Young diagrams and intersection numbers for toric manifolds associated with Weyl chambers

Hiraku Abe

Osaka City University Advanced Mathematical Institute
3-3-138 Sugimoto, Sumiyoshi-ku
Osaka 558-8585, Japan

hirakuabe@globe.ocn.ne.jp

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Abstract

We study intersection numbers of invariant divisors in the toric manifold associated with the fan determined by the collection of Weyl chambers for each root system of classical type and of exceptional type G_2. We give a combinatorial formula for intersection numbers of certain subvarieties which are naturally indexed by elements of the Weyl group. These numbers describe the ring structure of the cohomology of the toric manifold.

Keywords: Young diagrams; intersection numbers; toric varieties; structure constants

1 Introduction

Let Φ be a root system in the n-dimensional Euclidean space E with its inner product. We denote by $\Delta(\Phi)$ the fan determined by the collection of Weyl chambers in E^*, and consider the toric manifold X associated with $\Delta(\Phi)$. This toric manifold arises as the closure of a general orbit in the flag variety with respect to the standard torus action which makes X a regular semisimple Hessenberg variety ([4]). The Weyl group W naturally acts on the Weyl chambers and hence also on X. The representation of W on the cohomology $H^*(X;\mathbb{C})$ has been extensively studied by Procesi [9], Dolgachev-Lunts [5], and Stembridge [10]. For the classical root system of type A_n, Losev-Manin [8] described X as the moduli space of stable $(n+1)$-pointed chains of projective lines (cf. Batyrev-Blume [1]).

Let $\Pi = \{\alpha_1, \cdots, \alpha_n\} \subset \Phi$ be a set of simple roots, then we have a torus invariant non-singular subvariety X_u in X of codimension $|u(\Pi) \cap \Phi^-|$ such that the associated cohomology classes $\{[X_u]\}_{u\in W}$ form a module basis of the integral singular cohomology $H^*(X)$. The cohomology class $[X_u]$ is written as a monomial of torus invariant divisors

$D_{u\omega_i}$ of X for all coweights $u\omega_i$ satisfying $u\alpha_i \in \Phi^-$ where $\{\omega_1, \cdots, \omega_n\}$ is the set of fundamental coweights (see Section 2 and (17) for details). In this paper, we study the case for the root systems of classical type and of exceptional type G_2, and we give a combinatorial formula of the intersection numbers

$$(\mu_X, [w_0 X_{w_0 w}][X_u][X_v])$$

of three subvarieties X_u, X_v and $w_0 X_{w_0 w}$ for $u, v, w \in W$ where μ_X is the fundamental homology class of X and w_0 is the longest element. As an application, we will obtain a recursive formula for the structure constants $c_{u,v}^w$ in the expansion of the product

$$[X_u][X_v] = \sum_{w \in W} c_{u,v}^w [X_w] \quad \text{where} \quad c_{u,v}^w \in \mathbb{Z}$$

as discussed in Section 4.

Let us state our formula for $(\mu_X, [w_0 X_{w_0 w}][X_u][X_v])$ in the case of the classical root system of type A_n (the results for the classical root systems of type B_n, C_n, and D_n are stated in Section 5). In this case, the Weyl group W is the $(n+1)$-th permutation group $\mathfrak{S}_{n+1}$. For each $u \in \mathfrak{S}_{n+1}$, we let

$$D(u) := \{\{u(1), u(2), \cdots, u(i)\} \mid u(i) > u(i+1)\}, \tag{1}$$
$$A(u) := \{\{u(1), u(2), \cdots, u(i)\} \mid u(i) < u(i+1)\} \tag{2}$$

where each $\{u(1), u(2), \cdots, u(i)\}$ is a subset of $[n+1]$. We define a Young diagram λ_{uv}^w as follows. Assume $d(u) + d(v) = d(w)$, and the collection $D(u) \coprod D(v) \coprod A(w)$ forms a nested chain of subsets of $[n+1]$. In this case, λ_{uv}^w is defined to be the Young diagram consisting of the cardinalities of those chains ordered as a weakly decreasing sequence. Otherwise, $\lambda_{uv}^w = \emptyset$. For example, suppose $n = 4$ and, let $u = 12354$, $v = 31254$, and $w = 35421$. Then, we have that $D(u) = \{1235\}$, $D(v) = \{3, 3125\} = \{3, 1235\}$, and $A(w) = \{3\}$ where 1235 denotes the set $\{1, 2, 3, 5\}$ and we use the same notation for others. These sets forms a nested chain of subsets $3 \subset 3 \subset 1235 \subset 1235$, and hence we obtain $\lambda_{12354,31254}^{35421} = (4, 4, 1, 1)$.

For a Young diagram $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n)$ with n rows (i.e. $\lambda_n > 0$) fitting into the $n \times n$ square, we define $I(\lambda) \in \mathbb{Z}$ to be the following integer. Let s be the number of lower-right corners of λ, i.e., $s = |\{i \in [n] \mid \lambda_i > \lambda_{i+1}\}|$ where $\lambda_{n+1} := 0$. Write

$$\{i \in [n] \mid \lambda_i > \lambda_{i+1}\} = \{i_1, \cdots, i_s\}.$$

We impose the condition $i_1 < i_2 < \cdots < i_s$ to determine them uniquely. Observe that $i_s = n$. For example, if $n = 4$ and $\lambda = (4, 2, 2, 1)$, then $s = 3$ and $\{i_1, i_2, i_3\} = \{1, 3, 4\}$. For $r = 1, \cdots, s$, define

$$a_r := i_r - i_{r-1} - 1, \quad b_r := \lambda_{i_r} - \lambda_{i_{r+1}} - 1, \quad c_r := \lambda_{i_r} + i_r - n - 1$$

where we write $i_0 = 0$, and let

$$y_r := \binom{a_r}{c_r}\binom{b_r}{c_r} \quad \text{for } r = 1, \cdots, s.$$

We use the convention $\binom{x}{y} = 0$ unless $0 \leqslant y \leqslant x$. By shading each lower-right corner of λ, the pictorial meanings of a_r, b_r, and c_r become clear (as shown in Figure 1). Namely, a_r is the number of boxes between the north side of the shaded box and the the upper-left corner placed above, b_r is the similar number for the horizontal segment of the corner, and c_r is the number of boxes between the north side of the shaded box and the crossing point of the vertical segment and the anti-diagonal line where we count negatively if that part of the vertical segment is above the anti-diagonal.

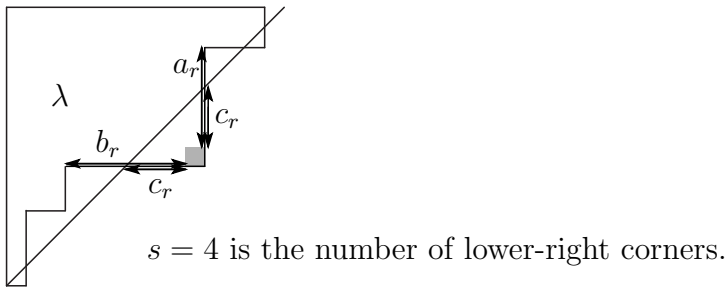


Figure 1: Three numbers a_r, b_r and c_r

Now, let

$$I(\lambda) := (-1)^{n+s} y_1 \cdots y_s,$$

and put $I(\emptyset) = 0$. The following is our main statement for type A_n.

Theorem 1. *$(\mu_X, [w_0 X_{w_0 w}][X_u][X_v]) = I(\lambda^w_{uv})$ for any $u, v, w \in \mathfrak{S}_{n+1}$ where μ_X is the fundamental homology class of X.*

We will prove Theorem 1 by computing general intersection numbers of invariant divisors of X in Section 3 and 4. Section 5 is devoted to the classical root systems of type B_n, C_n and D_n.

2 Preliminaries

Let Φ be a root system in the n-dimensional Euclidean space E with its inner product. Let $M \subset E$ be the root lattice of Φ and $N \subset E^*$ be the coweight lattice of Φ. Then M is the dual lattice of N with respect to the natural pairing.

We choose a set of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_n\} \subset \Phi$, and let $\Pi^* := \{\omega_1, \cdots, \omega_n\} \subset E^*$ be the dual basis of Π defined by $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$, i.e., $\omega_1, \cdots, \omega_n$ are the *fundamental coweights*. For each $u \in W$, denote

$$\sigma_u := \mathrm{cone}(u\omega_1, \cdots, u\omega_n) = \{\textstyle\sum_{i=1}^n \lambda_i u\omega_i \mid \lambda_i \geqslant 0\}.$$

These cones $\{\sigma_u\}_{u\in W}$ form a non-singular complete fan $\Delta(\Phi)$ in E^* by including all their faces. The set of minimal generators of these cones are the set of *coweights*:

$$\Phi^* = \bigcup_{v\in W} \{v\omega_1, \cdots, v\omega_n\}.$$

For each element $u \in W$, the maximal cones containing a minimal generator $u\omega_i$ are σ_v for $v \in W$ such that $u\omega_i = v\omega_j$ for some j. There is a cone of $\Delta(\Phi)$ generated by minimal generators $x_1, \cdots, x_k \in \Phi^*$ if and only if there exists $u \in W$ such that each x_i can be written as $u\omega_j$ for some j.

We consider the toric manifold $X = X(\Phi)$ associated with the fan $\Delta(\Phi)$. For root systems Φ and Φ', it is easily verified that $X(\Phi) \cong X(\Phi')$ as toric varieties (in the sense of [3] Sec. 3.3.) if and only if $\Phi \cong \Phi'$ as root systems (in the sense that their Cartan matrices are the same up to permuting the indexes). We refer to [1] and [7] for general properties of X. Let $D_x \subset X$ be the invariant divisor corresponding to the ray generated by $x \in \Phi^*$. The Poincaré dual $\tau_x := [D_x]$ gives us a cohomology class of degree 2 in the integral singular cohomology $H^*(X)$. The cohomology ring $H^*(X)$ is isomorphic to the face ring of the underlying simplicial complex of the fan $\Delta(\Phi)$ modulo some linear relations ([6]). More precisely, we have

$$H^*(X) = \mathbb{Z}[\tau_x \mid x \in \Phi^*]/I$$

where the ideal I is generated by $\tau_{x_1} \cdots \tau_{x_k}$ for which $x_1, \cdots, x_k$ do not generate a face of σ_u for some $u \in W$ and $\sum_{x\in\Phi^*} \langle x, \alpha\rangle \tau_x$ for any root α. Namely, we have the following equalities in $H^*(X)$:

$$\sum_{x\in\Phi^*} \langle x, \alpha\rangle \tau_x = 0 \quad \text{for any root } \alpha. \tag{3}$$

The above observation about rays of $\Delta(\Phi)$ implies that

Lemma 2. *We have $\tau_{x_1} \cdots \tau_{x_k} = 0$ unless there exists $u \in W$ such that each x_i can be written as $u\omega_j$ for some j.*

Let μ_X be the fundamental homology class of X. For subvarieties $Z_1, \cdots, Z_k \subset X$, we call $(\mu_X, [Z_1] \cdots [Z_k])$ the *intersection number* of $Z_1, \cdots, Z_k$ where $[Z_i]$ denotes the Poincaré dual of Z_i. Note that the Weyl group W acts on the fan $\Delta(\Phi)$, and hence acts on the toric manifold X. We have $uX_x = X_{ux}$ for any $u \in W$ and $x \in \Phi^*$ which means that $(u^{-1})^*\tau_x = \tau_{ux}$. The next lemma says that intersection numbers for divisors $D_{u\omega_i}$ are invariant under the Weyl group action.

Lemma 3. *Let $x_1, \cdots, x_n \in \Phi^*$. Then for any $u \in W$, we have*

$$(\mu_X, \tau_{ux_1} \cdots \tau_{ux_n}) = (\mu_X, \tau_{x_1} \cdots \tau_{x_n}).$$

Proof. Observe that $\tau_{ux_1} \cdots \tau_{ux_n} = (u^{-1})^*(\tau_{x_1} \cdots \tau_{x_n})$. Both of $\tau_{ux_1} \cdots \tau_{ux_n}$ and $\tau_{x_1} \cdots \tau_{x_n}$ can be written as the cohomology class $[p]$ of a point p in X multiplied by some integer, and these integers are the corresponding intersection numbers. Since u preserves the orientation of X, we have $(u^{-1})^*([p]) = [u \cdot p] = [p]$ which proves the claim. □

For any $u \in W$, the product $\tau_{u\omega_1} \cdots \tau_{u\omega_n}$ is the Poincaré dual of a point in X since the invariant divisors $X_{u\omega_1}, \cdots, X_{u\omega_n}$ intersect transversally which means that

$$(\mu_X, \tau_{u\omega_1} \cdots \tau_{u\omega_n}) = 1. \tag{4}$$

We will compute the intersection number $(\mu_X, \tau_{x_1} \cdots \tau_{x_n})$ for arbitrary $x_1, \cdots, x_n \in \Phi^*$ for the root systems of classical type. By Lemma 2, we can assume that this number is of the form $(\mu_X, (\tau_{u\omega_{i_1}})^{m_1} \cdots (\tau_{u\omega_{i_s}})^{m_s})$ for some $1 \leqslant i_1 < \cdots < i_s \leqslant n$ and $1 \leqslant m_k \leqslant n$ ($k = 1, \cdots, s$) satisfying $m_1 + \cdots + m_s = n$ without loss of generality. We call the number m_k the *multiplicity* of $\tau_{u\omega_{i_k}}$. We compute this number by applying the linear relations (3) to reduce the multiplicities $m_1, \cdots, m_s$. Although Lemma 3 shows that this number is equal to $(\mu_X, (\tau_{\omega_{i_1}})^{m_1} \cdots (\tau_{\omega_{i_s}})^{m_s})$, we will need Lemma 3 again after applying the relations (3).

In the next section, we will consider the classical root system of type A_n, and compute the intersection numbers.

3 Intersection numbers for Type A_n

In this section, we compute the intersection numbers for the toric manifold X of type A_n. Let $E = \{x \in \mathbb{R}^{n+1} \mid x_1 + \cdots + x_{n+1} = 0\}$. The roots are $t_i - t_j \in E$ ($1 \leqslant i, j \leqslant n+1$) where $t_i \in \mathbb{R}^{n+1}$ is the i-th standard vector. We choose $\Pi = \{t_i - t_{i+1} \mid 1 \leqslant i \leqslant n\}$ as the set of simple roots, and write $\alpha_i = t_i - t_{i+1}$ for each i. The Weyl group $W = \mathfrak{S}_{n+1}$ is the $(n+1)$-th permutation group acting on E by $u(t_i - t_j) = t_{u(i)} - t_{u(j)}$ for each $u \in W$. The minimal generators $\omega_1, \cdots, \omega_n \in E^*$ of the fundamental Weyl chamber σ_{id} are

$$\omega_i = (e_1 + \cdots + e_i) - \frac{i}{n+1}(e_1 + \cdots + e_{n+1}) \quad \text{for} \quad i = 1, \cdots, n$$

where $\{e_i\}_i \subset (\mathbb{R}^{n+1})^*$ is the dual basis of $\{t_i\}_i \subset \mathbb{R}^{n+1}$.

Denoting by $2^{[n+1]}$ the set of all subsets of $[n+1] = \{1, \cdots, n+1\}$, we have a well-defined map $\Phi^* \to 2^{[n+1]}$ by sending $u\omega_i \mapsto \{u(1), \cdots, u(i)\}$. It is easy to see that this is an injection, and this establishes an identification

$$\Phi^* \quad \longleftrightarrow \quad \text{the set of non-empty proper subsets of } [n+1]. \tag{5}$$

In particular, the well-definedness implies that if $u\omega_i = v\omega_j$ then $i = j$. Now, for each $\emptyset \subsetneq S \subsetneq [n+1]$, we define $\tau_S := \tau_{u\omega_i}$ where $u\omega_i \in \Phi^*$ corresponds to S by this identification. Then, for $\emptyset \subsetneq S_1, \cdots, S_q \subsetneq [n+1]$ ($1 \leqslant q \leqslant n$), it follows by Lemma 2 that $\tau_{S_1} \cdots \tau_{S_q} = 0$ unless these sets form a nested chain of subsets, i.e. $S_1 \subset \cdots \subset S_q$ up to reordering.

With Lemma 3, it is easy to show the following invariance property of intersection numbers which implies that $(\mu_X, \tau_{S_1} \cdots \tau_{S_n})$ for $\emptyset \subsetneq S_1 \subset \cdots \subset S_n \subsetneq [n+1]$ is determined by the set of integers $1 \leqslant |S_1| \leqslant \cdots \leqslant |S_n| \leqslant n$.

Lemma 4. *Let $\emptyset \subsetneq S_1 \subset \cdots \subset S_n \subsetneq [n+1]$ and $\emptyset \subsetneq S'_1 \subset \cdots \subset S'_n \subsetneq [n+1]$. If $|S_i| = |S'_i|$ for all $i = 1, \cdots, n$, then $(\mu_X, \tau_{S_1} \cdots \tau_{S_n}) = (\mu_X, \tau_{S'_1} \cdots \tau_{S'_n})$.*

Motivated by this property, we compute intersection numbers in terms of Young diagrams consisting of the cardinalities of the sets corresponding to the given invariant divisors. The linear relations (3) are translated to

$$\sum_{\substack{\emptyset \subsetneq S \subsetneq [n+1] \\ k \in S, l \notin S}} \tau_S - \sum_{\substack{\emptyset \subsetneq S \subsetneq [n+1] \\ k \notin S, l \in S}} \tau_S = 0 \qquad \text{for each } k, l \in [n+1]. \tag{6}$$

In the following, we write $\tau_\emptyset = \tau_{[n+1]} = 1$. This equality together with the above observation about $\tau_{S_1} \cdots \tau_{S_q}$ being 0 implies the next lemma.

Lemma 5. *Let $\emptyset \subset A \subsetneq B \subsetneq C \subset [n+1]$. For any $b \in B \backslash A$ and $c \in C \backslash B$, we have*

$$\tau_A {\tau_B}^2 \tau_C = - \sum_{\substack{A \subsetneq B' \subsetneq C \\ B' \neq B, b \in B', c \notin B'}} \tau_A \tau_B \tau_{B'} \tau_C.$$

For a Young diagram fitting into the $n \times n$ square, we write the *dotted anti-diagonal line* shifted down half the length of a single box from the standard anti-diagonal. (See Figure 2).

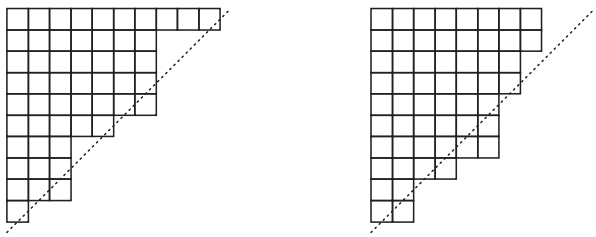


Figure 2: Young diagrams and the dotted anti-diagonal line

Let $\emptyset \subsetneq S_1 \subset \cdots \subset S_n \subsetneq [n+1]$, and denote by λ the Young diagram consisting of $\lambda_i = |S_{n+1-i}|$ for each i. We write $\lambda_{n+1} = 0$. Let s be the number of the lower-right corners of λ, that is,

$$s := |\{i \in [n] \mid \lambda_i > \lambda_{i+1}\}|.$$

Proposition 6. (Vanishing property) $(\mu_X, \tau_{S_1} \cdots \tau_{S_n}) = 0$ *unless each step of the zigzag line of the lower-right corners of λ crosses the dotted anti-diagonal.*

Proof. We suppose that there is a step of the zigzag line of λ which does not cross the dotted anti-diagonal, and show $(\mu_X, \tau_{S_1} \cdots \tau_{S_n}) = 0$ by induction on $k := n - s$. Since there is no such case for $k = 0$, we consider the case $k = 1$. In this case, there is a unique vertical segment of length 2 in the zigzag line of λ. If there is a (unique) horizontal segment of length 2, then the vertical and horizontal segments are not adjacent because of our assumption. By applying Lemma 5 for the square corresponding to this vertical segment, it follows that the intersection number is zero since there is no summand.

For the general case, take a vertical segment of length $\geqslant 2$. Let us say that this vertical segment contains S_i and S_{i+1} (i.e. $S_i = S_{i+1}$). We separate this square in $\tau_{S_1} \cdots \tau_{S_n}$ by Lemma 5. Let λ' be the Young diagram corresponding to a summand in the right-hand-side. Then the zigzag line of λ' has a step which does not cross the dotted anti-diagonal. In fact, if the vertical segment does not cross the dotted anti-diagonal, then this segment survives as a non-crossing segment of length at least 1, and if it does then we can find another vertical segment which does not, and this segment is preserved for each λ' in the summands. Now the induction hypothesis shows that each term will vanish after taking the intersection number, and we get $(\mu_X, \tau_{S_1} \cdots \tau_{S_n}) = 0$. $\square$

Let $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n)$ be a Young diagram with n rows (i.e. $\lambda_n > 0$) fitting into the $n \times n$ square. Let $I(\lambda) \in \mathbb{Z}$ be the one defined in Section 1. We here recall the definition for the convenience of the reader. Let s be the number of lower-right corners of λ, i.e., $s = |\{i \in [n] \mid \lambda_i > \lambda_{i+1}\}|$ where $\lambda_{n+1} := 0$. Write

$$\{i \in [n] \mid \lambda_i > \lambda_{i+1}\} = \{i_1, \cdots, i_s\}.$$

We impose the condition $i_1 < i_2 < \cdots < i_s$ to determine them uniquely. Observe that $i_s = n$. For $r = 1, \cdots, s$, define

$$a_r := i_r - i_{r-1} - 1, \quad b_r := \lambda_{i_r} - \lambda_{i_{r+1}} - 1, \quad c_r := \lambda_{i_r} + i_r - n - 1 \tag{7}$$

where we write $i_0 = 0$, and let

$$y_r := \binom{a_r}{c_r}\binom{b_r}{c_r} \quad \text{for } r = 1, \cdots, s. \tag{8}$$

See Figure 1 for the pictorial meaning of these numbers. We use the convention $\binom{x}{y} = 0$ unless $0 \leqslant y \leqslant x$. Now, let

$$I(\lambda) := (-1)^{n+s} y_1 \cdots y_s. \tag{9}$$

The next is the main theorem of this section.

Theorem 7. *If $S_1, \cdots, S_n$ form a nested chain of subsets, then we have*

$$(\mu_X, \tau_{S_1} \cdots \tau_{S_n}) = I(\lambda)$$

where μ_X is the fundamental homology class and λ is the Young diagram consisting of $|S_1|, \cdots, |S_n|$ reordered as a weakly decreasing sequence. Otherwise, the intersection number is zero.

Proof. Recall that λ is the Young diagram defined by $\lambda_i = |S_{n+1-i}|$ for $i = 1, \cdots, n$. We denote $J(\lambda) := (\mu_X, \tau_{S_1} \cdots \tau_{S_n})$, and we show that $J(\lambda) = I(\lambda)$. Observe that the condition $0 \leqslant c_r \leqslant b_r$ for all $r = 1, \cdots, s$ is equivalent to the condition that each step of the zigzag line of the corners of λ crosses the dotted anti-diagonal. If this condition is not satisfied, then both of $J(\lambda)$ and $I(\lambda)$ are zero. Hence, in the following, we can assume that this condition holds.

We prove the claim by induction on $k := n - s$. For the case $k = 0$, we have $\lambda_i = |S_{n+1-i}| = n+1-i$ for all $1 \leqslant i \leqslant n$. So we have $J(\lambda) = 1$ by (4). Since $y_1 = \cdots = y_n = 1$ in this case, we have $I(\lambda) = 1$, and the claim follows. For a general case, there is a lower-right corner (say r-th corner from the top) of λ whose vertical line has length $\geqslant 2$. Then, Lemma 4 and Proposition 6 combined together show that

$$J(\lambda) = \begin{cases} -\binom{b_{r-1}}{c_{r-1}} J(\lambda') - \binom{b_r}{c_r} J(\lambda'') & (\text{if } \lambda \neq \lambda', \lambda'') \\ -\binom{b_{r-1}}{c_{r-1}} J(\lambda') & (\text{if } \lambda \neq \lambda', \lambda = \lambda'') \\ -\binom{b_r}{c_r} J(\lambda'') & (\text{if } \lambda = \lambda', \lambda \neq \lambda'') \end{cases} \tag{10}$$

where a_r, b_r, and c_r are those for λ, and the Young diagrams λ' and λ'' are given by

$$\lambda'_j = \begin{cases} n+1-j & \text{if } j = i_{r-1}+1, \\ \lambda_j & \text{otherwise,} \end{cases} \qquad \lambda''_j = \begin{cases} n+1-j & \text{if } j = i_r, \\ \lambda_j & \text{otherwise.} \end{cases}$$

(See Figure 3.) Note that there are no cases that $\lambda = \lambda' = \lambda''$ since our vertical line has length $\geqslant 2$.

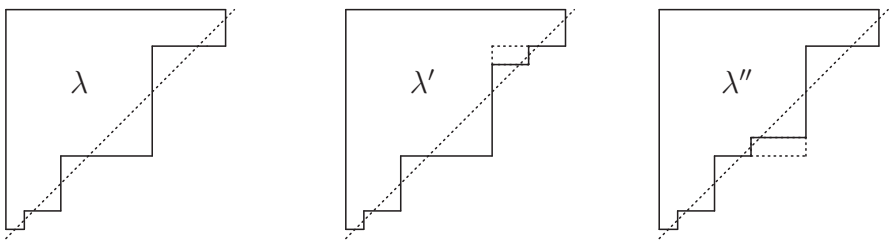


Figure 3: The Young diagrams λ, λ' and λ''

If $\lambda \neq \lambda'$, by the induction hypothesis, we have

$$J(\lambda') = (-1)^{n+(s+1)} y_1 \cdots y_{r-2} \cdot \binom{a_{r-1}}{c_{r-1}} \cdot 1 \cdot \binom{a_r - 1}{c_r} \binom{b_r}{c_r} \cdot y_{r+1} \cdots y_s. \tag{11}$$

Similarly, if $\lambda \neq \lambda''$, we have

$$J(\lambda'') = (-1)^{n+(s+1)} y_1 \cdots y_{r-1} \cdot \binom{a_r - 1}{c_r - 1} \cdot 1 \cdot y_{r+1} \cdots y_s. \tag{12}$$

Observe that the right-hand-sides of (11) and (12) vanish when $\lambda = \lambda'$ and $\lambda = \lambda''$, respectively. Hence, the right-hand-side of the equation (10) can be written as

$$\begin{aligned} J(\lambda) &= -\binom{b_{r-1}}{c_{r-1}} \cdot (-1)^{n+s+1} y_1 \cdots y_{r-2} \binom{a_{r-1}}{c_{r-1}} \binom{a_r - 1}{c_r} \binom{b_r}{c_r} y_{r+1} \cdots y_s \\ &\quad - \binom{b_r}{c_r} \cdot (-1)^{n+s+1} y_1 \cdots y_{r-2} y_{r-1} \binom{a_r - 1}{c_r - 1} y_{r+1} \cdots y_s \\ &= (-1)^{n+s} y_1 \cdots y_s = I(\lambda). \end{aligned}$$

□

4 The ring structure of the cohomology

The cohomology ring $H^*(X)$ of the toric manifold X associated with the fan $\Delta(A_n)$ is given by the face ring of $\Delta(A_n)$ modulo the linear relations (3) ([6]). As an application of Theorem 7, we describe the ring structure of the cohomology $H^*(X)$ in terms of an additive basis.

Recall that $D_{u\omega_i}$ for some $i \in [n]$ and permutation $u \in \mathfrak{S}_{n+1}$ is the invariant divisor of X associated with the ray generated by $u\omega_i \in N$. Let

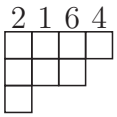
$$X_u := \bigcap_i D_{u\omega_i}$$

for each permutation $u \in \mathfrak{S}_{n+1}$ where i runs over all descents in u. Here, a descent in u is a number $i \in [n]$ which satisfies $u(i) > u(i+1)$, and we denote by $d(u)$ the number of descents in u. Denote by $[X_u] \in H^{2d(u)}(X)$ the Poincaré dual of X_u, then we have

$$[X_u] = \prod_i \tau_{\{u(1),\cdots,u(i)\}}$$

where i runs over all descents in u since invariant divisors of X intersect transversely. $\{[X_u]\}_{u \in \mathfrak{S}_{n+1}}$ forms a module basis of $H^*(X)$ (See [7] or [1] for combinatorial proofs and [4] for a geometric proof). The class $[X_u]$ can be expressed by a Young diagram consisting of the descents in u with the numbers in the nested chain of subsets in $D(u)$ (see (1) for the definition) written above the diagram so that each column represents the written number above it. This expression effectively encodes the descents in u and the information of the chain of subsets. (See Figure 4.) Denoting $Y^u := w_0 X_{w_0 u} = \cap X_{u[i]}$ where the intersection runs over all ascents i in u, the similar expression works for $[Y^u]$ and the chain of subsets in $A(u)$. Here, w_0 is the longest element of $\mathfrak{S}_{n+1}$.

For $u, v, w \in \mathfrak{S}_{n+1}$, we have the Young diagram λ_{uv}^w constructed in Section 1. Recall that μ_X is the fundamental homology class of X. The following corollary provides the combinatorial rule to compute the intersection number of X_u, X_v, and Y^w in X.

$[X_{216435}]$: 

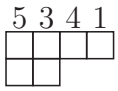
$[Y^{534162}]$: 

Figure 4: Two examples for $n = 5$ in one-line notations

Corollary 8. $(\mu_X, [Y^w][X_u][X_v]) = I(\lambda^w_{uv})$.

For example, for $n = 4$, we have

$$(\mu_X, [Y^{35421}][X_{12354}][X_{31254}]) = 2.$$

In Figure 5, we left the numbers on the Young diagram so that we can see the nested chain of subsets appeared in the construction of λ^w_{uv}.

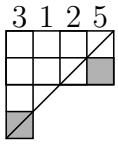
$\lambda^{35421}_{12354,31254} =$ 

Figure 5:

Since $\{[X_u]\}_{u\in\mathfrak{S}_{n+1}}$ forms a module basis of $H^*(X)$, we can consider the expansion coefficients of the product

$$[X_u][X_v] = \sum_w c^w_{uv}[X_w]. \tag{13}$$

For example, these coefficients for the product $[X_{s_i}][X_{s_j}]$ can be calculated directly if $|i-j| \geqslant 1$ where s_i is the simple reflection exchanging i and $i+1$. In fact, we have

$$[X_{s_i}][X_{s_j}] = \begin{cases} [X_{s_is_j}](= [X_{s_js_i}]) & \text{if } |i-j| \geqslant 2, \\ 0 & \text{if } |i-j| = 1. \end{cases} \tag{14}$$

since $\{1, \cdots, i-1, i+1\}$ and $\{1, \cdots, i, i+2\}$ do not form a chain of subsets.

Let us describe each structure constant $c^w_{u,v}$ in terms of intersection numbers computed above. Since a Weyl chamber $\sigma_u = \mathrm{cone}(u\omega_1, \cdots, u\omega_n)$ is a maximal cone of the fan, σ_u is identified with a fixed point of the canonical torus action on X denoted by $p_u \in X$ where p_u is the intersection $\cap_{i=1}^n D_{u\omega_i}$. Then from the definition of $X_{w'}$, one can show that $p_u \in X_{w'}$ implies $u \geqslant w'$ (e.g. [2]; Theorem 2.6.3) where $>$ is the Bruhat order. If $Y^w \cap X_{w'} \neq \emptyset$, then $Y^w \cap X_{w'}$ must contain a fixed point since it is an intersection of invariant divisors of X, and hence it follows that $w \geqslant w'$. From this observation, we see that $Y^w \cap X_{w'} = \emptyset$ unless $w \geqslant w'$. Also, it is easy to see that Y^w and $X_{w'}$

intersect transversally when $w = w'$. Recalling that the class $[Y^w][X_{w'}]$ is supported on the intersection $Y^w \cap X_{w'}$, we obtain

$$(\mu_X, [Y^w][X_{w'}]) = \begin{cases} 0 & \text{unless } w \geqslant w' \text{ and } d(w) = d(w'), \\ 1 & \text{if } w = w'. \end{cases} \tag{15}$$

See [4] for a proof using a cellular decomposition of X. Let $\mathcal{I}$ be the matrix whose (u, v)-component is given by $\mathcal{I}^u_{\ v} = (\mu_X, [Y^u][X_v]) = I(\lambda^u_{v\,\mathrm{id}})$ for all $u, v \in W$. This matrix $\mathcal{I}$ is invertible over $\mathbb{Z}$ because of (15). Now, each coefficient c^w_{uv} in (13) is a linear transform of the intersection numbers $I(\lambda^w_{uv})$;

$$c^w_{uv} = \sum_{w'} (\mathcal{I}^{-1})^w_{\ w'} I(\lambda^{w'}_{uv}). \tag{16}$$

We note that it suffices to take the sum for w' satisfying $d(w) = d(w')$ and $w \geqslant w'$ since $(\mathcal{I}^{-1})^w_{\ w'}$ is also upper-triangular in the sense of the right-hand-side of (15).
So the formula (16) exhibits the upper-triangularity of c^w_{uv} in the sense that $c^w_{uv} = 0$ unless $u, v \leqslant w$ since $I(\lambda^w_{uv})$ satisfies the same property.

The transition formula (16) together with (15) provides us a recursive formula for the structure constants c^w_{uv} which is manifestly integral;

$$c^w_{uv} = I(\lambda^w_{uv}) - \sum_{w > w'} \mathcal{I}^w_{\ w'} c^{w'}_{uv}.$$

Note again that it is enough to take the sum for all w' satisfying $d(w) = d(w')$ and $w > w'$. From this recursion, we recover (14), and we can compute the expansion of $[X_{s_i}]^2$. For example, if $n = 3$, we obtain

$$[X_{2134}]^2 = [X_{2431}] - [X_{4213}] - [X_{3421}] - [X_{3241}] - [X_{3214}].$$

5 Other classical types

Note that the argument in the previous section can be naturally generalized to arbitrary root systems by considering the non-singular subvariety

$$X_u = \bigcap_i D_{u\omega_i} \tag{17}$$

for each $u \in W$ where $D_{u\omega_i}$ is the invariant divisor of X corresponding to the ray generated by $u\omega_i$ and i runs over all i satisfying $u(\alpha_i) \in \Phi^-$. Here, Φ^- is the set of negative roots. It follows that the Poincaré duals $\{[X_u]\}_{u \in W}$ form an additive basis of the integral cohomlogy $H^*(X)$ (see [4]).

Remark 9. *The collections* $\{c^w_{u,v}\}_{u,v,w \in W}$ *and* $\{(\mu_X, [Y^w][X_u][X_v])\}_{u,v,w \in W}$ *are independent on the choice of the simple roots* Π.

5.1 Intersection numbers for type B_n

For the classical root system of type B_n, the roots are $\{t_i - t_j,\ \pm(t_i + t_j),\ \pm t_i \in E \mid 1 \leqslant i \neq j \leqslant n\}$ where $E = \mathbb{R}^n$. We choose $\Pi = \{t_i - t_{i+1},\ t_n \mid 1 \leqslant i \leqslant n-1\}$ as a set of simple roots, and write $\alpha_i = t_i - t_{i+1} (1 \leqslant i \leqslant n-1)$, $\alpha_n = t_n$. The Weyl group $\widetilde{\mathfrak{S}}_n$ is the n-th signed permutation group. Letting $t_{-i} := -t_i$ for all $1 \leqslant i \leqslant n$, $u \in \widetilde{\mathfrak{S}}_n$ acts on E by $ut_i = t_{u(i)}$. The minimal generators $\omega_1, \cdots, \omega_n \in E^*$ of the fundamental Weyl chamber are $\omega_i = e_1 + \cdots + e_i$ for $i = 1, \cdots, n$.

Let $[\pm n] = \{1, \cdots, n, -1, \cdots, -n\}$. For $S \in 2^{[\pm n]}$, consider a condition

$$\text{for any } i \in [\pm n], \text{ if } i \in S \text{ then } -i \notin S. \tag{$*$}$$

We have a well-defined map $\Phi^* \to 2^{[\pm n]}$ by $u\omega_i \mapsto \{u(1), \cdots, u(i)\}$. This leads us to an identification

$$\Phi^* \quad \longleftrightarrow \quad \text{the set of non-empty subsets of } [\pm n] \text{ satisfying } (*).$$

Now, for each $\emptyset \subsetneq S \subset [\pm n]$ satisfying $(*)$, we define $\tau_S := \tau_{u\omega_i}$ where $u\omega_i \in \Phi^*$ corresponds to S by this identification. For $\emptyset \subsetneq S_1, \cdots, S_q \subset [\pm n]$ $(1 \leqslant q \leqslant n)$ satisfying $(*)$, we have that $\tau_{S_1} \cdots \tau_{S_q} = 0$ unless these sets form a nested chain of subsets, as in the case for type A_n.

For each $k \in [\pm n]$, let $B \subset [\pm n]$ satisfy $(*)$, $k \in B$, and $|B| = n$. From the linear relation (3) for the root $\alpha = t_k$, we can deduce that

$$\tau_B{}^2 = -\sum_{\substack{k \in B' \\ B' \subsetneq B}} \tau_{B'} \tau_B$$

where the sum is taken over all $\emptyset \subsetneq B' \subset [\pm n]$ satisfying $(*)$ with the prescribed conditions. Similarly, for each $k, l \in [\pm n]$, let $B \subset [\pm n]$ satisfy $(*)$, $k \in B$, and $\pm l \notin B$ (hence $1 \leqslant |B| \leqslant n-1$). Then from (3) for the root $\alpha = t_k - t_l$, we obtain

$$\tau_B{}^2 = -\sum_{\substack{k \in B',\ \pm l \notin B' \\ B' \neq B}} \tau_{B'} \tau_B - \sum_{k, -l \in B'} 2\tau_{B'} \tau_B.$$

Observe that the second summand will vanish after multiplying τ_A and τ_C for $A \subset B \backslash \{k\}$ and $B \coprod \{l\} \subset C$ where we write $\tau_\emptyset = 0$. So these two equations can be used to prove the separation rule similar to Lemma 5, and we obtain the same type of vanishing property as in Proposition 6. Now the argument in the proof of Theorem 7 also works for this case, and it follows that

Theorem 10. *If $\emptyset \subsetneq S_1, \cdots, S_n \subset [\pm n]$ satisfying $(*)$ form a nested chain of subsets, then we have*

$$(\mu_X, \tau_{S_1} \cdots \tau_{S_n}) = 2^{n-\lambda_1} I(\lambda)$$

where μ_X is the fundamental homology class of X and λ is the Young diagram consisting of $|S_1|, \cdots, |S_n|$ reordered as a weakly decreasing sequence and I is the function defined in (9). Otherwise, the intersection number is zero.

Let $\alpha_i := t_i - t_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $\alpha_n := t_n$. For each signed permutation $u \in \widetilde{\mathfrak{S}}_n$, an element $i \in [n]$ satisfies $u(\alpha_i) \in \Phi^-$ if and only if

(D-1) if $i \leqslant n-1$, then $u(i) > u(i+1)$ with the same sign or $u(i) < u(i+1)$ with different signs,

(D-2) if $i = n$, then $u(i) < 0$.

Similarly, consider the conditions

(A-1) if $i \leqslant n-1$, then $u(i) < u(i+1)$ with the same sign or $u(i) > u(i+1)$ with different signs,

(A-2) if $i = n$, then $u(i) > 0$.

Denoting

$$D(u) := \{u[i] \mid i \text{ satisfies (D)}\} \quad \text{and} \quad A(u) := \{u[i] \mid i \text{ satisfies (A)}\},$$

we define a Young diagram $\lambda^w_{u,v}$ in the manner described in the last section. Note that we put $I(\emptyset) = 0$ as a convention.

Now, for signed permutations $u, v, w \in \widetilde{\mathfrak{S}}_n$, the intersection number of Y^w, X_u, and X_v in X of type B_n is given by the following.

Corollary 11. *For signed permutations $u, v, w \in \widetilde{\mathfrak{S}}_n$, we have*

$$(\mu_X, [Y^w][X_u][X_v]) = 2^{n-(\lambda^w_{u,v})_1} I(\lambda^w_{uv})$$

where I is the function defined in (9).

For example, for $n = 4$ with the convention $\bar{k} = -k$, Corollary 11 computes

$$(\mu_X, [Y^{2\bar{3}\bar{1}\bar{4}}][X_{2\bar{3}14}][X_{2\bar{3}14}]) = 4.$$

(See Figure 6.)

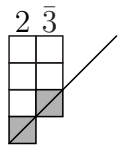
$\lambda^{2\bar{3}\bar{1}\bar{4}}_{2\bar{3}14,2\bar{3}14} =$ 

Figure 6: An example for type B_n

5.2 Intersection numbers for type C_n

For the classical root system of type C_n, the roots are $\{t_i - t_j,\ \pm(t_i + t_j),\ \pm 2t_i \in E \mid 1 \leqslant i \neq j \leqslant n\}$ where $E = \mathbb{R}^n$. We choose $\Pi = \{t_i - t_{i+1},\ 2t_n \mid 1 \leqslant i \leqslant n-1\}$ as a set of simple roots, and write $\alpha_i = t_i - t_{i+1} (1 \leqslant i \leqslant n-1)$, $\alpha_n = 2t_n$. The Weyl group $\widetilde{\mathfrak{S}}_n$ is the n-th signed permutation group as above. The minimal generators $\omega_1, \cdots, \omega_n$ of the fundamental Weyl chamber are $\omega_i = e_1 + \cdots + e_i$ for $i = 1, \cdots, n-1$ and $\omega_n = \frac{1}{2}(e_1 + \cdots + e_n)$.

We have a well-defined map $\Phi^* \to 2^{[\pm n]}$ by $v\omega_i \mapsto \{v(1), \cdots, v(i)\}$, and obtain an identification Φ^* and the set of non-empty subsets of $[\pm n]$ satisfying $(*)$. For $\emptyset \subsetneq S_1, \cdots, S_q \subsetneq [\pm n]$ $(1 \leqslant q \leqslant n)$ satisfying $(*)$, we have that $\tau_{S_1} \cdots \tau_{S_q} = 0$ unless these sets form a nested chain of subsets where τ_S is defined as in Section 5.1.

For each $k \in [\pm n]$, let $B \subset [\pm n]$ satisfy $(*)$, $k \in B$, and $|B| = n$. Then (3) for the root $\alpha = 2t_k$ shows that

$$\tau_B{}^2 = -\sum_{\substack{k \in B' \\ B' \subsetneq B}} 2\tau_{B'}\tau_B$$

where the sum is taken over all $\emptyset \subsetneq B' \subset [\pm n]$ satisfying $(*)$ with the prescribed conditions. For each $k, l \in [\pm n]$, let $B \subset [\pm n]$ satisfy $(*)$, $k \in B$, and $\pm l \notin B$ (hence $1 \leqslant |B| \leqslant n-1$). Then from (3) for the root $\alpha = t_k - t_l$, we obtain

$$\tau_B{}^2 = -\sum_{\substack{k \in B',\ \pm l \notin B' \\ B' \neq B}} \tau_{B'}\tau_B - \sum_{\substack{k \in B',\ -l \in B' \\ |B'| \neq n}} 2\tau_{B'}\tau_B - \sum_{\substack{k \in B',\ -l \in B' \\ |B'| = n}} \tau_{B'}\tau_B.$$

With a similar observation made for type B_n, we again have the same type of vanishing property as in Proposition 6. Hence, we obtain

Theorem 12. *If $\emptyset \subsetneq S_1, \cdots, S_n \subsetneq [\pm n]$ satisfying $(*)$ form a nested chain of subsets, then we have*

$$(\mu_X, \tau_{S_1} \cdots \tau_{S_n}) = 2^{n-\lambda_1+m-1} I(\lambda)$$

where μ_X is the fundamental homology class of X and λ is the Young diagram consisting of the numbers $|S_1|, \cdots, |S_n|$ reordered as a weakly decreasing sequence and I is the function defined in (9) and m is the number of rows of λ of length n. Otherwise, the intersection number is zero.

For signed permutations $u, v, w \in \widetilde{\mathfrak{S}}_n$, let $\lambda^w_{u,v}$ be the Young diagram defined in Section 5.1. The intersection number of Y^w, X_u, and X_v in X of type C_n is given by the following.

Corollary 13. *For signed permutations $u, v, w \in \widetilde{\mathfrak{S}}_n$, we have*

$$(\mu_X, [Y^w][X_u][X_v]) = 2^{n-(\lambda^w_{u,v})_1+m-1} I(\lambda^w_{uv})$$

where I is the function defined in (9) and and m is the number of rows of $\lambda^w_{u,v}$ of length n.

5.3 Intersection numbers for type D_n

For the classical root system of type D_n, the roots are $\{t_i - t_j,\ \pm(t_i + t_j) \in E \mid 1 \leqslant i \neq j \leqslant n\}$ where $E = \mathbb{R}^n$. We choose $\Pi = \{t_i - t_{i+1},\ t_{n-1} + t_n \mid 1 \leqslant i \leqslant n-1\}$ as a set of simple roots, and write $\alpha_i = t_i - t_{i+1} (1 \leqslant i \leqslant n-2)$, $\alpha_{n-1} = t_{n-1} + t_n$, $\alpha_n = t_{n-1} - t_n$. The Weyl group $\widetilde{\mathfrak{S}}_n^+$ is the n-th *even signed permutation group* defined by

$$\widetilde{\mathfrak{S}}_n^+ := \{w \in \widetilde{\mathfrak{S}}_n \mid \text{ the number of } i \text{ with } w(i) < 0 \text{ is even}\}$$

where $\widetilde{\mathfrak{S}}_n$ is the n-th signed permutation group. The minimal generators $\omega_1, \cdots, \omega_n \in E^*$ of the fundamental Weyl chamber are $\omega_i = e_1 + \cdots + e_i$ for $i = 1, \cdots, n-2$, $\omega_{n-1} = \frac{1}{2}(e_1 + \cdots + e_{n-1} + e_n)$ and $\omega_n = \frac{1}{2}(e_1 + \cdots + e_{n-1} - e_n)$.

For $S \in 2^{[\pm n]}$, consider a condition

$$|S| \neq n-1, \text{ and if } i \in S \text{ then } -i \notin S \text{ for any } i \in [\pm n]\ . \tag{$**$}$$

We have a well-defined map $\Phi^* \to 2^{[\pm n]}$ given by

$$\begin{aligned} u\omega_i &\mapsto u[i] = \{u(1), \cdots, u(i)\} \quad \text{for } 1 \leqslant i \leqslant n-2, \\ u\omega_{n-1} &\mapsto u[n]_+ = \{u(1), \cdots, u(n-1), u(n)\}, \\ u\omega_n &\mapsto u[n]_- = \{u(1), \cdots, u(n-1), -u(n)\}. \end{aligned}$$

where $[n]_+ = \{1, 2, \cdots, n-1, n\}$ and $[n]_- = \{1, 2, \cdots, n-1, -n\}$.

It follows that this map $\Phi^* \to 2^{[\pm n]}$ is an injection. In fact, we cannot have

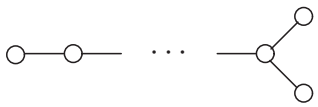
$$\{u(1), \cdots, u(n-1), u(n)\} = \{v(1), \cdots, v(n-1), -v(n)\}$$

for any $u, v \in \widetilde{\mathfrak{S}}_n^+$ since the number of negative integers in the left hand side and the right-hand-side are different, and so $u\omega_{n-1}$ and $v\omega_n$ are never mapped to the same element. The other cases are left to the reader. So we can make an identification

$$\Phi^* \quad \longleftrightarrow \quad \text{the set of non-empty subsets of } [\pm n] \text{ satisfying } (**). \tag{18}$$

Hence, for each $\emptyset \subsetneq S \subset [\pm n]$ satisfying $(**)$, we define $\tau_S := \tau_{u\omega_i}$ where $u\omega_i \in \Phi^*$ corresponds to S by this identification.

Let us denote by $\mathcal{C}$ the set of chains of subsets $\{S_i^\circ\}_i$ of $[\pm n]$ of the following form: there exists $u \in \widetilde{\mathfrak{S}}_n^+$ such that $S_i^\circ = u[i]$ for $1 \leqslant i \leqslant n-2$, $S_{n-1}^\circ = u[n]_+$, and $S_n^\circ = u[n]_-$. Note that $\{S_i^\circ\}_i$ does not have a set of order $n-1$ and satisfies the same inclusion relation shown in Figure 7. A *subchain* $\{S_i\}_i$ of a chain $\{S_i^\circ\}_i$ in $\mathcal{C}$ is a sequence satisfying $S_j \in \{S_i^\circ\}_i$ for $1 \leqslant j \leqslant n$ and $S_j \subset S_{j'}$ for $1 \leqslant j \leqslant j' \leqslant n$ unless $|S_j| = |S_{j'}| = n$. For $\emptyset \subsetneq S_1, \cdots, S_q \subsetneq [\pm n]$ $(1 \leqslant q \leqslant n)$ satisfying $(**)$, we have $\tau_{S_1} \cdots \tau_{S_q} = 0$ unless the sequence forms a subchain of a chain in $\mathcal{C}$ up to reordering. Let $\{S_i\}_i$ be a subchain of a chain in $\mathcal{C}$. For S_i satisfying $|S_i| = n$, we say that S_i is *even* (resp. *odd*) if the number of negative elements of S_i is even (resp. odd). Recall that μ_X is the fundamental homology class of X. The following is Lemma 3 for type D_n.



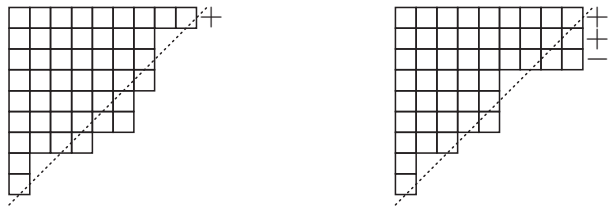
$$[1] \subset [2] \subset \cdots \subset [n-2] \begin{cases} \subset \{1, \cdots, n-1, n\} \\ \subset \{1, \cdots, n-1, -n\} \end{cases}$$

Figure 7: The Dynkin diagram and a maximal chain of subsets for type D_n

Lemma 14. *Let $\{S_i\}_i$ and $\{S'_i\}_i$ be subchains of some chains in $\mathcal{C}$. If $|S_i| = |S'_i|$ for $i = 1, \cdots, n$ and the number of even S_i's and the number of even S'_i's are the same, then $(\mu_X, \tau_{S_1} \cdots \tau_{S_n}) = (\mu_X, \tau_{S'_1} \cdots \tau_{S'_n})$.*

Let $\{S_i\}_i$ be a subchain of a chain in $\mathcal{C}$. We denote by λ the *signed Young diagram* consisting of $\lambda_i = |S_{n+1-i}|$ for $i = 1, \cdots, n$ where the label of λ is defined as follows: if we have $\lambda_i = n$, then we label this row by $+$ (resp. $-$) if S_{n+1-i} is even (resp. odd). Recall from Section 3 that the *dotted anti-diagonal line* drawn on the Young diagram is the dotted line shifted down half the length of a single box from the standard anti-diagonal. Our first aim is to prove the following.

Proposition 15. (The vanishing property) $(\mu_X, \tau_{S_1} \cdots \tau_{S_n}) = 0$ *unless each step of the zigzag line of the lower-right corners of λ crosses the dotted anti-diagonal.*



For each $k, l \in [\pm n]$, let $B \subset [\pm n]$ satisfy $(**)$, $k \in B$, and $\pm l \notin B$ (hence $|B| \leqslant n-2$). By the linear relation (3) for the root $\alpha = t_k - t_l$, it follows that

$$\tau_B{}^2 = - \sum_{\substack{k \in B',\ \pm l \notin B' \\ B' \neq B}} \tau_{B'} \tau_B - \sum_{\substack{k \in B',\ -l \in B' \\ |B'| \neq n}} 2\tau_{B'} \tau_B - \sum_{\substack{k \in B',\ -l \in B' \\ |B'| = n}} \tau_{B'} \tau_B \tag{19}$$

where the sum is taken over all $\emptyset \subsetneq B' \subset [\pm n]$ satisfying $(**)$ with the prescribed conditions. If $A \subsetneq B$ with $k \notin A$, and if $B \subsetneq C$ with $l \in C$ then,

$$\tau_A \tau_B{}^2 \tau_C = - \sum_{\substack{k \in B',\ \pm l \notin B' \\ A \subsetneq B' \subsetneq C,\ B' \neq B}} \tau_A \tau_{B'} \tau_B \tau_C \quad - \quad \delta_{|C|,n} \tau_A \tau_B \tau_{(l,-l)C} \tau_C. \tag{20}$$

where $\delta_{|C|,n}$ is the Kronecker delta. If $|C| = n$, then after multiplying (20) by $\tau_{\overline{C}}$ where $\overline{C} = (p, -p)C$ for some $p \in C \backslash \{l\}$, we obtain

$$\tau_A \tau_B{}^2 \tau_C \tau_{\overline{C}} = - \sum_{\substack{k \in B',\ \pm l, \pm p \notin B' \\ A \subsetneq B' \subsetneq C,\ B' \neq B}} \tau_A \tau_{B'} \tau_B \tau_C \tau_{\overline{C}}. \tag{21}$$

Let λ as above. We denote

$$m_+(\lambda) := |\{i \mid \lambda_i = n \text{ and the label of } \lambda_i \text{ is } +\}|,$$
$$m_-(\lambda) := |\{i \mid \lambda_i = n \text{ and the label of } \lambda_i \text{ is } -\}|.$$

Lemma 16. *Suppose that one of the following holds:*

(i) $m_+(\lambda) = m_-(\lambda) = 1$

(ii) $(m_+(\lambda), m_-(\lambda))$ *is equal to* $(1,0)$ *or* $(0,1)$,

(iii) $m_+(\lambda) = m_-(\lambda) = 0$.

Then $(\mu_X, \tau_{S_1} \cdots \tau_{S_n}) = 0$ *unless each step of the zigzag line of the corners of* λ *crosses the dotted anti-diagonal.*

Proof. The claim for the case (i) can be proved by (20) and (21) as in the proof of Proposition 6. For the case (ii), the same argument works together with (20), since we have already proved the claim for the case (i). Now, the case (iii) is shown again by the same proof used for Proposition 6 together with (19), (20), and the case (ii). □

For each $k, l \in [\pm n]$, let $\emptyset \subsetneq B \subsetneq [\pm n]$ satisfy $(**)$, $k, l \in B$, and $|B| = n$. If $A \subset B \backslash \{k, l\}$, then from the linear relation (3) for the root $\alpha = t_k + t_l$, we obtain

$$\begin{aligned} {\tau_A \tau_B}^2 = &- \sum_{k \in B',\ \pm l \notin B'} \tau_A \tau_{B'} \tau_B - \sum_{\pm k \notin B',\ l \in B'} \tau_A \tau_{B'} \tau_B \\ &- \sum_{\substack{k,l \in B', \\ |B'| \neq n}} 2 \tau_A \tau_{B'} \tau_B - \sum_{\substack{k,l \in B',\ B' \neq B, \\ |B'| = n}} \tau_A \tau_{B'} \tau_B \end{aligned} \tag{22}$$

where we denote $\tau_\emptyset = 1$. Especially if $A = B \backslash \{k, l\}$, then $|A| = n - 2$ and we have

$${\tau_A \tau_B}^2 = - \sum_{\substack{k,l \in B',\ B' \neq B, \\ |B'| = n}} \tau_A \tau_{B'} \tau_B = 0. \tag{23}$$

The second equality follows since an element of B' which is neither k nor l has to be -1 times an element of B, which implies that $A \not\subset B'$. On the other hand, letting $\overline{B} = (-k, k)B$, we obtain from (22) that

$${\tau_A \tau_B}^2 \tau_{\overline{B}} = - \sum_{\substack{\pm k \notin B',\ l \in B' \\ A \subsetneq B' \subsetneq B}} \tau_A \tau_{B'} \tau_B \tau_{\overline{B}}. \tag{24}$$

Lemma 17. *Suppose that one of the following holds:*

(i) $m_+(\lambda), m_-(\lambda) \geqslant 1$,

(ii) $m_+(\lambda) \geqslant 1$ *and* $m_-(\lambda) = 0$,

(iii) $m_+(\lambda) = 0$ *and* $m_-(\lambda) \geqslant 1$.

Then $(\mu_X, \tau_{S_1} \cdots \tau_{S_n}) = 0$ *unless each step of the zigzag line of the lower-right corners of* λ *crosses the dotted anti-diagonal.*

Proof. The claim for the case (i) follows from (24) and the case (i) of Lemma 16 by induction on $m_+(\lambda) + m_-(\lambda)$. Let us consider the case (ii). We prove the claim by induction on $m_+(\lambda)$. For the case $m_+(\lambda) = 1$, the claim follows from the case (ii) of Lemma 16. For the general case, the induction hypothesis and the claim for the case (i) shows our claim by applying (22) to reduce the multiplicity for τ_{S_n} in $\tau_{S_1} \cdots \tau_{S_n}$. The claim for the case (iii) can be proved similarly. □

Now, Proposition 15 follows from Lemma 17 and the case (iii) of Lemma 16.

For a signed Young diagram λ with n rows fitting into the $n \times n$-square, let

$$m := |\{i \mid \lambda_i = n\}| = m_+(\lambda) + m_-(\lambda)$$

be the number of rows of λ of length n. Recall that the numbers a_r, b_r, c_r, and y_r are defined in (7) and (8). We now define

$$\widetilde{y}_1 := \begin{cases} 2^{(n-\lambda_1-1)(1-m)} \binom{a_1}{c_1} \binom{b_1}{c_1} & \text{if } m \leqslant 1, \\ -\binom{b_1-1}{c_1-1} & \text{if } m \geqslant 2 \text{ and } m_+(\lambda)m_-(\lambda) \neq 0, \\ (2^{a_1} - a_1 - 1)\binom{b_1}{c_1} + \binom{b_1-1}{c_1} & \text{if } m \geqslant 2 \text{ and } m_+(\lambda)m_-(\lambda) = 0 \end{cases}$$

Theorem 18. *If* $\emptyset \subsetneq S_1, \cdots, S_n \subsetneq [\pm n]$ *satisfying* $(**)$ *form a subchain of a chain in* $\mathcal{C}$*, then we have*

$$(\mu_X, \tau_{S_1} \cdots \tau_{S_n}) = (-1)^{n+s} \widetilde{y}_1 y_2 \cdots y_s.$$

where μ_X *is the fundamental homology class of* X *and* λ *is the signed Young diagram consisting of* $|S_1|, \cdots, |S_n|$ *reordered as a weakly decreasing sequence and* m *is the number of rows of* λ *of length* n*. Otherwise, the intersection number is zero.*

Remark 19. *In each case, the given number vanishes unless each step of the zigzag line of the lower-right corners of* λ *crosses the dotted anti-diagonal.*

Proof of Theorem 18. We compute the intersection number

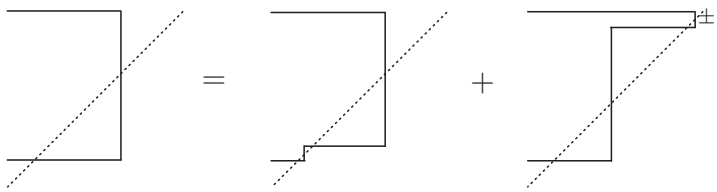
$$J(\lambda) := (-1)^{n-s}(\mu_X, \tau_{S_1} \cdots \tau_{S_n})$$

with sign where we can assume that each step of the zigzag line of the corners of λ crosses the dotted anti-diagonal by Proposition 15 and the remark above. We first prove the

case (i). If $m_+(\lambda) = 1$ and $m_-(\lambda) = 0$ (or $m_+(\lambda) = 1$ and $m_-(\lambda) = 0$), then it follows that $J(\lambda) = y_1 \cdots y_s$. This can be proved by induction similar to that used in the proof of Theorem 7 because of the separating properties (20). So, let us consider the case of $m_+(\lambda) = m_-(\lambda) = 0$. In this case, Proposition 15 shows that the separation rule (19) replaces the square $\tau_{S_n}^2$ in $\tau_{S_1} \cdots \tau_{S_n}$ to

$$\sum_{\substack{k \in S',\ \pm l \notin S' \\ S' \subsetneq S_n,\ |B'| \neq n}} \tau_{S'} \tau_{S_n} + \sum_{\substack{k \in S',\ -l \in S' \\ |S'| = n}} \tau_{S'} \tau_{S_n}$$

for some $k \in S_n$ and $\pm l \in S_n$ when we compute the intersection number $J(\lambda)$ with sign. Namely, this replacement can be pictured as



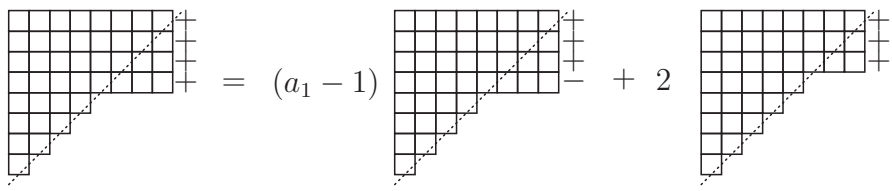
where we omit the coefficients in the picture. Hence, with the claim for the previous case, we get $J(\lambda) = 2^{n-\lambda_1-1} y_1 \cdots y_s$ as in the case of type B_n.

Let us consider the case (ii-a). We prove the claim by induction on the sum of the multiplicities for S_i's satisfying $|S_i| \neq n$. The base case has $\lambda_i = n + 1 - i$ for all $\lambda_i \neq n$, so it is obvious that $-J(\lambda) = (-1)^{n-s-1}(\mu_X, \tau_{S_1} \cdots \tau_{S_n})$ is equal to 1 by iterating (24). For the general case, we apply (20) and (21) to some square ${\tau_{S_i}}^2$ with $|S_i| \neq n$ in $J(\lambda)$. If $i_2 < i$, the computation with (20) works as in the proof of Theorem 7. If $i_1 < i \leqslant i_2$, then the right-hand-side of (21) applied to $-J(\lambda)$ can be calculated as follows by the induction hypothesis:

$$\binom{b_2}{c_2} \cdot \binom{b_1 - 1}{c_1 - 1} \binom{a_2 - 1}{c_2 - 1} y_3 \cdots y_s + \binom{b_1 - 1}{c_1 - 1} \cdot \binom{a_2 - 1}{c_2} \binom{b_2}{c_2} y_3 \cdots y_s$$

which is equal to $\binom{b_1-1}{c_1-1} y_2 \cdots y_s$, and the claim follows.

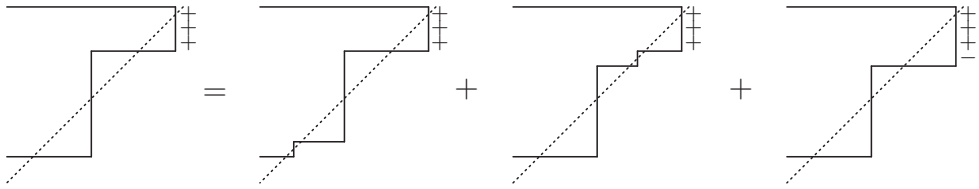
Let us consider the case (ii-b). We can assume $m = m_+(\lambda) (= a_1 + 1)$ without loss of generality. We first consider the case that $\lambda_i = n+1-i$ for all $\lambda_i \neq n$. Note that $\binom{b_1-1}{c_1} = 0$ in this case. For this special case, we prove the claim by induction on $m = m_+(\lambda)$. For the case $m = a_1 + 1 = 2$, the intersection number is zero by (23), and the claim follows since $2^{a_1} - a_1 - 1 = 0$. For the general case, we have an inductive formula by (22), namely,



where the first coefficient $a_1 - 1$ comes from the choices of an element in $B' \backslash (A \cup \{k, l\})$ turned to be negative, and the second coefficient 2 comes from the two summands corresponding to $k \in B', \pm l \notin B'$ and $\pm k \notin B', l \in B'$. Noticing that the intersection number for the first summand is equal to 1 as we already saw, it follows that

$$J(\lambda) = \sum_{i=0}^{a_1} 2^i (a_1 - 1 - i) = 2^{a_1} - a_1 - 1.$$

We now prove the claim (ii-b) by induction on the sum of the multiplicities for S_i satisfying $|S_i| \neq n$. The base case $\lambda_i = n + 1 - i$ for all $\lambda_i \neq n$ is proved above. For the general case, we apply (20). If $i_2 < i$, the computation with (20) again works as in the proof of Theorem 7. If $i_1 < i \leqslant i_2$, we also apply (20) to a square ${S_i}^2$ in $-J(\lambda)$. Namely, we have



with omitting the coefficients. The right-hand-side can be calculated by the induction hypothesis and the claim for the case (ii-a), and $J(\lambda) = (-1)^{n-s}(\mu_X, \tau_{S_1} \cdots \tau_{S_n})$ is

$$\begin{aligned}
&\binom{b_2}{c_2} \cdot \left\{ (2^{a_1} - a_1 - 1) \binom{b_1}{c_1} \binom{a_2 - 1}{c_2 - 1} y_3 \cdots y_s + \binom{b_1 - 1}{c_1} \binom{a_2 - 1}{c_2 - 1} y_3 \cdots y_s \right\} \\
&\quad + \binom{b_1}{c_1} \cdot \left\{ (2^{a_1} - a_1 - 1) \binom{a_2 - 1}{c_2} \binom{b_2}{c_2} y_3 \cdots y_s + \ 0 \right\} \\
&\quad + \binom{b_1 - 1}{c_1 + 1 - 1} \binom{a_2 - 1}{c_2} \binom{b_2}{c_2} y_3 \cdots y_s \\
&= (2^{a_1} - a_1 - 1) \binom{b_1}{c_1} y_2 \cdots y_s + \binom{b_1 - 1}{c_1} y_2 \cdots y_s.
\end{aligned}$$

□

For example, for $n = 5$ with the convention $\bar{k} = -k$, we can calculate

$$(\mu_X, {\tau_{\{\bar{1}\}}}^2 \tau_{\{\bar{1},3,4,5,\bar{2}\}}{}^3) = -4$$

by the case (ii-b) of Theorem 18 (See Figure 8).

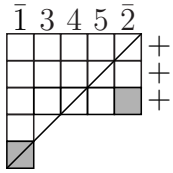


Figure 8: The Young diagram corresponding to ${\tau_{\{\bar{1}\}}}^2{\tau_{\{\bar{1},3,4,5,\bar{2}\}}}^3$

For each even signed permutation $u \in \widetilde{\mathfrak{S}}_n^+$, an element $i \in [n]$ satisfies $u(\alpha_i) \in \Phi^-$ if and only if

(D-1) if $i \leqslant n-2$, then $u(i) > u(i+1)$ with the same sign, or $u(i) < u(i+1)$ with different signs,

(D-2) if $i = n-1$, then $u(n-1), u(n) < 0$, or $u(n-1)$ and $u(n)$ have different signs and the absolute value of the negative one is less than the positive one,

(D-3) if $i = n$, then $u(n-1) > u(n)$ with the same sign, or $u(n-1) < u(n)$ with different signs.

Consider the similar condition

(A-1) if $i \leqslant n-2$, then $u(i) < u(i+1)$ with the same sign or $u(i) > u(i+1)$ with different signs,

(A-2) if $i = n-1$, then $u(n-1), u(n) > 0$, or $u(n-1)$ and $u(n)$ have different signs and the absolute value of the negative one is greater than the positive one,

(A-3) if $i \neq n$, then $u(n-1) < u(n)$ with the same sign or $u(n-1) > u(n)$ with different signs,

Denote

$$\begin{aligned} D(u) :=& \{u[i] \mid i \leqslant n-2 \text{ and } i \text{ satisfies (D)}\} \\ & \cup \{u[n]_+ \mid i = n-1 \text{ satisfies (D)}\} \cup \{u[n]_- \mid i = n \text{ satisfies (D)}\} \\ A(u) :=& \{u[i] \mid i \text{ satisfies (A)}\} \\ & \cup \{u[n]_+ \mid i = n-1 \text{ satisfies (A)}\} \cup \{u[n]_- \mid i = n \text{ satisfies (A)}\} \end{aligned}$$

where

$$[n]_+ = \{1, 2, \cdots, n-1, n\} \quad \text{and} \quad [n]_- = \{1, 2, \cdots, n-1, -n\}$$

(cf. Figure 7). We define a signed Young diagram $\lambda_{u,v}^w$ for $u, v, w \in \widetilde{\mathfrak{S}}_n^+$ in the manner described in Section 4. Note that we put $I(\emptyset) = 0$ as a convention.

Now, the intersection number of Y^w, X_u and X_v in X of type D_n is given by the following.

Corollary 20. *For even signed permutations $u, v, w \in \widetilde{\mathfrak{S}}_n^+$, we have*

$$(\mu_X, [Y^w][X_u][X_v]) = \widetilde{I}(\lambda_{uv}^w)$$

where $\widetilde{I} = (-1)^{n+s}\widetilde{y}_1 y_2 \cdots y_s$ is the function described in Theorem 18.

For example, for $n = 5$ with the convention $\bar{k} = -k$, the Young diagram $\lambda_{\bar{1}\bar{3}45\bar{2},\bar{1}345\bar{2}}^{\bar{1}\bar{2}543}$ is the one in Figure 8, and hence we obtain

$$(\mu_X, [Y^{\bar{1}\bar{2}543}][X_{\bar{1}345\bar{2}}][X_{\bar{1}345\bar{2}}]) = -4.$$

6 On exceptional types

In this section, we include the computation of intersection numbers of invariant divisors of the toric manifold X for the root system of exceptional type G_2. For other exceptional types F_4, E_6, E_7, and E_8, it would be interesting to find combinatorial objects which effectively compute the intersection numbers of invariant divisors.

Let $E = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$. The roots are

$$\begin{aligned} &\pm(t_1 - t_2),\ \pm(t_1 - t_3),\ \pm(t_2 - t_3), \\ &\pm(2t_1 - t_2 - t_3),\ \pm(2t_2 - t_1 - t_3),\ \pm(2t_3 - t_1 - t_2) \end{aligned}$$

where $t_i \in \mathbb{R}^3$ is the i-th standard vector. We choose $\Pi = \{t_1 - t_2, -2t_1 + t_2 + t_3\}$ as the set of simple roots, and write $\alpha_1 = t_1 - t_2$ and $\alpha_2 = -2t_1 + t_2 + t_3$. The Weyl group W is the dihedral group of order 12 which is identified with the subgroup

$$W_{G_2} := \{u \in \widetilde{\mathfrak{S}}_3 \mid u(1),\ u(2), \text{ and } u(3) \text{ have the same sign}\}$$

of the 3rd signed permutation group. Under this identification, the action of the Weyl group on E is written as the natural action of W_{G_2} on the indexes i of t_i; $u \cdot t = t_{u(1)}$ for $u \in W_{G_2}$ where $t_{-i} := -t_i$ $(1 \leqslant i \leqslant 3)$. This action of W_{G_2} preserves Φ. The minimal generators $\omega_1, \omega_2 \in E^*$ of the fundamental Weyl chamber σ_{id} are

$$\omega_1 = e_3 - e_2,\ \omega_2 = \frac{1}{3}(2e_3 - e_1 - e_2)$$

where $\{e_i\}_i \subset (\mathbb{R}^3)^*$ is the dual basis of $\{t_i\}_i \subset \mathbb{R}^3$.

Denoting by $2^{[\pm 3]}$ the set of all subsets of $[\pm 3] = \{1, 2, 3, -1, -2, -3\}$, we have a well-defined map $\Phi^* \to 2^{[n+1]}$ by sending

$$e_{u(3)} - e_{u(2)} \mapsto \{u(3), -u(2)\}, \quad \frac{1}{3}(2e_{u(3)} - e_{u(1)} - e_{u(2)}) \mapsto \{u(3)\}$$

for $u \in W_{G_2}$. This is an injection, and hence we can identify Φ^* with the following subset of $2^{[\pm 3]}$;

$$\mathcal{S} := \{3\bar{2}, \bar{3}2, 3\bar{1}, \bar{3}1, 2\bar{1}, \bar{2}1, 3, \bar{3}, 2, \bar{2}, 1, \bar{1}\}$$

where $\bar{k} = -k$ for $1 \leqslant k \leqslant 3$ and each sequence ab in $\mathcal{S}$ is the set $\{a, b\}$, i.e. $3\bar{2} = \{3, -2\}$ for example. Now, for each $S \in \mathcal{S}$, we have $\tau_S := \tau_{u\omega_i} \in H^2(X)$ where $u\omega_i \in \Phi^*$ corresponds to S by this identification. Then, for $S_1, S_2 \in \mathcal{S}$, it follows by Lemma 2 that $\tau_{S_1}\tau_{S_2} = 0$ unless these sets form a nested chain of subsets, i.e. $S_1 \subset S_2$ or $S_1 \supset S_2$.

The linear relations (3) for $\alpha = \alpha_1, \alpha_2$ are translated to

$$\begin{aligned}
&\tau_{3\bar{2}} + \tau_{\bar{3}1} + 2\tau_{\bar{2}1} + \tau_{\bar{2}} + \tau_1 = \tau_{\bar{3}2} + \tau_{3\bar{1}} + 2\tau_{2\bar{1}} + \tau_2 + \tau_{\bar{1}}, \\
&3\tau_{3\bar{1}} + 3\tau_{2\bar{1}} + \tau_3 + \tau_2 + 2\tau_{\bar{1}} = 3\tau_{\bar{3}1} + 3\tau_{\bar{2}1} + \tau_{\bar{3}} + \tau_{\bar{2}} + 2\tau_1,
\end{aligned}$$

respectively. From these relations together with the above observation about the vanishing of $\tau_{S_1}\tau_{S_2}$, we see that

$$\tau_{3\bar{2}}\tau_3 = 1,\ \tau_{3\bar{2}}\tau_{3\bar{2}} = -1,\ \tau_3\tau_3 = -3.$$

Now, let

$$I_{G_2}(2,1) := 1,\ I_{G_2}(1,1) := -3,\ I_{G_2}(2,2) := -1$$

where $(2,1)$, $(1,1)$, and $(2,2)$ are Young diagrams with 2 rows. Now the next claim follows from Lemma 3; if $S_1, S_2 \in \mathcal{S}$ form a nested chain of subsets, then we have

$$(\mu_X, \tau_{S_1}\tau_{S_2}) = I_{G_2}(\lambda) \tag{25}$$

where μ_X is the fundamental homology class and λ is the Young diagram consisting of $|S_1|$ and $|S_2|$ reordered as a weakly decreasing sequence. Otherwise, the intersection number is zero.

Finally, we list the presentations of $[X_u]$ as monomials of τ_S for all $u \in W_{G_2}$ in one-line notations;

$$\begin{aligned}
&[X_{123}] = 1, \quad [X_{213}] = \tau_{3\bar{1}}, [X_{132}] = \tau_{2\bar{3}}, [X_{231}] = \tau_{1\bar{3}}, [X_{312}] = \tau_2,\ \ [X_{321}] = \tau_1, \\
&[X_{\bar{1}\bar{2}\bar{3}}] = \tau_{\bar{3}2}\tau_{\bar{3}},\ [X_{\bar{2}\bar{1}\bar{3}}] = \tau_{\bar{3}},\ [X_{\bar{1}\bar{3}\bar{2}}] = \tau_{\bar{2}},\ [X_{\bar{2}\bar{3}\bar{1}}] = \tau_{\bar{1}},\ [X_{\bar{3}\bar{1}\bar{2}}] = \tau_{\bar{2}1},\ [X_{\bar{3}\bar{2}\bar{1}}] = \tau_{\bar{1}2}.
\end{aligned}$$

Since we have $[Y^u] = (w_0^{-1})^*[X_{w_0u}] = w_0^*[X_{w_0u}]$ where $w_0 = \bar{1}\bar{2}\bar{3}$ is the longest permutation, we obtain the list of $[Y^u]$;

$$\begin{aligned}
&[Y^{\bar{1}\bar{2}\bar{3}}] = 1, \quad [Y^{\bar{2}\bar{1}\bar{3}}] = \tau_{\bar{3}1}, [Y^{\bar{1}\bar{3}\bar{2}}] = \tau_{\bar{2}3}, [Y^{\bar{2}\bar{3}\bar{1}}] = \tau_{\bar{1}3}, [Y^{\bar{3}\bar{1}\bar{2}}] = \tau_{\bar{2}},\ \ [Y^{\bar{3}\bar{2}\bar{1}}] = \tau_{\bar{1}}, \\
&[Y^{123}] = \tau_{3\bar{2}}\tau_3,\ [Y^{213}] = \tau_3,\ [Y^{132}] = \tau_2,\ [Y^{231}] = \tau_1,\ [Y^{312}] = \tau_{2\bar{1}},\ [Y^{321}] = \tau_{1\bar{2}}.
\end{aligned}$$

With these lists, we can compute intersection numbers $(\mu_X, [Y^w][X_u][X_v])$ for all $u, v, w \in W_{G_2}$ by (25).

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