 to the pentagon $u_2u_1u_5u_4u_3$ with $\partial(v, u_2) = \ell - 1$ and $\partial(v, u_5) = \ell + 1$, we conclude that $u_2u_1u_5u_4u_3$ has shape $\ell - 1, \ell, \ell + 1, \ell + 1, \ell$ with respect to $v$. In particular $\partial(v, u_4) = \ell + 1$ and hence $v \in B(x, u_4) = B(x, u_2)$. This is a contradiction to $v \in C(x, u_2)$. Consequently we have $C(x, u_2) \subseteq C(x, u_1)$ and hence $C(x, u_1) = C(x, u_2)$ as desired.

By substituting $u_4$ to $u_1$, $u_3$ to $u_2$ in the last paragraph and consider the shape of the pentagon $u_3u_4u_5u_1u_2$ with respect to $v' \in C(x, u_3) - C(x, u_4)$, similarly we have $C(x, u_4) = C(x, u_3)$.

It remains to show $C(x, u_2) = C(x, u_4)$. It suffices to show $C(x, u_2) \subseteq C(x, u_4)$. Suppose to the contrary that there exists $u \in C(x, u_2) - C(x, u_4)$. With the similar arguments in the previous paragraphs, we have $u \in A(x, u_4)$ and then $B(u_4, x) = B(u_4, u)$ by Lemma 11. Hence $\partial(u, u_5) = \ell + 1$ since $u_5 \in B(u_4, x) = B(u_4, u)$. Applying Lemma 10 to the pentagon $u_2u_1u_5u_4u_3$ with $\partial(u, u_2) = \ell - 1$ and $\partial(u, u_5) = \ell + 1$, we conclude that $u_2u_1u_5u_4u_3$ has shape $\ell - 1, \ell, \ell + 1, \ell + 1, \ell$ with respect to $u$. In particular $\partial(u, u_4) = \ell + 1$ and hence $u \in B(x, u_4)$. This is a contradiction since $u \in A(x, u_4)$.

Pick a vertex $w \in C(x, u_1) = C(x, u_2) = C(x, u_3) = C(x, u_4)$. Since $\partial(x, w) = 1$ and $\partial(x, u_5) = \ell + 1$, we have $\partial(w, u_5) = \ell + 2, \ell + 1$ or $\ell$. Since $\partial(u_4, u_5) = 1$ and $\partial(u_4, w) = \ell - 1$, we have $\partial(w, u_5) = \ell, \ell - 1$ or $\ell - 2$. Consequently we have $\partial(w, u_5) = \ell$. Then $u_1u_2u_3u_4u_5$ is a pentagon of shape $\ell - 1, \ell - 1, \ell - 1, \ell - 1, \ell$ with respect to $w$, which is a contradiction to the inductive hypothesis.

□

**Lemma 13.** *Fix integers $1 \leqslant i \leqslant d \leqslant D-1$, and suppose $\Gamma$ contains no parallelograms of any length up to $d+1$. Let $x$ be a vertex and $u_1u_2u_3u_4u_5$ be a pentagon of shape $i, i-1, i, i-1, i$ or of shape $i, i-1, i, i-1, i-1$ with respect to $x$ in $\Gamma$. Then $B(x, u_1) = B(x, u_3)$.*

*Proof.* It suffices to show $B(x, u_3) \subseteq B(x, u_1)$ since $|B(x, u_3)| = |B(x, u_1)| = b_i$. Pick $u \in B(x, u_3)$, this implies $\partial(u, u_3) = i+1$. Since $\partial(u_3, u_2) = 1$ and $\partial(u_3, u) = i+1$, we have $\partial(u_2, u) = i+2, i+1$, or $i$. Since $\partial(x, u) = 1$ and $\partial(x, u_2) = i-1$, we have $\partial(u_2, u) = i, i-1$, or $i-2$. Consequently we have $\partial(u, u_2) = i$. Substituting $u_4$ to $u_2$ in the above arguments, we similarly have $\partial(u, u_4) = i$. Next we consider $\partial(u, u_1)$. Note that $\partial(u, u_1) = i+1, i$ or $i-1$ since $\partial(x, u) = 1$ and $\partial(x, u_1) = i$. We show that $\partial(u, u_1) = i+1$ by excluding the other two cases in the following.

(1) Suppose $\partial(u, u_1) = i-1$, then the pentagon $u_1u_2u_3u_4u_5$ has shape $i-1, i, i+1, i+1, i$ with respect to $u$ by Lemma 10. In particular we have $\partial(u, u_4) = i+1$, which is a contradiction to $\partial(u, u_4) = i$ obtained in the last paragraph.

(2) Suppose $\partial(u, u_1) = i$. Since $\partial(u_1, u_5) = 1$ and $\partial(u_1, u) = i$, we have $\partial(u, u_5) = i+1, i$, or $i-1$. If $\partial(u, u_5) = i$, then the pentagon $u_4u_5u_1u_2u_3$ has shape $i, i, i, i, i+1$ with respect to $u$, which is a contradiction to Lemma 12. If $\partial(u, u_5) = i-1$, then the pentagon $u_5u_4u_3u_2u_1$ has shape $i-1, i, i+1, i, i$ with respect to $u$, which is a contradiction to Lemma 10. Consequently we have $\partial(u, u_5) = i+1$. For the case $u_1u_2u_3u_4u_5$ having shape $i, i-1, i, i-1, i-1$ with respect to $x$, we have $\partial(u, u_5) \leqslant \partial(x, u_5) + 1 = i$, which is a contradiction to $\partial(u, u_5) = i+1$. For the other case $u_1u_2u_3u_4u_5$ having shape $i, i-1, i, i-1, i$ with respect to $x$, $\partial(x, u_5) = i$ and hence $u_5u_1xu$ is a parallelogram of length $i+1$, also a contradiction.

Hence $\partial(u, u_1) = i+1$, or equivalently $u \in B(x, u_1)$. This proves $B(x, u_3) \subseteq B(x, u_1)$ as desired. □

The following lemma rules out a class of pentagons of certain shapes with respect to a given vertex.

**Lemma 14.** *Fix integers $1 \leqslant i \leqslant d \leqslant D-1$, and suppose $\Gamma$ contains no parallelograms of any length up to $d+1$. Let $x$ be a vertex in $\Gamma$. Then there is no pentagon of shape $i, i, i, i+1, i+1$ with respect to $x$ in $\Gamma$.*

*Proof.* We prove this lemma by induction on $i$. The case $i = 1$ holds otherwise we have a pentagon of shape $1, 1, 1, 2, 2$ with respect to $x$. In particular we have three vertices $x, u_1, u_2$ with $\partial(x, u_1) = \partial(x, u_2) = \partial(u_1, u_2) = 1$, which is a contradiction to the initial assumption $a_1 = 0$.

Suppose the assertion holds for any $i < \ell \leqslant d$. For the case $i = \ell$, suppose to the contrary that $u_1u_2u_3u_4u_5$ is a pentagon of shape $\ell, \ell, \ell, \ell+1, \ell+1$ with respect to $x$. Pick $v \in C(x, u_1)$ and note that hence $\partial(u_1, v) = \ell - 1$. Since $\partial(x, v) = 1$ and $\partial(x, u_5) = \ell + 1$, we have $\partial(v, u_5) = \ell+2, \ell+1$, or $\ell$. Since $\partial(u_1, u_5) = 1$ and $\partial(u_1, v) = \ell - 1$, we have $\partial(v, u_5) = \ell, \ell-1$, or $\ell-2$. Consequently we have $\partial(v, u_5) = \ell$.

Next we consider $\partial(v, u_3)$. Note that $\partial(x, v) = 1$ and $\partial(x, u_3) = \ell$, hence $\partial(v, u_3) = \ell+1, \ell$, or $\ell-1$. We show that $\partial(v, u_3) = \ell - 1$ by excluding the other two cases in the following.

(1) If $\partial(v, u_3) = \ell + 1$, then $v \in B(x, u_3)$. Note that $u_2 \in A(u_1, x)$ and $u_3 \in A(u_2, x)$, hence we have $B(x, u_1) = B(x, u_2) = B(x, u_3)$ by Lemma 11. Then $v \in B(x, u_3) = B(x, u_2) = B(x, u_1)$, which is a contradiction to $v \in C(x, u_1)$.

(2) If $\partial(v, u_3) = \ell$, we have $\partial(v, u_4) = \ell + 1, \ell$, or $\ell - 1$ since $\partial(u_3, u_4) = 1$. We also have $\partial(v, u_4) = \ell + 2, \ell + 1$, or $\ell$ since $\partial(x, u_4) = \ell + 1$ and $\partial(x, v) = 1$. Consequently we have $\partial(v, u_4) = \ell + 1$ or $\ell$. For the case $\partial(v, u_4) = \ell + 1$, applying Lemma 10 to the pentagon $u_1u_5u_4u_3u_2$ with $\partial(u_1, v) = \ell - 1$ and $\partial(v, u_4) = \ell + 1$, we have that the pentagon $u_1u_5u_4u_3u_2$ is of shape $\ell - 1, \ell, \ell + 1, \ell + 1, \ell$ with respect to $v$. In particular, $\partial(v, u_3) = \ell + 1$ which contradicts $\partial(v, u_3) = \ell$. For the case $\partial(v, u_4) = \ell$, $xvu_3u_4$ is a parallelogram of length $\ell + 1$, a contradiction to our initial assumption.

Next we consider $\partial(v, u_4)$. Since $\partial(u_3, u_4) = 1$ and $\partial(u_3, v) = \ell - 1$, we have $\partial(v, u_4) = \ell, \ell - 1$, or $\ell - 2$. Since $\partial(x, v) = 1$ and $\partial(x, u_4) = \ell + 1$, we have $\partial(v, u_4) = \ell + 2, \ell + 1$, or $\ell$. Consequently we have $\partial(v, u_4) = \ell$.

Finally we consider $\partial(v, u_2)$. Since $\partial(x, v) = 1$ and $\partial(x, u_2) = \ell$, we have $\partial(v, u_4) = \ell + 1, \ell$, or $\ell - 1$. Since $\partial(u_1, u_2) = 1$ and $\partial(u_1, v) = \ell - 1$, we have $\partial(v, u_2) = \ell, \ell - 1$, or $\ell - 2$. Consequently we have $\partial(v, u_2) = \ell$ or $\ell - 1$. If $\partial(v, u_2) = \ell - 1$, the pentagon $u_1u_2u_3u_4u_5$ is of shape $\ell - 1, \ell - 1, \ell - 1, \ell, \ell$ with respect to $v$. This is a contradiction to the induction hypothesis. Hence $\partial(v, u_2) = \ell$.

We conclude that the pentagon $u_5u_1u_2u_3u_4$ is of shape $\ell, \ell - 1, \ell, \ell - 1, \ell$ with respect to $v$. By Lemma 13, we have $B(v, u_2) = B(v, u_5)$. Since $\partial(x, u_5) = \ell + 1$ and $\partial(v, u_5) = \ell$, we have $x \in B(v, u_5)$. Since $\partial(x, u_2) = \ell$ and $\partial(v, u_2) = \ell$, we have $x \notin B(v, u_2)$. Consequently we have $x \in B(v, u_5) - B(v, u_2)$, which is a contradiction to $B(v, u_2) = B(v, u_5)$.

□

# 4 D-bounded Property and Nonexistence of Parallelograms

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geqslant 3$. Fix an integer $1 \leqslant d \leqslant D - 1$. Throughout this section, we assume that $\Gamma$ satisfies the following conditions.

**Assumption:**

(i) The intersection numbers satisfy $a_1 = 0$, $a_2 \neq 0, c_2 = 1$, and

(ii) $\Gamma$ contains no parallelograms of any length up to $d + 1$.

We shall prove the $d$-bounded property of $\Gamma$ in this section. By the definition of strongly closed subgraphs, the following proposition is easily seen.

**Proposition 15.** *Suppose $\Delta \subseteq X$ is a strongly closed subgraph of $\Gamma$ and $ux_1vx_2x_3$ or $ux_1x_2vx_3$ is a pentagon in $\Gamma$. If $u, v \in \Delta$, then $x_1, x_2, x_3$ are all in $\Delta$.*

*Proof.* Since $a_1 = 0$, it is easily seen that $\partial(u, v) = 2$ and $u, x_i, v$ is weak-geodetic for $i = 1, 2, 3$. □

We then give a definition.

**Definition 16.** For any vertex $x \in X$ and any subset $\Pi \subseteq X$, define $[x, \Pi]$ to be the set

$$\{v \in X \mid \text{there exists } y' \in \Pi, \text{ such that the sequence } x, v, y' \text{ is geodetic }\}.$$

For any $x, y \in X$ with $\partial(x, y) = d'$, set

$$\Pi_{xy} := \{y' \in \Gamma_{d'}(x) \mid B(x, y) = B(x, y')\}$$

and

$$\Delta(x, y) = [x, \Pi_{xy}].$$

For convenience, we also use $\Delta(x, y)$ to denote the subgraph of $\Gamma$ induced on $\Delta(x, y)$. Note that $\Delta(x, y)$ contains $x, y$ and $\Gamma_{d'}(x) \cap \Delta(x, y) = \Pi_{xy}$. We can also easily see the following proposition.

**Proposition 17.** *For $x, y, z, w \in X$ and $w \in \Delta(x, y)$, if $x, z, w$ is geodetic, then $z \in \Delta(x, y)$.*

*Proof.* Suppose $\partial(x, y) = d'$, $\partial(x, w) = i$ and $\partial(x, z) = j$. Then $\partial(z, w) = i - j$. By the construction of Definition 16, there exists $y' \in \Pi_{xy}$ such that $x, w, y'$ is geodetic. Hence $\partial(w, y') = d' - i$. Note that $\partial(z, y') \leqslant \partial(z, w) + \partial(w, y') = d' - j$, and $\partial(z, y') \geqslant \partial(x, y') - \partial(x, z) = d' - j$. So $\partial(z, y') = d' - j$ and thus $x, z, y'$ is geodetic. Hence $z \in \Delta(x, y)$. □

For any $1 \leqslant j \leqslant d$, we define the following three kinds of conditions:

($B_j$) For any vertices $x, y \in X$ with $\partial(x, y) = j$, $\Delta(x, y)$ is a regular strongly closed subgraph of $\Gamma$ with valency $a_j + c_j$ and diameter $j$.

($W_j$) For any vertices $x, y \in X$ with $\partial(x, y) = j$, $\Delta(x, y)$ is strongly closed with respect to $x$.

($R_j$) For any vertices $x, y \in X$ with $\partial(x, y) = j$, $\Delta(x, y)$ is a regular subgraph of $\Gamma$ with valency $a_j + c_j$.

By Definition 2, ($B_j$) holds for each $1 \leqslant j \leqslant d$ implies that $\Gamma$ is $d$-bounded since we can choose $\Delta(x, y)$ as the desired strongly closed subgraphs. By referring to Theorem 8, we know that for a subgraph $\Omega$ of $\Gamma$, if $\Omega$ is regular and $\Omega$ is strongly closed with respect to some vertex $x \in \Omega$, then $\Omega$ is strongly closed and is a distance-regular subgraph of $\Gamma$. Thus if ($W_\ell$) and ($R_\ell$) hold for some $1 \leqslant \ell \leqslant d$, then ($B_\ell$) holds. Consequently ($W_j$) and ($R_j$) hold for all $1 \leqslant j \leqslant d$ provides a sufficient condition for the $d$-bounded property of $\Gamma$. We plan to prove Theorem 5 through the above deduction, that is, to prove ($W_j$) and ($R_j$) hold for all $1 \leqslant j \leqslant d$ under the assumptions in the beginning of this section. We use induction on $j$ to achieve our objective. To adequately proceed the induction process, the following lemmas are required.

**Lemma 18.** *Fix integers $i, d'$ with $1 \leqslant i < d' \leqslant d$ and let $x, y \in X$ with $\partial(x,y) = d'$. Suppose for all $\ell \in \{i+1, i+2, \dots d'\}$, if vertex $z' \in \Delta(x,y) \cap \Gamma_\ell(x)$, we have the following (i), (ii).*

*(i) $A(z', x) \subseteq \Delta(x,y)$.*

*(ii) For any vertex $w' \in \Gamma_\ell(x) \cap \Gamma_2(z')$ with $B(x, w') = B(x, z')$, we have $w' \in \Delta(x,y)$.*

*Then for any $z \in \Delta(x,y) \cap \Gamma_i(x)$, $A(z,x) \subseteq \Delta(x,y)$.*

*Proof.* Let $v \in A(z,x)$. Pick $u \in \Delta(x,y) \cap \Gamma_{i+1}(x) \cap \Gamma_1(z)$. Let $uu_2u_3vz$ be a pentagon of $\Gamma$ for some $u_2, u_3 \in X$. Note that $uu_2u_3vz$ cannot have shape $i+1, i, i-1, i, i$, shape $i+1, i+2, i+1, i, i$ by Lemma 10, cannot have shape $i+1, i, i, i, i$ by Lemma 12, and cannot have shape $i+1, i+1, i, i, i$ by Lemma 14 with respect to $x$. Hence $uu_2u_3vz$ has shape $i+1, i+1, i+1, i, i$ or $i+1, i, i+1, i, i$ with respect to $x$. In the first case we have $u_2 \in A(u,x)$, $u_3 \in A(u_2, x)$, and this implies $u_2, u_3 \in \Delta(x,y)$ by the assumption (i). Then $v \in \Delta(x,y)$ by Proposition 17 since $x, v, u_3$ is geodetic. In the latter case we have $B(x,u) = B(x,u_3)$ by Lemma 13, and consequently $u_3 \in \Delta(x,y)$ by the assumption (ii). Then $v \in \Delta(x,y)$ by Proposition 17 since $x, v, u_3$ is geodetic.

□

**Lemma 19.** *Fix integers $i, d'$ with $1 \leqslant i < d' \leqslant d$ and let $x, y \in X$ with $\partial(x,y) = d'$. Suppose $(W_j)$, $(R_j)$ and thus $(B_j)$ hold in $\Gamma$ for all $j < d'$, and for all $\ell \in \{i+1, i+2, \dots d'\}$, if vertex $z' \in \Delta(x,y) \cap \Gamma_\ell(x)$, we have the following (i), (ii).*

*(i) $A(z', x) \subseteq \Delta(x,y)$.*

*(ii) For any vertex $w' \in \Gamma_\ell(x) \cap \Gamma_2(z')$ with $B(x, w') = B(x, z')$, we have $w' \in \Delta(x,y)$.*

*Then for any $z \in \Delta(x,y) \cap \Gamma_i(x)$ and $w \in \Gamma_i(x) \cap \Gamma_2(z)$ with $B(x,w) = B(x,z)$, we have $w \in \Delta(x,y)$.*

*Proof.* Let $z \in \Delta(x,y) \cap \Gamma_i(x)$. First we note that $(B_i)$ holds since $1 \leqslant i < d'$, hence $\Delta(x,z)$ is a regular strongly closed subgraph of diameter $i$.

Suppose to the contrary that there exists $w \in \Gamma_i(x) \cap \Gamma_2(z)$ with $B(x,w) = B(x,z)$ such that $w \notin \Delta(x,y)$. Since $B(x,w) = B(x,z)$, we have $\Pi_{xz} = \Pi_{xw}$ and thus $\Delta(x,z) = \Delta(x,w)$ by the construction in Definition 16.

Note that $|C(w,z)| = 1$ since $\partial(w,z) = 2$ and $c_2 = 1$. Let $v_2$ be the unique vertex in $C(w,z)$.

**Claim 19.1.** $\partial(x, v_2) = i - 1$.

*Proof of Claim 19.1.* Let $zv_2wv_4v_5$ be a pentagon for some $v_4, v_5 \in X$. Note that this pentagon exists since we can choose $v_4 \in A(w,z)$ with the assumption $a_2 \neq 0$, and we can choose $v_5 \in C(v_4, z)$ where $v_5 \neq v_2$ with the assumption $a_1 = 0$. Since $\partial(x,z) = i$ and $\partial(z, v_2) = 1$, we have $\partial(x, v_2) = i+1, i$, or $i-1$. We prove this claim by excluding the other two cases.

(1) Suppose $\partial(x, v_2) = i + 1$. Since $w \in \Delta(x, w) = \Delta(x, z)$ and $z \in \Delta(x, z)$, we have that $v_2, v_4, v_5 \in \Delta(x, z)$ by Proposition 15. In particular, $\partial(x, v_2) \leqslant i$ since $\Delta(x, z)$ is of diameter $i$. This is a contradiction.

(2) Suppose $\partial(x, v_2) = i$, that is, $v_2 \in A(z, x)$, then $v_2 \in \Delta(x, y)$ by Lemma 18. Since $\partial(x, v_2) = \partial(x, w) = i$, we have $w \in A(v_2, x)$. Applying Lemma 18 again by viewing $v_2$ as the role of $z$, we have $w \in \Delta(x, y)$. This contradicts our assumption that $w \notin \Delta(x, y)$. Hence $\partial(x, v_2) = i - 1$.

Let $u$ be a vertex in $\Delta(x, y) \cap \Gamma_{i+1}(x) \cap \Gamma_1(z)$. Let $y_3 \in A(u, v_2)$ and $y_4 \in C(y_3, v_2)$.

**Claim 19.2.** The pentagon $v_2zuy_3y_4$ has shape $i - 1, i, i + 1, i + 1, i$ with respect to $x$. Moreover the pentagon is contained in $\Delta(x, y)$.

*Proof of Claim 19.2.* The shape of the pentagon $v_2zuy_3y_4$ is determined by Lemma 10. Since $\partial(x, y_3) = i + 1$, we have $y_3 \in A(u, x)$ and we can conclude that $y_3 \in \Delta(x, y)$ by the assumption (i). We can also conclude that the remaining $v_2$ and $y_4$ are in $\Delta(x, y)$ by Proposition 17 since $x, v_2, y_3$ and $x, y_4, y_3$ are both geodetic.

If $w = y_4$ then $w \in \Delta(x, y)$ by Claim 19.2. This contradicts our assumption that $w \notin \Delta(x, y)$. Hence $w \neq y_4$ and we have $\partial(w, y_4) = 2$ by excluding the other possible case $\partial(w, y_4) = 1$ under the assumption $a_1 = 0$. Let $w_3 \in A(y_4, w)$ and $w_4 \in C(w_3, w)$.

**Claim 19.3.** The pentagon $v_2y_4w_3w_4w$ has shape $i - 1, i, i + 1, i + 1, i$ with respect to $x$ and $\{w_3, w_4\} \cap \{y_3, u\} = \emptyset$.

*Proof of Claim 19.3.* Recall that $\Delta(x, w) = \Delta(x, z)$ is strongly closed of diameter $i$ since $(B_i)$ holds. Also note that $v_2 \in \Delta(x, z)$ since $x, v_2, z$ is geodetic. Since $\partial(w, w_4) = 1$ and $\partial(x, w) = i$, we have $\partial(x, w_4) = i - 1, i$, or $i + 1$.

If $\partial(x, w_4) = i-1$ or $i$, then $x, w_4, w$ is weak-geodetic. Since $\Delta(x, w)$ is strongly closed, we have $w_4 \in \Delta(x, w) = \Delta(x, z)$. This forces $y_4 \in \Delta(x, z)$ by applying Proposition 15 to the pentagon $v_2y_4w_3w_4w$ with $v_2, w_4 \in \Delta(x, z)$. By applying Proposition 15 again to the pentagon $zv_2y_4y_3u$ with $z, y_4 \in \Delta(x, z)$, we have $y_3 \in \Delta(x, z)$. This is a contradiction since $\Delta(x, z)$ has diameter $i$ and $\partial(x, y_3) = i + 1 > i$. Hence $\partial(x, w_4) = i + 1$ and $v_2ww_4w_3y_4$ has shape $i - 1, i, i + 1, i + 1, i$ with respect to $x$ by Lemma 10.

Since $\partial(x, w_3) = \partial(x, w_4) = i + 1$ and $\partial(w_3, w_4) = 1$, we have $w_4 \in A(w_3, x)$. By the assumption (i), if $w_3 \in \Delta(x, y)$ then $w_4 \in \Delta(x, y)$. Recall that $y_3$ and $u$ are both in $\Delta(x, y)$ by Claim 19.2. Therefore if $\{w_3, w_4\} \cap \{y_3, u\} \neq \emptyset$, we can conclude that $w_4 \in \Delta(x, y)$ for any case. Since $x, w, w_4$ is geodetic, we have $w \in \Delta(x, y)$ by Proposition 17. This is a contradiction to our assumption that $w \notin \Delta(x, y)$.

The two pentagons $v_2zuy_3y_4$ and $v_2y_4w_3w_4w$ are shown in Figure 1.

**Claim 19.4.** $B(x, y_3) \neq B(x, w_3)$.

*Proof of Claim 19.4.* Note that $\partial(y_3, w_3) = 2$ since $\partial(y_4, w_3) = 1$, $\partial(y_4, y_3) = 1$, and $a_1 = 0$. Suppose to the contrary that $B(x, y_3) = B(x, w_3)$. Recall that $y_3 \in \Delta(x, y)$ by Claim 19.2. Hence we have $w_3 \in \Delta(x, y)$ by the assumption (ii). Since $\partial(x, w_3) = \partial(x, w_4) = i+1$ and $\partial(w_3, w_4) = 1$, we have $w_4 \in A(w_3, x)$. We then have $w_4 \in \Delta(x, y)$ by the assumption (i).

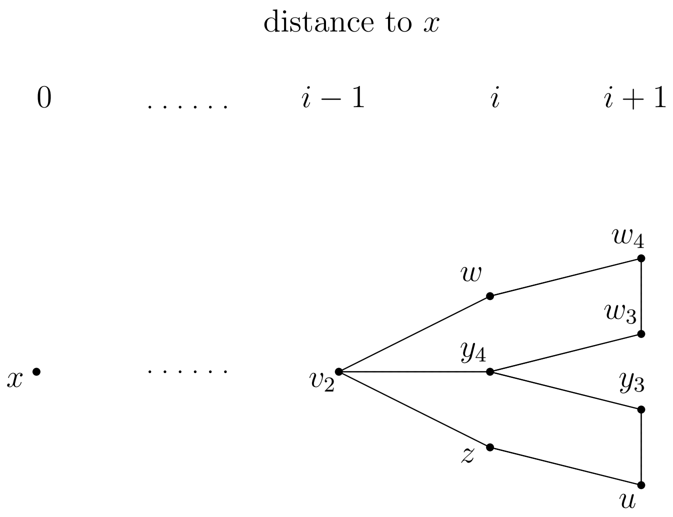


Figure 1: Two pentagons in the proof of Lemma 19.

Since $x, w, w_4$ is geodetic, we have $w \in \Delta(x, y)$ by Proposition 17. This is a contradiction to our assumption that $w \notin \Delta(x, y)$.

Let $p_3 \in A(y_3, w_3)$ and $p_4 \in C(p_3, w_3)$. Note that these two vertices exist since $\partial(y_3, w_3) = 2, a_2 \neq 0$, and $c_2 = 1$.

**Claim 19.5.** The pentagon $y_4y_3p_3p_4w_3$ has shape $i, i+1, i+2, i+2, i+1$ with respect to $x$.

*Proof of Claim 19.5.* Since $\partial(p_3, y_3) = 1$ and $\partial(x, y_3) = i+1$, we have $\partial(x, p_3) = i, i+1$ or $i+2$. We show that $\partial(x, p_3) = i+2$ by excluding the other two cases in the following.

(1) Suppose $\partial(x, p_3) = i+1$, then $\partial(x, p_4) = i+2, i+1$, or $i$ since $\partial(p_3, p_4) = 1$.

If $\partial(x, p_4) = i+2$, then the pentagon $y_4y_3p_3p_4w_3$ should have shape $i, i+1, i+2, i+2, i+1$ with respect to $x$ by Lemma 10. This is a contradiction to the assumption $\partial(x, p_3) = i+1$ for this case.

If $\partial(x, p_4) = i+1$, then $\partial(x, y_3) = \partial(x, p_3) = \partial(x, p_4) = \partial(x, w_3) = i+1$. Hence $p_3 \in A(y_3, x), p_4 \in A(p_3, x)$, and $w_3 \in A(p_4, x)$. By applying Lemma 11 three times, we have $B(x, y_3) = B(x, p_3) = B(x, p_4) = B(x, w_3)$. This is a contradiction to Claim 19.4.

If $\partial(x, p_4) = i$, then the pentagon $y_3y_4w_3p_4p_3$ should have shape $i+1, i, i+1, i, i+1$ with respect to $x$. By Lemma 13, we have $B(x, y_3) = B(x, w_3)$. This is also a contradiction to Claim 19.4.

(2) Suppose $\partial(x, p_3) = i$, then $\partial(x, p_4) = i-1, i$, or $i+1$ since $\partial(p_3, p_4) = 1$.

If $\partial(x, p_4) = i-1$, then we immediately get a contradiction from $\partial(x, p_4) = i-1, \partial(x, w_3) = i+1$, and $\partial(w_3, p_4) = 1$.

If $\partial(x, p_4) = i$, the pentagon $y_3y_4w_3p_4p_3$ should have shape $i+1, i, i+1, i, i$ with respect to $x$. By Lemma 13, we have $B(x, y_3) = B(x, w_3)$. This is a contradiction to Claim 19.4.

If $\partial(x, p_4) = i+1$, the pentagon $w_3y_4y_3p_3p_4$ should have shape $i+1, i, i+1, i, i+1$ with respect to $x$. By Lemma 13, we have $B(x, y_3) = B(x, w_3)$. This is also a contradiction to

Claim 19.4.

We conclude that $\partial(x, p_3) = i + 2$. In particular, the pentagon $y_4y_3p_3p_4w_3$ has shape $i, i+1, i+2, i+2, i+1$ with respect to $x$ by Lemma 10.

Now we have three pentagons and their shapes with respect to $x$ as shown in Figure 2.

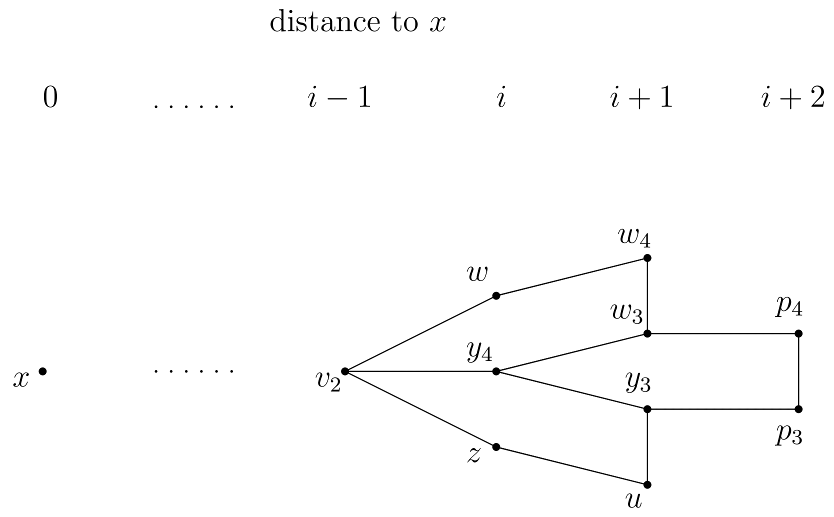


Figure 2: Three pentagons in the proof of Lemma 19.

**Claim 19.6.** $B(x, y_4) \neq B(x, z)$ and thus $B(x, y_4) - B(x, z) \neq \emptyset$.

*Proof of Claim 19.6.* Suppose to the contrary that $B(x, y_4) = B(x, z)$. By the construction in Definition 16, we have $\Delta(x, y_4) = \Delta(x, z)$, which is a strongly closed subgraph of diameter $i$ since $(B_i)$ holds. By applying Proposition 15 to the pentagon $zv_2y_4y_3u$ with $z, y_4 \in \Delta(x, z)$, we have $y_3 \in \Delta(x, z)$. This is a contradiction since $\partial(x, y_3) = i + 1$ and $\Delta(x, z)$ is of diameter $i$. The fact $B(x, y_4) - B(x, z) \neq \emptyset$ is easily seen by further observe that $|B(x, y_4)| = |B(x, z)| = b_i$, which implies that $B(x, y_4) \nsubseteq B(x, z)$.

Pick $p \in B(x, y_4) - B(x, z)$. Note that hence $\partial(p, y_4) = i + 1$.

**Claim 19.7.** $\partial(p, z) = i$.

*Proof of Claim 19.7.* Note that $\partial(p, z) = i$ or $i - 1$ since $p \notin B(x, z)$ and $\partial(p, x) = 1$. We exclude the case $\partial(p, z) = i - 1$ in the following.

Suppose $\partial(p, z) = i - 1$. Then $zv_2y_4y_3u$ is a pentagon of shape $i - 1, i, i + 1, i + 1, i$ with respect to $p$ by Lemma 10. More precisely, $\partial(p, z) = i - 1, \partial(p, v_2) = i, \partial(p, y_4) = i + 1, \partial(p, y_3) = i + 1$, and $\partial(p, u) = i$.

Next we show that $\partial(p, p_3) = i + 2$. Since $\partial(p, y_3) = i + 1$ and $\partial(p_3, y_3) = 1$, we have $\partial(p, p_3) = i+2$, $i+1$, or $i$. Since $\partial(x, p_3) = i+2$ and $\partial(x, p) = 1$, we have $\partial(p, p_3) = i+3$, $i + 2$, or $i + 1$. Consequently we have $\partial(p, p_3) = i + 2$ or $i + 1$. If $\partial(p, p_3) = i + 1$ then $xpy_3p_3$ is a parallelogram of length $i + 2 \leqslant d + 1$, a contradiction to our initial assumption that no parallelogram of length up to $d + 1$ exists. Hence $\partial(p, p_3) = i + 2$.

Next we show that $\partial(p, w_3) = i + 2$. We know that $\partial(p, w_3) = i, i + 1$ or $i + 2$ since $\partial(x, w_3) = i + 1$ and $\partial(x, p) = 1$. If $\partial(p, w_3) = i$, then he pentagon $w_3p_4p_3y_3y_4$ has shape $i, i + 1, i + 2, i + 2, i + 1$ with respect to $p$ by Lemma 10. In particular $\partial(p, y_3) = i + 2$, a contradiction to $\partial(p, y_3) = i + 1$. If $\partial(p, w_3) = i + 1$, we have $\partial(p, p_4) = i + 2$ or $i + 1$ since $\partial(p, w_3) = i + 1$, $\partial(p, p_3) = i + 2$, and $p_4$ is the common neighbor of $p_3$ and $w_3$. If $\partial(p, p_4) = i + 2$, the pentagon $w_3y_4y_3p_3p_4$ has shape $i + 1, i + 1, i + 1, i + 2, i + 2$ with respect to $p$, a contradiction to Lemma 14. If $\partial(p, p_4) = i + 1$, the pentagon $p_4w_3y_4y_3p_3$ has shape $i + 1, i + 1, i + 1, i + 1, i + 2$ with respect to $p$, a contradiction to Lemma 12. Hence $\partial(p, w_3) = i + 2$.

We finally consider the shape of the pentagon $v_2y_4w_3w_4w$ with respect to $p$ and get a contradiction. Since $\partial(x, p) = 1$ and $\partial(x, v_2) = i - 1$, we have $\partial(p, v_2) = i, i - 1$, or $i - 2$. Since $\partial(y_4, v_2) = 1$ and $\partial(y_4, p) = i + 1$, we have $\partial(p, v_2) = i + 2, i + 1$, or $i$. Consequently $\partial(p, v_2) = i$. Hence $v_2y_4w_3w_4w$ is a pentagon of shape $i, i + 1, i + 2, i + 2, i + 1$ with respect to $p$ by Lemma 10. In particular $\partial(p, w) = i + 1$, which implies $p \in B(x, w)$, a contradiction to our assumptions $B(x, z) = B(x, w)$ and $p \in B(x, y_4) - B(x, z)$.

**Claim 19.8.** $\partial(p, w) = i$.

*Proof of Claim 19.8.* Most of the following arguments are similar as the ones in the previous Claim 19.7, so we omit some details. Since $\partial(x, p) = 1$ and $\partial(x, w) = i$, we have $\partial(p, w) = i + 1, i$, or $i - 1$. We exclude the other two cases in the following.

(1) Suppose $\partial(p, w) = i + 1$, then $p \in B(x, w) = B(x, z)$. This is a contradiction to our assumption $p \in B(x, y_4) - B(x, z)$.

(2) Suppose $\partial(p, w) = i - 1$. First we have that the pentagon $wv_2y_4w_3w_4$ is of shape $i - 1, i, i + 1, i + 1, i$ with respect to $p$ by Lemma 10.

Next we show that then $\partial(p, p_4) = i + 2$. To avoid $xpw_3p_4$ to be a parallelogram of length $i + 2 \leqslant d + 1$, we have $\partial(p, p_4) = i + 2$.

Then we show that $\partial(p, y_3) = i+2$. By applying Lemma 10, Lemma 12, and Lemma 14 to the shape of the pentagon $y_4w_3p_4p_3y_3$ with respect to $p$, we have that $\partial(p, y_3) = i + 2$.

We finally consider the shape of the pentagon $v_2y_4y_3uz$ with respect to $p$ and get a contradiction. Consequently $v_2y_4y_3uz$ is a pentagon of shape $i, i+1, i+2, i+2, i+1$ with respect to $p$ by Lemma 10, which is a contradiction to $\partial(p, z) = i$.

**Claim 19.9.** $\partial(p, u) = \partial(p, w_4) = i + 1$.

*Proof of Claim 19.9.* Since $\partial(p, z) = \partial(x, z) = i$, we have $p \in A(x, z)$ and thus $B(z, x) = B(z, p)$ by Lemma 11. In particular $u \in B(z, p)$ and hence $\partial(p, u) = i + 1$. Similarly, $\partial(p, w_4) = i + 1$.

**Claim 19.10.** $\partial(p, y_3) = i$.

*Proof of Claim 19.10.* Since $\partial(x, y_3) = i+1$ and $\partial(x, p) = 1$, we have $\partial(p, y_3) = i+2, i+1$, or $i$. We exclude the other two cases in the following.

(1) Suppose $\partial(p, y_3) = i + 2$, then $p \in B(x, y_3)$ since $\partial(x, y_3) = i + 1$ and $\partial(x, p) = 1$. Since $\partial(x, y_3) = \partial(x, u) = i + 1$ and $\partial(u, y_3) = 1$, we have $y_3 \in A(u, x)$ and hence

$B(x,u) = B(x,y_3)$ by Lemma 11. Then we have $p \in B(x,u)$, which implies $\partial(p,u) = i+2$. This is a contradiction to Claim 19.9.

(2) Suppose $\partial(p,y_3) = i+1$. We first show that $\partial(p,p_3) = i+2$. By applying Lemma 11 we have $B(y_3,x) = B(y_3,p)$. Then as $p_3 \in B(y_3,x) = B(y_3,p)$, $\partial(p,p_3) = i+2$.

Next we show that $\partial(p,w_3) = i+2$. Applying Lemma 12, Lemma 14 to the pentagon $w_3y_4y_3p_3p_4$ and considering its shape with respect to $p$, we find $\partial(p,w_3) \neq i+1$. Applying Lemma 10 to the pentagon $w_3p_4p_3y_3y_4$, we find $\partial(p,w_3) \neq i$. Thus $\partial(p,w_3) = i+2$.

Recall that $\partial(p,w_4) = i+1$ by Claim 19.9. Then $pxw_4w_3$ is a parallelogram of length $i+2 \leqslant d+1$. This contradicts our initial assumption that no parallelogram of length up to $d+1$ exists.

**Claim 19.11.** $\partial(p,w_3) = i$.

*Proof of Claim 19.11.* Since $\partial(x,w_3) = i+1$ and $\partial(x,p) = 1$, we have $\partial(p,w_3) = i+2, i+1$, or $i$. We exclude the other two cases in the following.

(1) Suppose $\partial(p,w_3) = i+2$. Since $\partial(x,w_3) = \partial(x,w_4) = i+1$, we have $w_4 \in A(w_3,x)$ and hence $B(x,w_4) = B(x,w_3)$ by Lemma 11. Then $p \in B(x,w_3) = B(x,w_4)$, which implies $\partial(p,w_4) = i+2$ since $\partial(x,w_4) = i+1$. This is a contradiction to Claim 19.9.

(2) Suppose $\partial(p,w_3) = i+1$. Since $\partial(p_4,w_3) = 1$, we have $\partial(p,p_4) = i+2, i+1$, or $i$. Since $\partial(x,p) = 1$ and $\partial(x,p_4) = i+2$, we have $\partial(p,p_4) = i+3, i+2$, or $i+1$. Consequently we have $\partial(p,p_4) = i+2$ or $i+1$.

If $\partial(p,p_4) = i+2$, recall that $\partial(p,y_3) = i$ by Claim 19.10. Then the pentagon $y_3p_3p_4w_3y_4$ has shape $i, i+1, i+2, i+2, i+1$ with respect to $p$ by Lemma 10. In particular $\partial(p,w_3) = i+2$, which contradicts the assumption $\partial(p,w_3) = i+1$.

If $\partial(p,p_4) = i+1$, then $xpw_3p_4$ is a parallelogram of length $i+2 \leqslant d+1$. This contradicts our initial assumption that no parallelogram of length up to $d+1$ exists.

**Claim 19.12.** The pentagon $p_4w_3y_4y_3p_3$ has shape $i+1, i, i+1, i, i+1$ with respect to $p$.

*Proof of Claim 19.12.* Since $\partial(x,p_3) = i+2$ and $\partial(x,p) = 1$, we have $\partial(p,p_3) = i+3, i+2$, or $i+1$. Since $\partial(p_3,y_3) = 1$ and $\partial(p,y_3) = i$ by Claim 19.10, we have $\partial(p,p_3) = i+1, i$, or $i-1$. Consequently we have $\partial(p,p_3) = i+1$. Similarly we have $\partial(p,p_4) = i+1$ since $\partial(p,w_3) = i$ by Claim 19.11.

Recall that $\partial(p,y_4) = i+1$ since $p \in B(x,y_4) - B(x,z)$. Sum up Claim 19.10, Claim 19.11 and the above arguments, we conclude that the pentagon $p_4w_3y_4y_3p_3$ has shape $i+1, i, i+1, i, i+1$ with respect to $p$.

Applying Lemma 13 to the pentagon $p_4w_3y_4y_3p_3$ yields that $B(p,p_4) = B(p,y_4)$. Since $\partial(x,p_4) = i+2$ and $\partial(p,p_4) = i+1$ by Claim 19.11, we have $x \in B(p,p_4) = B(p,y_4)$. Hence $\partial(x,y_4) = \partial(p,y_4) + 1 = i+2$. This is a contradiction since $\partial(x,y_4) = i$.

Consequently, $w \in \Delta(x,y)$ and this completes the proof.

□

**Lemma 20.** *Fix integer $d'$ with $1 < d' \leqslant d$ and let $x, y \in X$ with $\partial(x, y) = d'$. Suppose $(W_j)$, $(R_j)$ and thus $(B_j)$ hold in $\Gamma$ for all $j < d'$. Then for any vertex $z \in \Delta(x, y) \cap \Gamma_\ell(x)$ where $1 \leqslant \ell \leqslant d'$, we have the following (i), (ii).*

*(i)* $A(z, x) \subseteq \Delta(x, y)$.

*(ii) For any vertex $w \in \Gamma_\ell(x) \cap \Gamma_2(z)$ with $B(x, w) = B(x, z)$, we have $w \in \Delta(x, y)$.*

*In particular $(W_{d'})$ holds.*

*Proof.* We prove (i), (ii) by induction on $d' - \ell$. For the case $d' - \ell = 0$, i.e. $\ell = d'$, we have $z \in \Pi_{xy}$. Hence (i), (ii) follows by Lemma 11 and the construction of $\Pi_{xy}$ in Definition 16.

Suppose for all $\ell$ with $0 \leqslant d' - \ell < d' - i$, i.e. $\ell \in \{i + 1, i + 2, \ldots d'\}$, if vertex $z' \in \Delta(x, y) \cap \Gamma_\ell(x)$, we have the following (a), (b).

(a) $A(z', x) \subseteq \Delta(x, y)$.

(b) For any vertex $w' \in \Gamma_\ell(x) \cap \Gamma_2(z')$ with $B(x, w') = B(x, z')$, we have $w' \in \Delta(x, y)$.

Then (i), (ii) hold for $\ell = i$, i.e. $d' - \ell = d' - i$, by Lemma 18 and Lemma 19. Then we conclude that (i), (ii) hold for all $0 \leqslant d' - \ell \leqslant d' - 1$, i.e. $1 \leqslant \ell \leqslant d'$, by induction.

In particular, we have $A(z, x) \subseteq \Delta(x, y)$ by (i), and we also have $C(z, x) \subseteq \Delta(x, y)$ by Proposition 17. Hence $(W_{d'})$ holds by (2). □

The following proposition proves $(R_{d'})$ and hence completes the preparation for the proof of Theorem 5.

**Lemma 21.** *Fix integer $d'$ with $1 < d' \leqslant d$ and let $x, y \in X$ with $\partial(x, y) = d'$. Suppose $(W_j)$, $(R_j)$ and thus $(B_j)$ hold in $\Gamma$ for all $j < d'$. Then $\Delta(x, y)$ is regular with valency $a_{d'} + c_{d'}$.*

*Proof.* Set $\Delta = \Delta(x, y)$. Clearly for any $v \in \Delta$, the construction ensures us that $\partial(x, v) \leqslant d'$. Hence $B(y', x) \cap \Delta = \emptyset$ for any $y' \in \Pi_{xy}$. Applying Lemma 20, we have $|\Gamma_1(y') \cap \Delta| = a_{d'} + c_{d'}$ for any $y' \in \Pi_{xy}$.

Next we show $|\Gamma_1(x) \cap \Delta| = a_{d'} + c_{d'}$. Note that $y \in \Delta \cap \Gamma_{d'}(x)$ by the construction of $\Delta$. For any $z \in C(x, y) \cup A(x, y)$,

$$\partial(x, z) + \partial(z, y) \leqslant \partial(x, y) + 1.$$

This implies $x, z, y$ is a weak-geodetic sequence, then $z \in \Delta$ since $\Delta$ is strongly closed with respect to $x$ by Lemma 20. Hence $C(x, y) \cup A(x, y) \subseteq \Delta$. Suppose $B(x, y) \cap \Delta \neq \emptyset$ and let $t \in B(x, y) \cap \Delta$. Then there exists $y' \in \Pi_{xy}$ such that $x, t, y'$ is a geodetic sequence by Definition 16. This implies $t \in C(x, y')$, a contradiction to $B(x, y) = B(x, y')$. Hence $B(x, y) \cap \Delta = \emptyset$ and $\Gamma_1(x) \cap \Delta = C(x, y) \cup A(x, y)$. This proves $|\Gamma_1(x) \cap \Delta| = a_{d'} + c_{d'}$.

Since each vertex in $\Delta$ appears in a sequence of vertices $x = x_0, x_1, \ldots, x_{d'}$ in $\Delta$, where $\partial(x, x_\ell) = \ell$, $\partial(x_{\ell-1}, x_\ell) = 1$ for $1 \leqslant \ell \leqslant d'$, and $x_{d'} \in \Pi_{xy}$, it suffices to show

$$|\Gamma_1(x_i) \cap \Delta| = a_{d'} + c_{d'} \tag{3}$$

for $1 \leqslant i \leqslant d'-1$. For each integer $1 \leqslant i \leqslant d'$, we show

$$|\Gamma_1(x_{i-1}) \setminus \Delta| \leqslant |\Gamma_1(x_i) \setminus \Delta| \tag{4}$$

by the 2-way counting of the number of the pairs $(z, s)$ with $z \in \Gamma_1(x_{i-1}) \setminus \Delta$, $s \in \Gamma_1(x_i) \setminus \Delta$ and $\partial(z, s) = 2$.

For a fixed $s \in \Gamma_1(x_i) \setminus \Delta$, we have $\partial(s, x_{i-1}) = 2$ since $a_1 = 0$. Hence such a $z$ must be one of the $a_2$ vertices in $A(x_{i-1}, s)$. The number of such pairs $(z, s)$ is thus at most $|\Gamma_1(x_i) \setminus \Delta| a_2$.

On the other hand, we show this number is $|\Gamma_1(x_{i-1}) \setminus \Delta| a_2$ exactly. Fix $z \in \Gamma_1(x_{i-1}) \setminus \Delta$. Note that $\partial(x_i, z) = 2$ since $a_1 = 0$. Hence the condition "$s \in \Gamma_1(x_i)$ with $\partial(z, s) = 2$" is equivalent to "$s \in A(x_i, z)$". We shall prove $s \notin \Delta$ for any $s \in A(x_i, z)$. Recall that $\Delta$ is strongly closed with respect to $x$ by Lemma 20, which implies $C(x_{i-1}, x) \subseteq \Delta$ and $A(x_{i-1}, x) \subseteq \Delta$. Then $z \in B(x_{i-1}, x)$ and hence $\partial(x, z) = i$.

Suppose to the contrary that there exists $s \in A(x_i, z) \cap \Delta$. Let $w \in C(s, z)$. Note that $w \neq x_i$ since $a_1 = 0$. Since $\partial(x_i, x) = i$ and $\partial(x_i, s) = 1$, we have $\partial(x, s) = i+1, i$, or $i-1$.

We first show that $\partial(x, s) = i$ or $i-1$. If $\partial(x, s) = i+1$, applying Lemma 10 to the pentagon $x_{i-1}x_i swz$ with $\partial(x, x_{i-1}) = i-1$ and $\partial(x, s) = i+1$, we see that the pentagon $x_{i-1}x_i swz$ has shape $i-1, i, i+1, i+1, i$ with respect to $x$. In particular, $\partial(x, w) = i+1$ and hence $w \in A(s, x)$. Then we have $w \in \Delta$ by Lemma 20(i). Note that $\partial(x, w) = i+1$ and $\partial(x, z) = i$, which implies that $x, z, w$ is a geodetic sequence. Then we have $z \in \Delta$ by Proposition 17, a contradiction to $z \in \Gamma_1(x_{i-1}) \setminus \Delta$.

We next show that $\partial(x, w) = i$ or $i-1$. Since $\partial(z, x) = i$ and $\partial(z, w) = 1$, we have $\partial(x, w) = i+1, i$, or $i-1$. If $\partial(x, w) = i+1$, the pentagon $x_{i-1}zwsx_i$ has shape $i-1, i, i+1, i+1, i$ with respect to $x$ by Lemma 10. In particular $\partial(x, s) = i+1$, which is a contradiction to $\partial(x, s) = i$ or $i-1$ constructed in the last paragraph.

If $\partial(x, w) = \partial(x, s) = i$, then $s \in A(x_i, x)$, $w \in A(s, x)$, and $z \in A(w, x)$. Applying Lemma 20(i) three times we have $z \in \Delta$, which is a contradiction to $z \in \Gamma_1(x_{i-1}) \setminus \Delta$. Hence $\partial(x, w) \leqslant i-1$ or $\partial(x, s) \leqslant i-1$. For the case $\partial(x, s) = i-1$ and $\partial(x, w) = i$ we consider the shape of the pentagon $zx_{i-1}x_i sw$ with respect to $x$. For the case $\partial(x, s) = i$ and $\partial(x, w) = i-1$, or the case $\partial(x, s) = i-1$ and $\partial(x, w) = i-1$, we consider the shape of the pentagon $x_i x_{i-1} zws$ with respect to $x$. Applying Lemma 13 to each of the these three cases we have $B(x, z) = B(x, x_i)$ and then $z \in \Delta$ by Lemma 20(ii), a contradiction to $z \in \Gamma_1(x_{i-1}) \setminus \Delta$.

From the above counting, we have

$$|\Gamma_1(x_{i-1}) \setminus \Delta| a_2 \leqslant |\Gamma_1(x_i) \setminus \Delta| a_2 \tag{5}$$

for $1 \leqslant i \leqslant d'$. Eliminating the nonzero $a_2$ from (5), we find (4) or equivalently

$$|\Gamma_1(x_{i-1}) \cap \Delta| \geqslant |\Gamma_1(x_i) \cap \Delta| \tag{6}$$

for $1 \leqslant i \leqslant d'$. We have shown previously that $|\Gamma_1(x_0) \cap \Delta| = |\Gamma_1(x_{d'}) \cap \Delta| = a_{d'} + c_{d'}$. Hence (3) follows from (6).

□

**Proof of Theorem 5.** For $1 \leqslant j \leqslant d$, we prove $(W_j)$ and $(R_j)$ by induction on $j$. Since $a_1 = 0$, there are no edges in $\Gamma_1(x)$ for any vertex $x \in X$.

For $j = 1$, then $\Pi_{xy} = \{y\}$ since for any other $y' \in \Gamma_1(x)$, $y' \in B(x, y)$ but $y' \notin B(x, y')$. Consequently $\Delta(x, y) = \{x, y\}$ is an edge; in particular $\Delta(x, y)$ is regular with valency $1 = a_1 + c_1$ and is strongly closed with respect to $x$ since $a_1 = 0$. This proves $(R_1)$ and $(W_1)$.

For $j \geqslant 2$, assume $(W_j)$, $(R_j)$ and thus $(B_j)$ hold for all $1 \leqslant j < d' \leqslant d$. By Lemma 20 and Lemma 21, we have that $(W_{d'})$, $(R_{d'})$ and thus $(B_{d'})$ hold.

Then we have $(B_j)$ holds for $1 \leqslant j \leqslant d$. By the deduction in the paragraph before Lemma 18, the proof is completed. □

Combining Theorem 4 and Theorem 5, the Proof of Theorem 6 can be completed.

**Proof of Theorem 6.** ((i) ⇒ (ii)) By Theorem 9, we see that $\Gamma$ contains no parallelograms of any length up to $d + 1$. Suppose that $\Gamma$ is $d$-bounded for $d \geqslant 2$. Let $\Omega \subseteq \Delta$ be two regular strongly closed subgraphs of diameters $1, 2$ respectively. Since $\Omega$ and $\Delta$ have different valency $b_0 - b_1$ and $b_0 - b_2$ respectively by Theorem 8, we have $b_1 > b_2$.

((ii) ⇒ (i)) Under the assumptions Theorem 6(ii) (hence $b_1 > b_2$) and $a_2 \neq 0$, consider the following four cases.

(a) $a_1 = 0$ and $c_2 > 1$: This case follows by Theorem 4 (i).

(b) $a_1 = 0$ and $c_2 = 1$: This case follows by Theorem 5.

(c) $a_1 \neq 0$ and $c_2 > 1$ : This case follows by Theorem 4 (ii).

(d) $a_1 \neq 0$ and $c_2 = 1$ : Note that by equation (1), $a_1 + b_1 + c_1 = k = a_2 + b_2 + c_2$. Since $c_1 = c_2 = 1$ and $b_1 > b_2$, this case is equivalent to the case $a_2 > a_1 \geqslant c_2 = 1$. Then the result follows by Theorem 4 (iii). □

## 5 Classical parameters

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geqslant 3$. $\Gamma$ is said to have *classical parameters* $(D, b, \alpha, \beta)$ whenever the intersection numbers of $\Gamma$ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right) \qquad \text{for } \ 0 \leqslant i \leqslant D, \tag{7}$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right) \qquad \text{for } \ 0 \leqslant i \leqslant D, \tag{8}$$

where

$$\begin{bmatrix} i \\ 1 \end{bmatrix}_b := \begin{cases} 1 + b + b^2 + \cdots + b^{i-1} & \text{if } i > 0, \\ 0 & \text{if } i \leqslant 0. \end{cases}$$

Applying (1) with (7) and (8), we have

$$a_i = \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(\beta - 1 + \alpha\left(\begin{bmatrix} D \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b - \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right)\right) \tag{9}$$

$$= \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(a_1 - \alpha\left(\begin{bmatrix} i \\ 1 \end{bmatrix}_b + \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b - 1\right)\right) \tag{10}$$

for $0 \leqslant i \leqslant D$.

Classical parameters were introduced in [2, Chapter 6]. Graphs with such parameters yield $P$- and $Q$-polynomial association schemes. Bannai and Ito proposed the classification of such schemes in [1].

The following theorem is a combination of [11, Theorem 2.12] and [14, Lemma 7.3(ii)].

**Theorem 22.** *([11, Theorem 2.12], [14, Lemma 7.3(ii)]) Let $\Gamma$ denote a distance-regular graph with classical parameters $(D, b, \alpha, \beta)$, where $b < -1$ and $D \geqslant 3$. Then $\Gamma$ contains no parallelograms of any length.*

The following two lemmas are given in [13].

**Lemma 23.** *([13, Corollary 3.7]) Let $\Gamma$ denote a distance-regular graph with classical parameters $(D, b, \alpha, \beta)$ and $D \geqslant 3$. Suppose $\Gamma$ contains no parallelogram of length 2 and $a_1 > -b - 1$. Then*

$$c_2 = b + 1.$$

**Lemma 24.** *([13, Theorem 4.2]) Let $\Gamma$ denote a distance-regular graph with classical parameters $(D, b, \alpha, \beta)$ and $D \geqslant 4$. Suppose $\Gamma$ is $D$-bounded and $a_1 \leqslant -b - 1$. Then*

$$\beta = \alpha \frac{1 + b^D}{1 - b}.$$

By Theorem 22, Lemma 23 and Lemma 24, we have the following theorem.

**Theorem 25.** *Let $\Gamma$ denote a distance-regular graph with classical parameters $(D, b, \alpha, \beta)$ where $b < -1$. Suppose that $\Gamma$ is $D$-bounded with $D \geqslant 4$. Then*

$$\beta = \alpha \frac{1 + b^D}{1 - b}. \tag{11}$$

*Proof.* Since $b < -1$ and $D \geqslant 3$, we have that $\Gamma$ contains no parallelograms of any length by Theorem 22. Note that $c_2 = b + 1$ implies $b > -1$. If $a_1 > -b - 1$ in $\Gamma$, then we get a contradiction by Lemma 23. Hence $a_1 \leqslant -b - 1$ and (11) follows by Lemma 24. □

The following is a proof of Theorem 7 which demonstrates an application of Theorem 6.

**Proof of Theorem 7.** Let $\Gamma$ denote a distance-regular graph with classical parameters $(D, b, \alpha, \beta) = (D, -2, -2, ((-2)^{D+1} - 1)/3)$, where $D \geqslant 4$. Then $\Gamma$ contains no parallelograms of any length by Theorem 22. By (7), (9) and (10) we have $c_2 = 1$ and $a_2 = 2 > 0 = a_1$. Hence $\Gamma$ is $D$-bounded by Theorem 6 since $b_1 > b_2$. By (11), $\beta = ((-2)^{D+1} - 2)/3$, which is a contradiction. □

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