The complete cd-index of Boolean lattices

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Abstract

Let $[u, v]$ be a Bruhat interval of a Coxeter group such that the Bruhat graph $BG(u, v)$ of $[u, v]$ is isomorphic to a Boolean lattice. In this paper, we provide a combinatorial explanation for the coefficients of the complete **cd**-index of $[u, v]$. Since in this case the complete **cd**-index and the **cd**-index of $[u, v]$ coincide, we also obtain a new combinatorial interpretation for the coefficients of the **cd**-index of Boolean lattices. To this end, we label an edge in $BG(u, v)$ by a pair of nonnegative integers and show that there is a one-to-one correspondence between such sequences of nonnegative integer pairs and Bruhat paths in $BG(u, v)$. Based on this labeling, we construct a flip $\mathcal{F}$ on the set of Bruhat paths in $BG(u, v)$, which is an involution that changes the ascent-descent sequence of a path. Then we show that the flip $\mathcal{F}$ is compatible with any given reflection order and also satisfies the flip condition for any **cd**-monomial M. Thus by results of Karu, the coefficient of M enumerates certain Bruhat paths in $BG(u, v)$, and so can be interpreted as the number of certain sequences of nonnegative integer pairs. Moreover, we give two applications of the flip $\mathcal{F}$. We enumerate the number of **cd**-monomials in the complete **cd**-index of $[u, v]$ in terms of Entringer numbers, which are refined enumerations of Euler numbers. We also give a refined enumeration of the coefficient of $\mathbf{d}^n$ in terms of Poupard numbers, and so obtain new combinatorial interpretations for Poupard numbers and reduced tangent numbers.

Keywords: complete **cd**-index, **cd**-index, Boolean lattice, Bruhat graph

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1 Introduction

Let W be a Coxeter group and $u, v \in W$ such that $u < v$ in the Bruhat order. The complete **cd**-index $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ of the interval $[u, v]$ is a nonhomogeneous polynomial in the noncommuting variables $\mathbf{c}$ and $\mathbf{d}$, which was introduced by Billera and Brenti [2] and conjectured to have nonnegative coefficients, see also Billera [1]. The complete **cd**-index $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ encodes compactly the ascent-descent sequences of the Bruhat paths in the Bruhat graph $BG(u, v)$ of $[u, v]$, and its combinatorial invariance is equivalent to the combinatorial invariance of the celebrated Kazhdan-Lusztig and R-polynomials, see [2,12].

In [2], Billera and Brenti also showed that $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ is a generalization of the **cd**-index $\psi_{u,v}(\mathbf{c}, \mathbf{d})$ of $[u, v]$ in the sense that $\psi_{u,v}(\mathbf{c}, \mathbf{d})$ is the sum of the highest degree terms of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$. For the definition of the **cd**-index of an Eulerian poset, see, e.g., Stanley [16]. Purtill [15] gave a combinatorial interpretation for the **cd**-index $\psi_{B_n}(\mathbf{c}, \mathbf{d})$ of the Boolean lattice B_n by showing that $\psi_{B_n}(\mathbf{c}, \mathbf{d})$ is the sum of the **cd**-variation monomials of augmented André permutations on $[n] := \{1, 2, \ldots, n\}$, and then derived a recursive formula for $\psi_{B_n}(\mathbf{c}, \mathbf{d})$. Besides, $\psi_{B_n}(\mathbf{c}, \mathbf{d})$ is also a refined enumeration of simsun permutations, which were first introduced by Simion and Sundaram [17,18], see also Hetyei [9].

In this paper, we give a combinatorial interpretation for the coefficients of the complete **cd**-index of $[u, v]$, where $BG(u, v)$ is isomorphic to the Boolean lattice B_n. Since in this case all the edges in $BG(u, v)$ are covering relations, $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ has no lower degree terms. Hence we have

$$\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d}) = \psi_{u,v}(\mathbf{c}, \mathbf{d}) = \psi_{B_n}(\mathbf{c}, \mathbf{d}).$$

Thus we also obtain a new combinatorial interpretation for the coefficients of the **cd**-index of the Boolean lattice B_n.

To this end, we label a directed edge in $BG(u, v)$, say $x \to y$, by a pair of nonnegative integers (i, j), where i (resp. j) is the number of edges $y \to z$ ($y < z \leqslant v$) such that the reflection $y^{-1}z$ is larger (resp. smaller) than the reflection $x^{-1}y$ in a given reflection order $\mathcal{O}$. Then we show that there is a one-to-one correspondence between the Bruhat paths in $BG(u, v)$ and the sequences of nonnegative integer pairs

$$((i_1, j_1), (i_2, j_2), \ldots, (i_n, j_n)),$$

such that $i_k + j_k = n - k$ for $1 \leqslant k \leqslant n$.

Based on this labeling, we construct a flip $\mathcal{F}$ on the set of Bruhat paths in $BG(u, v)$, which is an involution that changes the ascent-descent sequence of a path. We show that the flip $\mathcal{F}$ is compatible with the reflection order $\mathcal{O}$ and also satisfies the flip condition for any **cd**-monomial M defined by Karu [11]. Then the coefficient of M enumerates certain Bruhat paths in $BG(u, v)$. Such paths are called valid paths and their corresponding sequences are called valid sequences. Therefore the coefficient of M is the number of certain valid paths in $BG(u, v)$ or certain valid sequences of nonnegative integer pairs.

We give two applications of the flip $\mathcal{F}$. Let E_n denote the Euler number, i.e., the number of up-down (or alternating) permutations on $[n]$. Denote by $E_n(k)$ the number of up-down permutations of length n beginning with k ($1 \leqslant k \leqslant n$). Clearly, we have

$E_n = \sum_{k=1}^{n} E_n(k)$. Purtill [15] showed that the number of **cd**-monomials in $\psi_{B_n}(\mathbf{c},\mathbf{d})$ is the Euler number E_n, that is,

$$\psi_{B_n}(1,1) = \widetilde{\psi}_{u,v}(1,1) = E_n.$$

We give a refined enumeration of $\widetilde{\psi}_{u,v}(1,1)$ in terms of the Entringer numbers $E_n(k)$. To be more specific, we show that the number of valid sequences of length n beginning with $(n-k,k-1)$ or the number of valid paths in $BG(u,v)$ with first edge labeled by $(n-k,k-1)$ is equal to $E_n(k)$.

As the second application, we give a refined enumeration of the coefficient of $\mathbf{d}^n$ in $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ in terms of the Poupard numbers $P_n(k)$ $(1 \leqslant k \leqslant 2n+1)$, where $BG(u,v)$ is isomorphic to B_{2n+1}. The paths in $BG(u,v)$ corresponding to the monomial $\mathbf{d}^n$ are called alternating paths. We show that the number of alternating paths with first edge labeled by $(2n-k+1,k-1)$ is equal to the Poupard number $P_n(k)$. Since by [8], the reduced tangent number t_n satisfies $t_n = \sum_{k=1}^{2n+1} P_n(k)$, we deduce that the coefficient of $\mathbf{d}^n$ is the reduced tangent number t_n. Therefore we obtain new combinatorial interpretations for the Poupard numbers and reduced tangent numbers.

The organization of this paper is as follows. In Section 2, we give some basic notation and definitions on Coxeter groups and the complete **cd**-index. We also recall some results of Karu. In Section 3, we first construct a flip $\mathcal{F}$ on $[u,v]$ and show that this flip is compatible with the given reflection order and satisfies the flip condition. Then we provide a combinatorial interpretation for the coefficient of $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$. In Section 4, We give a refined enumeration of the number of **cd**-monomials in $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$. Finally, in Section 5, we give a refined enumeration of the coefficient of $\mathbf{d}^n$ in $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$.

2 Preliminary

Let (W,S) be a Coxeter system, and let $T = \{wsw^{-1} \mid s \in S,\ w \in W\}$ be the set of reflections, see, e.g., Humphreys [10]. We use $\ell(w)$ to denote the length of $w \in W$. For $u,v \in W$, we say that $u \leqslant v$ in the Bruhat order if there exists a sequence of reflections $t_1,t_2,\ldots,t_r$ in T such that (i) $v = u\,t_1\,t_2\cdots t_r$ and (ii) $\ell(u\,t_1\cdots t_{i-1}) < \ell(u\,t_1\cdots t_i)$ for $1 \leqslant i \leqslant r$. We say that u is covered by v, if $u < v$ and $\ell(v) = \ell(u)+1$. Let $[u,v] = \{w \in W \mid u \leqslant w \leqslant v\}$ be the interval formed by u and v in the Bruhat order. The atoms of $[u,v]$ are the elements $w \in [u,v]$ such that w covers u.

The Bruhat graph $BG(W)$ of the Coxeter group W is a directed graph whose vertices are the elements of W and there is a directed edge from u to v, denoted by $u \to v$, if $v = ut$ for some reflection $t \in T$ and $\ell(u) < \ell(v)$. The interval $[u,v]$ forms a subgraph $BG(u,v)$ of the Bruaht graph of W. A Bruhat path of length n from u to v in $BG(u,v)$ is a sequence

$$x = (u = x_0 \to x_1 \to \cdots \to x_{n-1} \to x_n = v) \tag{1}$$

such that $t_i = x_{i-1}^{-1}x_i \in T$ for $1 \leqslant i \leqslant n$. We call $(t_1,t_2,\ldots,t_n)$ the reflection sequence of x, and call the first edge $u \to x_1$ of x a first edge of the interval $[u,v]$. Let $B_k(u,v)$ denote the set of Bruhat paths of length k from u to v, and let $B(u,v) = \bigcup_k B_k(u,v)$.

Recall that a reflection order $(\mathcal{O}, \prec)$ is a total order defined on the set of reflections, see Dyer [4]. The reverse of the order $\mathcal{O}$, denoted by $\overline{\mathcal{O}}$, is also a reflection order. Throughout this paper, we shall always use a given reflection order $(\mathcal{O}, \prec)$. We say that the path x in (1) is increasing if $t_1 \prec t_2 \prec \cdots \prec t_n$, and decreasing if $t_1 \succ t_2 \succ \cdots \succ t_n$. Dyer [4] showed that each Bruhat interval $[u, v]$ is shellable. That is, there is a unique increasing (resp. decreasing) path of length n, say x (resp. y), and the reflection sequence of x (resp. y) is the lexicographically smallest (resp. largest) among all the reflection sequences of paths in $B_n(u, v)$.

The following result is due to Dyer [5].

Theorem 1. *Let $x = (u \to x_1 \to \cdots \to x_{n-1} \to v)$ be an increasing path in $B_n(u, v)$, and $y = (u \to y_1 \to \cdots \to y_{m-1} \to v)$ be a decreasing path in $B_m(u, v)$. Then we have*

$$u^{-1}x_1 \prec u^{-1}y_1 \text{ and } y_{m-1}^{-1}v \prec x_{n-1}^{-1}v.$$

The ascent-descent sequence of the Bruhat path x is a monomial in the noncommuting variables $\mathbf{a}$ and $\mathbf{b}$ defined by

$$w(x) = w_1 w_2 \cdots w_{n-1},$$

where

$$w_i = \begin{cases} \mathbf{a}, & \text{if } t_i \prec t_{i+1}; \\ \mathbf{b}, & \text{if } t_i \succ t_{i+1}. \end{cases}$$

The $\mathbf{ab}$-index $\widetilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ of the interval $[u, v]$ is the polynomial obtained by summing the ascent-descent sequences of all the Bruhat paths from u to v:

$$\widetilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b}) = \sum_{x \in B(u,v)} w(x).$$

The complete $\mathbf{cd}$-index $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ of the interval $[u, v]$ is obtained by a change of variable in the $\mathbf{ab}$-index $\widetilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ of $[u, v]$. Let $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. Billera and Brenti [2] showed that $\widetilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b})$ can be expressed in terms of $\mathbf{c}$ and $\mathbf{d}$:

$$\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d}) = \widetilde{\psi}_{u,v}(\mathbf{a} + \mathbf{b}, \mathbf{ab} + \mathbf{ba}) = \widetilde{\phi}_{u,v}(\mathbf{a}, \mathbf{b}).$$

It can be shown that $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ does not depend on the reflection order.

Now we proceed to recall some definitions and results in [11].

For an $\mathbf{ab}$-monomial M, denote by $\overline{M}$ the $\mathbf{ab}$-monomial obtained by exchanging $\mathbf{a}$ and $\mathbf{b}$ in M. This operator is an involution on the noncommutative ring $\mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle$.

Definition 2. A flip $F = F_{u,v}$ on $[u, v]$ is defined to be an involution

$$F_{u,v} : B(u, v) \to B(u, v),$$

such that $w(F(x)) = \overline{w(x)}$ for any path $x \in B(u, v)$.

Fix a flip for every interval in the Bruhat graph of W. Let $1 \leqslant m \leqslant n$ and

$$x = (u = x_0 \to x_1 \to \cdots \to x_m \to x_{m+1} \to \cdots \to x_n \to x_{n+1} = v)$$

be a path in $B(u,v)$. After applying the flip $F_{x_m,v}$ to x, we obtain

$$y = (u = x_0 \to x_1 \to \cdots \to x_m \to y_{m+1} \to \cdots \to y_n \to y_{n+1} = v).$$

If $w(x) = \beta_1 \cdots \beta_m \cdots \beta_n$, then $w(y) = \beta_1 \cdots \beta_{m-1} \alpha_m \overline{\beta}_{m+1} \cdots \overline{\beta}_n$, where α_m can be either $\mathbf{a}$ or $\mathbf{b}$. Define

$$s_{m,\mathbf{a}}(x) = \begin{cases} 1, & \text{if } \beta_m = \mathbf{a}; \\ 0, & \text{otherwise.} \end{cases}$$

$$s_{m,\mathbf{b}}(x) = \begin{cases} 1, & \text{if } \beta_m = \mathbf{b}, \alpha_m = \mathbf{a}; \\ -1, & \text{if } \beta_m = \mathbf{a}, \alpha_m = \mathbf{b}; \\ 0, & \text{otherwise.} \end{cases}$$

Let the variables $\mathbf{a}, \mathbf{b}, \mathbf{c}$ have degree 1, and let the variable $\mathbf{d}$ have degree 2. Given a $\mathbf{cd}$-monomial $M(\mathbf{c}, \mathbf{d})$ of degree n, we can obtain a unique $\mathbf{ab}$-monomial $M(\mathbf{a}, \mathbf{ba})$ of degree n by substituting $\mathbf{a}$ for $\mathbf{c}$ and $\mathbf{ba}$ for $\mathbf{d}$ in $M(\mathbf{c}, \mathbf{d})$. Clearly, this is a one-to-one correspondence between $\mathbf{cd}$-monomials and $\mathbf{ab}$-monomials in which every $\mathbf{b}$ is followed by an $\mathbf{a}$.

Definition 3. Let $M(\mathbf{c}, \mathbf{d})$ be a $\mathbf{cd}$-monomial such that $M(\mathbf{a}, \mathbf{ba}) = \gamma_1 \gamma_2 \cdots \gamma_n$. Define

$$s_M(x) = \prod_{m=1}^{n} s_{m,\gamma_m}(x).$$

Note that $s_{m,\gamma_m}(x)$, and hence $s_M(x)$, depend on both the reflection order and the given flip. Denote by $\overline{s}_{m,\gamma_m}(x)$ the value of $s_{m,\gamma_m}(x)$ by using the reverse reflection order $\overline{\mathcal{O}}$, and let $\overline{s}_M(x) = \prod_{m=1}^{n} \overline{s}_{m,\gamma_m}(x)$.

Definition 4. A flip F is said to be compatible with the reflection order $\mathcal{O}$ if

$$s_M(x) = \overline{s}_M(F(x))$$

for any interval $[u,v]$, any path $x \in B(u,v)$ and any $\mathbf{cd}$-monomial M.

Theorem 5. *Assume that the flip F is compatible with the reflection order $\mathcal{O}$. For any $\mathbf{cd}$-monomial M of degree n, the coefficient of M in $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ is equal to*

$$\sum_{x \in B_{n+1}(u,v)} s_M(x).$$

If -1 does not appear in the above sum, then the coefficient of M is clearly nonnegative. Therefore Karu [11] introduced the following flip condition.

Definition 6. The flip condition holds for the interval $[u, v]$ and monomial M if for every path $x \in B(u, v)$ the following is satisfied. If $s_{m,\gamma_m}(x) = -1$ for some m, then there exists $k > m$ such that $s_{k,\gamma_k}(x) = 0$.

Definition 7. Let $M(\mathbf{c}, \mathbf{d})$ be a $\mathbf{cd}$-monomial of degree n with $M(\mathbf{a}, \mathbf{ba}) = \gamma_1\gamma_2 \cdots \gamma_n$. Define

$$T_M(u, v) = \{x \in B_{n+1}(u, v) \,|\, s_{m,\gamma_m}(x) = 1, \text{ for all } 1 \leqslant m \leqslant n\}.$$

From Theorem 5 we have

Corollary 8. *If the flip condition holds for the interval $[u, v]$ and monomial M, then the coefficient of M in $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ is equal to $|T_M(u, v)|$ and hence is nonnegative.*

In [11], Karu proved that when M contains at most one $\mathbf{d}$, that is, $M = \mathbf{c}^i$ or $M = \mathbf{c}^i\mathbf{d}\mathbf{c}^j$ $(i, j \geqslant 0)$, the flip condition holds by Theorem 1. Then the coefficient of M is nonnegative by Corollary 8. Recently, the authors showed that when $M = \mathbf{d}\mathbf{c}^i\mathbf{d}\mathbf{c}^j$ $(i, j \geqslant 0)$, the coefficient of M is also nonnegative, see [7].

3 Combinatorial interpretation of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$

In this section, we first give a labeling scheme for the edges in $BG(u, v)$ and then construct a flip $\mathcal{F}$ on the set of paths in $B(u, v)$ based on the labels of the paths. By using the flip $\mathcal{F}$, we provide a combinatorial interpretation for the coefficients of $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$.

Let $[u, v]$ be a Bruhat interval such that $BG(u, v)$ is isomorphic to the Boolean lattice B_n. In the following, we shall always refer to $[u, v]$ as such an interval if there is no further notification. Note that every edge in $BG(u, v)$ is a covering relation. In fact, if there is an edge $u_1 \to u_2$ in $BG(u, v)$ such that $u_2 = u_1 t$ for some $t \in T$ and $\ell(u_2) - \ell(u_1) > 1$, then we must have $\ell(u_2) - \ell(u_1) = 2k + 1$ for some $k \geqslant 1$. This implies $BG(u_1, u_2)$ would not be isomorphic to a Boolean lattice. Hence $B(u, v) = B_n(u, v)$, i.e., all the paths from u to v have length n.

Suppose that

$$x = (u = x_0 \to x_1 \to x_2 \to \cdots \to x_n = v)$$

is a path in $B(u, v)$. For $1 \leqslant k \leqslant n$, label the edge $x_{k-1} \to x_k$ by an integer pair (i_k, j_k), where i_k (resp. j_k) is the number of first edges $x_k \to z$ in the interval $[x_k, v]$ such that $x_k^{-1}z \succ x_{k-1}^{-1}x_k$ (resp. $x_k^{-1}z \prec x_{k-1}^{-1}x_k$). Call the sequence $((i_1, j_1), \ldots, (i_n, j_n))$ the label sequence of the path x.

Proposition 9. *Suppose that the atoms of $[u, v]$ are $u_1, u_2, \ldots, u_n$. Then the set of labels of the first edges $u \to u_k$ $(1 \leqslant k \leqslant n)$ of $[u, v]$ is*

$$\{(i, j) \,|\, i + j = n - 1, 0 \leqslant i \leqslant n - 1\}.$$

Moreover, the label of $u \to u_r$ is lexicographically smaller than the label of $u \to u_k$ if and only if $u^{-1}u_r \succ u^{-1}u_k$.

Proof. We make induction on n. By Theorem 1, it is easy to see that the proposition holds for $n = 2$. Now assume that $n > 2$. Since $[u, v]$ is shellable, there is a unique increasing (resp. decreasing) path x (resp. y), and the reflection sequence of x (resp. y) is the lexicographically smallest (resp. largest). Without loss of generality, we can assume

$$x = (u \to u_1 \to x_2 \to \cdots \to x_n = v)$$

and

$$y = (u \to u_n \to y_2 \to \cdots \to y_n = v).$$

So the reflection $u^{-1}u_1$ (resp. $u^{-1}u_n$) is the minimum (resp. maximum) among all the n first edges of $[u, v]$. Moreover, since x is increasing, $u^{-1}u_1 \prec u_1^{-1}x_2$ and $u_1^{-1}x_2$ is the minimum among all the $n-1$ first edges of $[u_1, v]$. Similarly, since y is decreasing, $u^{-1}u_n \succ u_n^{-1}y_2$ and $u_n^{-1}y_2$ is the maximum among all the $n-1$ first edges in $[u_n, v]$. Therefore, the label of the edge $u \to u_1$ is $(n-1, 0)$ and the label of the edge $u \to u_n$ is $(0, n-1)$.

Notice that u and the atoms $u_2, u_3, \ldots, u_{n-1}$ determine a Boolean lattice B_{n-2}. By induction, the set of labels of the first edges $u \to u_k$ $(2 \leqslant k \leqslant n-1)$ in the Boolean lattice B_{n-2} is

$$\{(s, r) \mid s + r = n - 3,\ 0 \leqslant s \leqslant n - 3\}.$$

Since $BG(u, v)$ is a Boolean lattice, for each atom u_k $(2 \leqslant k \leqslant n-1)$, there exists $z_k \in [u, v]$ such that $u_k \to z_k$ and $u_1 \to z_k$. Since $\ell(z_k) - \ell(u) = 2$, by [3, Lemma 2.7.3], the interval $[u, z_k]$ has exactly two paths. Since $u^{-1}u_1 \prec u_1^{-1}z_k$ and $[u, z_k]$ is shellable, we have $u^{-1}u_k \succ u_k^{-1}z_k$. Similarly, there exists $w_k \in [u, v]$ such that $u_k \to w_k$ and $u_n \to w_k$. Since $u^{-1}u_n \succ u_n^{-1}w_k$, we see that $u^{-1}u_k \prec u_k^{-1}w_k$. That is to say, if the edge $u \to u_k$ has label (s, r) in B_{n-2}, then its label would be $(s+1, r+1)$ in B_n. Therefore, in the Boolean lattice B_n, the set of labels of the first edges is

$$\{(i, j) \mid i + j = n - 1,\ 0 \leqslant i \leqslant n - 1\}.$$

It is clear that $u^{-1}u_1$ (resp. $u^{-1}u_n$) is the minimum (resp. maximum) among all the first edges of $[u, v]$ in the reflection order $\mathcal{O}$, and the label of $u \to u_1$ (resp. $u \to u_n$) is the largest (resp. smallest) in the lexicographic order. By induction, for $2 \leqslant k, r \leqslant n-1$, the label of $u \to u_k$ is lexicographically smaller than the label of $u \to u_r$ if and only if $u^{-1}u_k \succ u^{-1}u_r$ in B_{n-2}. Consequently, for $1 \leqslant k, r \leqslant n$, the label of $u \to u_k$ is lexicographically smaller than the label of $u \to u_r$ if and only if $u^{-1}u_k \succ u^{-1}u_r$ in B_n. This completes the proof. □

Remark 10. According to Proposition 9, if we arrange the labels of the first edges $u \to u_1, \ldots, u \to u_n$ of $[u, v]$ decreasingly in the lexicographic order, then the corresponding reflections $u^{-1}u_1, \ldots, u^{-1}u_n$ are arranged increasingly in the reflection order. Thus, without loss of generality, we can require the edge $u \to u_k$ to have label $(n-k, k-1)$ for $1 \leqslant k \leqslant n$.

Corollary 11. *There is a bijection between the Bruhat paths in $B(u,v)$ and sequences of nonnegative integer pairs $((i_1,j_1),\dots,(i_n,j_n))$ such that $i_k+j_k=n-k$ for $1\leqslant k\leqslant n$. In other words, a label sequence determines a unique path in $B(u,v)$ and vice versa.*

Proposition 12. *Let $u_{k-1}\to u_k\to u_{k+1}$ be two adjacent edges in $B(u,v)$ such that the edge $u_{k-1}\to u_k$ has label (i_k,j_k) and the edge $u_k\to u_{k+1}$ has label (i_{k+1},j_{k+1}). Then*

(1) $i_k>i_{k+1}$ if and only if $u_{k-1}^{-1}u_k\prec u_k^{-1}u_{k+1}$.

(2) $j_k>j_{k+1}$ if and only if $u_{k-1}^{-1}u_k\succ u_k^{-1}u_{k+1}$.

Proof. (1) Since there are i_k edges among the first edges of $[u_k,v]$ which are larger than $u_{k-1}^{-1}u_k$, by Remark 10, we see that the labels of these i_k edges are (i,j) such that $0\leqslant i\leqslant i_k-1$. It follows that $i_k>i_{k+1}$ if and only if $u_{k-1}^{-1}u_k\prec u_k^{-1}u_{k+1}$. (2) Notice that $i_k+j_k=i_{k+1}+j_{k+1}+1$, then $i_k>i_{k+1}$ if and only if $j_k\leqslant j_{k+1}$. Thus $j_k>j_{k+1}$ if and only if $u_{k-1}^{-1}u_k\succ u_k^{-1}u_{k+1}$. ☐

Now we can define a flip on $B(u,v)$ according to the labels of the edges.

Definition 13. Let

$$x=(u=x_0\to x_1\to x_2\to\cdots\to x_n=v)$$

be a path in $B(u,v)$ with label sequence $((i_1,j_1),(i_2,j_2),\dots,(i_n,j_n))$. Define

$$\mathcal{F}:B(u,v)\to B(u,v)$$

as follows.

(1) If $\min\{i_1,j_1\}\leqslant\min\{i_2,j_2\}$, then let

$$\mathcal{F}(x)=(u=x_0\to y_1\to y_2\to\cdots\to y_n=v)$$

such that the label sequence of $\mathcal{F}(x)$ is $((j_1,i_1),(j_2,i_2),\dots,(j_n,i_n))$.

(2) If $\min\{i_1,j_1\}>\min\{i_2,j_2\}$, then let

$$\mathcal{F}(x)=(u=x_0\to x_1\to y_2\to\cdots\to y_n=v)$$

such that the label sequence of $\mathcal{F}(x)$ is $((i_1,j_1),(j_2,i_2),\dots,(j_n,i_n))$.

By Proposition 12, it is easy to see that $\mathcal{F}$ is a flip on $B(u,v)$. For example, let $[u,v]$ be an interval such that $BG(u,v)$ is isomorphic to B_4, see Figure 1. By Remark 10, we can label the first edges $u\to u_1,u\to u_2,u\to u_3,u\to u_4$ of $[u,v]$ by $(3,0),(2,1),(1,2),(0,3)$ respectively. The first edges $u_1\to u_5,u_1\to u_6,u_1\to u_7$ of $[u_1,v]$ are labeled by $(2,0),(1,1),(0,2)$ respectively. The images of the flip $\mathcal{F}$ on some paths in $B(u,v)$ are listed below.

$$\begin{aligned}
\mathcal{F}&:(u\to u_1\to u_5\to v_1\to v)\mapsto(u\to u_4\to u_{10}\to v_4\to v),\\
\mathcal{F}&:(u\to u_2\to u_5\to v_1\to v)\mapsto(u\to u_2\to u_9\to v_4\to v),\\
\mathcal{F}&:(u\to u_2\to u_8\to v_1\to v)\mapsto(u\to u_3\to u_8\to v_4\to v),\\
\mathcal{F}&:(u\to u_3\to u_6\to v_3\to v)\mapsto(u\to u_3\to u_{10}\to v_3\to v).
\end{aligned}$$

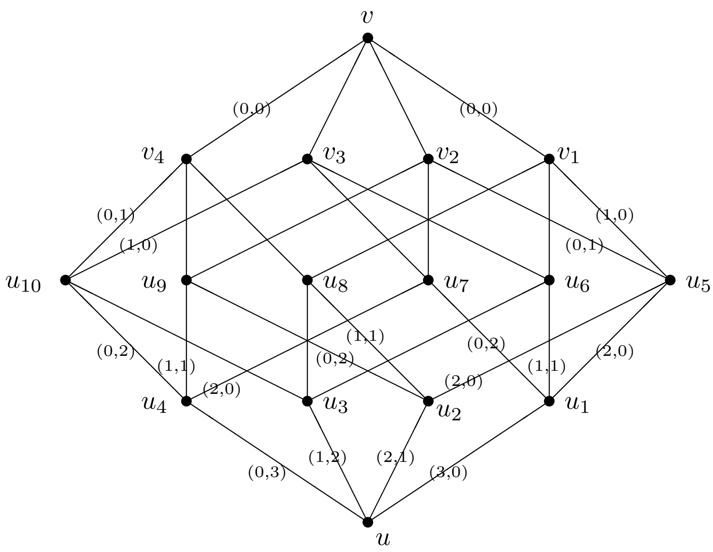

Figure 1: The Boolean lattice B_4.

Theorem 14. *The flip $\mathcal{F}$ is compatible with the reflection order $\mathcal{O}$ and also satisfies the flip condition for any* **cd***-monomial M.*

Proof. We first show that the flip $\mathcal{F}$ satisfies the flip condition for any **cd**-monomial M. We claim that $s_{m,\mathbf{b}}(x) \neq -1$ for any m, any **cd**-monomial M and any path x in $B(u,v)$. Suppose to the contrary that there exists a path x in $B(u,v)$ such that $s_{m,\mathbf{b}}(x) = -1$ for some integer m. Let

$$x = (u \to x_1 \to \cdots \to x_{m-1} \to x_m \to x_{m+1} \to x_{m+2} \to \cdots \to v). \tag{2}$$

Since each $\mathbf{b}$ follows by an $\mathbf{a}$ in $M(\mathbf{a}, \mathbf{ba})$, by the definitions of $s_{m,\mathbf{b}}(x)$ and $s_{m,\mathbf{a}}(x)$, we can assume that $x_{m-1}^{-1}x_m \prec x_m^{-1}x_{m+1} \prec x_{m+1}^{-1}x_{m+2}$. For $k = m, m+1, m+2$, let the label of the edge $x_{k-1} \to x_k$ be (i_k, j_k). Then by Proposition 12, we have $i_m > i_{m+1} > i_{m+2}$.

To calculate $s_{m,\mathbf{b}}(x)$, we need to flip the path x at x_m. Let

$$y = \mathcal{F}_{x_m,v}(x) = (u \to x_1 \to \cdots \to x_{m-1} \to x_m \to y_{m+1} \to y_{m+2} \to \cdots \to v). \tag{3}$$

If $x_{m+1} = y_{m+1}$ then we have $s_{m,\mathbf{b}}(x) = 0$, a contradiction. If $x_{m+1} \neq y_{m+1}$, then by the definition of the flip $\mathcal{F}$, we find that $\min\{i_{m+1}, j_{m+1}\} \leqslant \min\{i_{m+2}, j_{m+2}\}$. Since $i_{m+1} > i_{m+2}$, we have $j_{m+1} \leqslant i_{m+2}$ and the label of the edge $x_m \to y_{m+1}$ must be (j_{m+1}, i_{m+1}). Then we obtain $i_m > i_{m+1} > i_{m+2} \geqslant j_{m+1}$. Thus by Proposition 12, $x_{m-1}^{-1}x_m \prec x_m^{-1}y_{m+1}$, and so $s_{m,\mathbf{b}}(x) = 0$ again, which is a contradiction. This means $s_{m,\mathbf{b}}(x) \neq -1$ for any m. Thus the flip $\mathcal{F}$ satisfies the flip condition.

Now we proceed to show that the flip $\mathcal{F}$ is compatible with the reflection order $\mathcal{O}$. Let $M(\mathbf{c}, \mathbf{d})$ be a **cd**-monomial with $M(\mathbf{a}, \mathbf{ba}) = \gamma_1 \cdots \gamma_{n-1}$. It suffices to show that for any integer $m \in [n-1]$ and any path p in $B(u,v)$, we have $s_{m,\gamma_m}(p) = \overline{s}_{m,\gamma_m}(\mathcal{F}(p))$.

Assume that

$$p = (u \to p_1 \to \cdots \to p_{m-1} \to p_m \to p_{m+1} \to \cdots \to p_{n-1} \to v)$$

is a path in $B(u, v)$ with label sequence

$$((r_1, s_1), \ldots, (r_m, s_m), (r_{m+1}, s_{m+1}), \ldots, (r_n, s_n))$$

in the reflection order $\mathcal{O}$. And let

$$\mathcal{F}(p) = (u \to q_1 \to \cdots \to q_{m-1} \to q_m \to q_{m+1} \to \cdots \to q_{n-1} \to v).$$

By the definition of the flip $\mathcal{F}$, the label sequence of $\mathcal{F}(p)$ is

$$\begin{cases} ((r_1, s_1), (s_2, r_2), \ldots, (s_m, r_m), (s_{m+1}, r_{m+1}), \ldots, (s_n, r_n)), & \text{if } p_1 = q_1; \\ ((s_1, r_1), (s_2, r_2), \ldots, (s_m, r_m), (s_{m+1}, r_{m+1}), \ldots, (s_n, r_n)), & \text{if } p_1 \neq q_1. \end{cases}$$

It is easy to check that if $\gamma_m = \mathbf{a}$, then $s_{m,\mathbf{a}}(p) = \overline{s}_{m,\mathbf{a}}(\mathcal{F}(p))$. Now we consider the case $\gamma_m = \mathbf{b}$. To compute $s_{m,\mathbf{b}}(p)$, we need to flip the path p at p_m. Let

$$p' = \mathcal{F}_{p_m,v}(p) = (u \to p_1 \to \cdots \to p_{m-1} \to p_m \to p'_{m+1} \to \cdots \to p'_{n-1} \to v).$$

Then the label sequence of p' is

$$\begin{cases} ((r_1, s_1), \ldots, (r_m, s_m), (r_{m+1}, s_{m+1}), (s_{m+2}, r_{m+2}), \ldots, (s_n, r_n)), & \text{if } p_{m+1} = p'_{m+1}; \\ ((r_1, s_1), \ldots, (r_m, s_m), (s_{m+1}, r_{m+1}), (s_{m+2}, r_{m+2}), \ldots, (s_n, r_n)), & \text{if } p_{m+1} \neq p'_{m+1}. \end{cases}$$

To calculate $\overline{s}_{m,\mathbf{b}}(\mathcal{F}(p))$, we need to flip $\mathcal{F}(p)$ at q_m. Let

$$\mathcal{F}_{q_m,v}(\mathcal{F}(p)) = (u \to q_1 \to \cdots \to q_{m-1} \to q_m \to q'_{m+1} \to \cdots \to q'_{n-1} \to v).$$

Then the label sequence of $\mathcal{F}_{q_m,v}(\mathcal{F}(p))$ is

$$\begin{cases} ((r_1, s_1), (s_2, r_2), \ldots, (s_m, r_m), (s_{m+1}, r_{m+1}), \ldots, (r_n, s_n)), & \text{if } p_1 = q_1,\ p_{m+1} = p'_{m+1}; \\ ((r_1, s_1), (s_2, r_2), \ldots, (s_m, r_m), (r_{m+1}, s_{m+1}), \ldots, (r_n, s_n)), & \text{if } p_1 = q_1,\ p_{m+1} \neq p'_{m+1}; \\ ((s_1, r_1), (s_2, r_2), \ldots, (s_m, r_m), (s_{m+1}, r_{m+1}), \ldots, (r_n, s_n)), & \text{if } p_1 \neq q_1,\ p_{m+1} = p'_{m+1}; \\ ((s_1, r_1), (s_2, r_2), \ldots, (s_m, r_m), (r_{m+1}, s_{m+1}), \ldots, (r_n, s_n)), & \text{if } p_1 \neq q_1,\ p_{m+1} \neq p'_{m+1}. \end{cases}$$

It is not hard to check that the paths $\mathcal{F}(p')$ and $\mathcal{F}_{q_m,v}(\mathcal{F}(p))$ have the same label sequence. Therefore,

$$\mathcal{F}(p') = \mathcal{F}_{q_m,v}(\mathcal{F}(p)).$$

That is, if we flip the path p at p_m to obtain p', then under the flip $\mathcal{F}$, we shall flip $\mathcal{F}(p)$ at q_m to obtain $\mathcal{F}(p')$.

Now we are prepared to show that $s_{m,\mathbf{b}}(p) = \overline{s}_{m,\mathbf{b}}(\mathcal{F}(p))$. If $m = 1$ and $p_1 = q_1$, then $p' = \mathcal{F}(p)$ and $p = \mathcal{F}(p')$. It is obvious that $s_{1,\mathbf{b}}(p) = \overline{s}_{1,\mathbf{b}}(\mathcal{F}(p))$. If $m = 1$ and $p_1 \neq q_1$ or $m > 1$, according to the label sequences of the paths $p, p', \mathcal{F}(p)$ and $\mathcal{F}(p')$ one can check that: (i) If $p_{m+1} = p'_{m+1}$, then $s_{m,\mathbf{b}}(p) = \overline{s}_{m,\mathbf{b}}(\mathcal{F}(p)) = 0$. (ii) If $p_{m+1} \neq p'_{m+1}$, then we also have $s_{m,\mathbf{b}}(p) = \overline{s}_{m,\mathbf{b}}(\mathcal{F}(p))$. This completes the proof. □

We are ready to give a combinatorial interpretation for the coefficients of $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$.

Theorem 15. *Let M be a* $\mathbf{cd}$*-monomial of degree $n-1$ such that $M(\mathbf{a},\mathbf{ba})=\gamma_1\cdots\gamma_{n-1}$. Then the coefficient of M in $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ is equal to the number of sequences*

$$((i_1,j_1),(i_2,j_2),\ldots,(i_n,j_n))$$

satisfying the following conditions:

(1) For $1\leqslant m\leqslant n$, $i_m,j_m\geqslant 0$ and $i_m+j_m=n-m$.

(2) For $1\leqslant m\leqslant n-1$, if $\gamma_m=\mathbf{a}$ then $i_m>i_{m+1}$.

(3) For $1\leqslant m\leqslant n-2$, if $\gamma_m=\mathbf{b}$ then $i_m,j_m>j_{m+1}$ and $i_{m+2}\geqslant j_{m+1}$.

Proof. According to Theorem 5, Corollary 8 and Theorem 14, the coefficient of M is equal to the number of paths x in $B(u,v)$ such that $s_M(x)=1$. Since a path is uniquely determined by its label sequence, we turn to the characterization of the corresponding sequences of nonnegative integer pairs $((i_1,j_1),(i_2,j_2),\ldots,(i_n,j_n))$ with $i_k+j_k=n-k$ for $1\leqslant k\leqslant n$.

Let x be the path in (2) with the edge $x_{m-1}\to x_m$ labeled by (i_m,j_m) for $1\leqslant m\leqslant n$. If $\gamma_m=\mathbf{a}$, then $s_{m,\mathbf{a}}(x)=1$ only if $x_{m-1}^{-1}x_m\prec x_m^{-1}x_{m+1}$. By Proposition 12, we have $i_m>i_{m+1}$. If $\gamma_m=\mathbf{b}$, then $\gamma_{m+1}=\mathbf{a}$. By Proposition 12, we have $i_{m+1}>i_{m+2}$. Since $i_{m+1}+j_{m+1}=i_{m+2}+j_{m+2}+1$, we see that $j_{m+1}\leqslant j_{m+2}$. After applying $\mathcal{F}_{x_m,v}$ to x, we get the path y as in (3). We see that $s_{m,\mathbf{b}}(x)=1$ only if

$$x_{m-1}^{-1}x_m\succ x_m^{-1}x_{m+1},\quad x_{m-1}^{-1}x_m\prec x_m^{-1}y_{m+1}\quad\text{and}\quad x_{m+1}\neq y_{m+1}.$$

Then the label of the edge $x_m\to y_{m+1}$ is (j_{m+1},i_{m+1}). By Proposition 12 and the definition of the flip $\mathcal{F}$, we find

$$i_m,j_m>j_{m+1}\ \text{and}\ \min\{i_{m+1},j_{m+1}\}\leqslant\min\{i_{m+2},j_{m+2}\}.$$

Combining with the facts $i_{m+1}>i_{m+2}$ and $j_{m+1}\leqslant j_{m+2}$, we obtain

$$i_m,j_m>j_{m+1}\ \text{and}\ i_{m+1}>i_{m+2}\geqslant j_{m+1}. \tag{4}$$

This completes the proof. □

For example, let $[u,v]$ be a Bruhat interval such that $BG(u,v)$ is isomorphic to B_5. And let $M=\mathbf{d}^2$ with $M(\mathbf{a},\mathbf{ba})=\mathbf{baba}$. There are 4 sequences corresponding to M that satisfy the conditions in Theorem 15, namely,

$$((1,3),(3,0),(1,1),(1,0),(0,0)),\ ((2,2),(2,1),(1,1),(1,0),(0,0)),$$
$$((2,2),(3,0),(1,1),(1,0),(0,0)),\ ((3,1),(3,0),(1,1),(1,0),(0,0)).$$

Then the coefficient of M in $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ is 4. In fact,

$$\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})=\mathbf{c}^4+3\mathbf{c}^2\mathbf{d}+5\mathbf{cdc}+3\mathbf{dc}^2+4\mathbf{d}^2.$$

$m=$	1	2	3	4	5	6	7	E_n
$n=1$	1							1
2	1	0						1
3	1	1	0					2
4	2	2	1	0				5
5	5	5	4	2	0			16
6	16	16	14	10	5	0		61
7	61	61	56	46	32	16	0	272

Table 1: The Entringer Numbers $E_n(k)$.

4 Refined enumeration of $\widetilde{\psi}_{u,v}(1,1)$

In this section, we give a refined enumeration of the number of **cd**-monomials in $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ in terms of Entringer numbers.

Definition 16. A path x in $B(u,v)$ is said to be valid if there exists some **cd**-monomial M such that $s_M(x)=1$. A sequence $s=((i_1,j_1),(i_2,j_2),\ldots,(i_n,j_n))$ with $i_k,j_k\geqslant 0$ and $i_k+j_k=n-k$ $(1\leqslant k\leqslant n)$ is said to be valid if it corresponds to a valid path in $B(u,v)$. Equivalently, the sequence s is said to be valid if s satisfies: (i) $i_{n-1}>i_n$ and $i_k+j_k=n-k$ for $1\leqslant k\leqslant n$; (ii) For $1\leqslant k\leqslant n-2$, if $j_k>j_{k+1}$ then $i_k>j_{k+1}$ and $i_{k+1}>i_{k+2}\geqslant j_{k+1}$.

Now we enumerate the valid paths beginning with the same first edge. Let E_n be the Euler number, i.e., the number of up-down permutations on $[n]$. It is well known that

$$\tan u+\sec u=\sum_{n\geqslant 0}E_n\frac{u^n}{n!}.$$

Denote by $E_n(k)$ the number of up-down permutations on $[n]$ beginning with k $(1\leqslant k\leqslant n)$. It is clear that $E_n=\sum_{k=1}^{n}E_n(k)$. The numbers $E_n(k)$ are called Euler and Bernoulli numbers, or Entringer numbers, see [6]. The Entringer numbers $E_n(k)$ for $n,k\leqslant 7$ are displayed in Table 1.

It is easy to verify that $E_n(n)=0$ and

$$E_n(k)=\sum_{i=1}^{n-k}E_{n-1}(i),\ 1\leqslant k\leqslant n-1. \tag{5}$$

Theorem 17. *Suppose that the atoms of $[u,v]$ are $u_1,u_2,\ldots,u_n$, and the edge $u\to u_k$ has label $(n-k,k-1)$ for $1\leqslant k\leqslant n$. Then the number of valid paths in $B(u,v)$ with first edge $u\to u_k$ is the Entringer number $E_n(k)$. In other words, the number of valid sequences beginning with $(n-k,k-1)$ is $E_n(k)$.*

Proof. Let $T_n(k)$ denote the number of valid paths in $B(u,v)$ with first edge $u\to u_k$ labeled by $(n-k,k-1)$ for $1\leqslant k\leqslant n$. It is easy to check that $T_2(1)=1,T_2(2)=0$. Then it suffices to show that $T_n(k)$ satisfies the relation (5). We analyze when a valid path in $B(u_k, v)$ can be extended to a valid path in $B(u, v)$.

For $1 \leqslant k \leqslant n$, let

$$x' = (u_k \to u_k^i \to u_k^{ij} \to \cdots \to v)$$

be a valid path in $B(u_k, v)$. By Proposition 9, we can label the edges $u_k \to u_k^i$ ($1 \leqslant i \leqslant n-1$) and $u_k^i \to u_k^{ij}$ ($1 \leqslant j \leqslant n-2$) by $(n-i-1, i-1)$ and $(n-j-2, j-1)$ respectively. By induction, $T_{n-1}(t)$ is the number of valid paths in $B(u_k, v)$ beginning with $u_k \to u_k^t$ for $1 \leqslant t \leqslant n-1$. Moreover, $T_{n-1}(n-1) = 0$ and for $1 \leqslant r \leqslant n-2$,

$$T_{n-1}(r) = \sum_{i=1}^{n-r-1} T_{n-2}(i).$$

Now we extend the path x' to a path x in $B(u, v)$. Let

$$x = (u \to u_k \to u_k^i \to u_k^{ij} \to \cdots \to v).$$

Since x' is a valid path in $B(u_k, v)$, x is a valid path in $B(u, v)$ if and only if $s_{1,\mathbf{a}}(x) = 1$ or $s_{1,\mathbf{b}}(x) = 1$. To compute $s_{1,\mathbf{a}}(x)$ or $s_{1,\mathbf{b}}(x)$, we need to flip the path x at u_k. Let

$$y = \mathcal{F}_{u_k,v}(x) = (u \to u_k \to (u_k^i)' \to (u_k^{ij})' \to \cdots \to v).$$

Then x is a valid path in $B(u, v)$ if and only if $u^{-1}u_k \prec u_k^{-1}u_k^i$, or $u^{-1}u_k \succ u_k^{-1}u_k^i$ and $u^{-1}u_k \prec u_k^{-1}(u_k^i)'$ and $u_k^i \neq (u_k^i)'$. Since the edge $u \to u_k$ has label $(n-k, k-1)$, by (4), we deduce that x is a valid path in $B(u, v)$ if and only if

$$n - k > n - i - 1 \tag{6}$$

or

$$k - 1 > i - 1,\ n - k > i - 1 \text{ and } n - i - 1 > n - j - 2 \geqslant i - 1. \tag{7}$$

If (6) holds, then we have $k \leqslant i \leqslant n-1$. Hence by induction, the number of valid paths in $B(u, v)$ extended from the first edges $u_k \to u_k^{n-1}, \ldots, u_k \to u_k^k$ in $B(u_k, v)$ is

$$\sum_{i=k}^{n-1} T_{n-1}(i). \tag{8}$$

If (7) holds, then we get

$$1 \leqslant i \leqslant \alpha \quad \text{and} \quad i \leqslant j \leqslant n-i-1,$$

where $\alpha = \min\{n-k, k-1\}$. Thus by induction, the number of valid paths in $B(u_k, v)$ which begin with $u_k \to u_k^i$ ($1 \leqslant i \leqslant \alpha$) and can be extended to a valid path in $B(u, v)$ is

$$\begin{aligned} T'_{n-1}(i) &= \sum_{j=i}^{n-i-1} T_{n-2}(j) \\ &= T_{n-1}(1) - \sum_{r=1}^{i-1} (T_{n-2}(r) + T_{n-2}(n-r-1)). \end{aligned} \tag{9}$$

Combining (8), (9) and by induction, we derive that

$$
\begin{aligned}
T_n(k) &= \sum_{i=1}^{\alpha} T'_{n-1}(i) + \sum_{i=k}^{n-1} T_{n-1}(i) \\
&= \sum_{i=1}^{\alpha} \left(T_{n-1}(1) - \sum_{r=1}^{i-1} (T_{n-2}(r) + T_{n-2}(n-r-1)) \right) + \sum_{i=k}^{n-1} T_{n-1}(i) \\
&= \alpha T_{n-1}(1) - \sum_{i=1}^{\alpha} T_{n-1}(n-i) - \sum_{i=1}^{\alpha} (T_{n-1}(1) - T_{n-1}(i)) + \sum_{i=k}^{n-1} T_{n-1}(i) \\
&= \sum_{i=1}^{\alpha} (T_{n-1}(i) - T_{n-1}(n-i)) + \sum_{i=k}^{n-1} T_{n-1}(i). \qquad (10)
\end{aligned}
$$

Since $\alpha = n-k$ or $k-1$, and it is easy to check that

$$
\sum_{i=1}^{n-k} (T_{n-1}(i) - T_{n-1}(n-i)) = \sum_{i=1}^{k-1} (T_{n-1}(i) - T_{n-1}(n-i)),
$$

we find that

$$
\sum_{i=1}^{\alpha} (T_{n-1}(i) - T_{n-1}(n-i)) = \sum_{i=1}^{k-1} (T_{n-1}(i) - T_{n-1}(n-i)). \qquad (11)
$$

It follows from (10) and (11) that

$$
T_n(k) = \sum_{i=1}^{k-1} (T_{n-1}(i) - T_{n-1}(n-i)) + \sum_{i=k}^{n-1} T_{n-1}(i) = \sum_{i=1}^{n-k} T_{n-1}(i),
$$

as desired. This completes the proof. □

Corollary 18. *The number of valid paths in $B(u,v)$ or the number of valid sequences of length n is equal to the Euler number E_n. In other words, $\widetilde{\psi}_{u,v}(1,1) = \psi_{B_n}(1,1) = E_n$.*

It is worth mentioning that Billera [1] conjectured that for all lower Bruhat intervals $[e,v]$, $\widetilde{\psi}_{e,v}(1,1) \leqslant \psi_{B_{\ell(v)}}(1,1)$. In our words, this conjecture asserts that for all lower Bruhat intervals $[e,v]$ such that $\ell(v) = n$, if there exists a flip on the set of paths in $B(e,v)$ satisfies the flip condition, then the number of valid paths in $B(e,v)$ is less than or equal to E_n.

5 The coefficient of $\mathbf{d}^n$

In this section, we interprete the coefficient of $\mathbf{d}^n$ in terms of the Poupard numbers.

			0	1	0			
		0	1	2	1	0		
	0	4	8	10	8	4	0	
0	34	68	94	104	94	68	34	0

Table 2: The Poupard triangle $P_n(k)$.

The Poupard numbers $P_n(k)$ $(1 \leqslant k \leqslant 2n+1)$ are defined recursively as follows, see Poupard [14] or Foata and Han [8]. Let $P_1(1) = 0, P_1(2) = 1, P_1(3) = 0$, and $P_n(1) = 0$, $P_n(2) = \sum_{j=1}^{2n-1} P_{n-1}(j)$ for $n \geqslant 2$. For $3 \leqslant k \leqslant 2n+1$,

$$P_n(k) = 2P_n(k-1) - P_n(k-2) - 2P_{n-1}(k-2). \tag{12}$$

By [8, Corollary 4.3], we have

$$P_n(k) = P_n(2n-k+2) \ \text{ for } \ 1 \leqslant k \leqslant n.$$

The first few lines of the Poupard triangle are listed in Table 2.

Recall that the numbers t_n appearing in the Taylor expansion

$$\begin{aligned}\sqrt{2}\tan(u/\sqrt{2}) &= \sum_{n\geqslant 0} t_n \frac{u^{2n+1}}{(2n+1)!}\\ &= 1\frac{u}{1!} + 1\frac{u^3}{3!} + 4\frac{u^5}{5!} + 34\frac{u^7}{7!} + 496\frac{u^9}{9!} + 11056\frac{u^{11}}{11!} + \cdots\end{aligned}$$

are called the reduced tangent numbers. It is easy to see that $t_n = E_{2n+1}/2^n$, where E_{2n+1} are the Euler numbers. By [8, Theorem 1.1], we have

$$t_n = \sum_{k=1}^{2n+1} P_n(k).$$

Definition 19. A valid path x in $B(u,v)$ of length $2n+1$ is said to be alternating if $w(x) = \mathbf{baba}\cdots\mathbf{ba}$. The label sequence $s = ((i_1,j_1),(i_2,j_2),\ldots,(i_{2n+1},j_{2n+1}))$ of an alternating path is called an alternating sequence. Equivalently, the sequence s is said to be alternating if s is valid and $j_{2r-1} > j_{2r}$ and $i_{2r} > i_{2r+1}$ for $1 \leqslant r \leqslant n$.

Theorem 20. *Let $[u,v]$ be a Bruhat interval such that $BG(u,v)$ is isomorphic to the Boolean lattice B_{2n+1}. Suppose that the atoms of $[u,v]$ are $u_1, u_2, \ldots, u_{2n+1}$, and the edge $u \to u_k$ $(1 \leqslant k \leqslant 2n+1)$ has label $(2n-k+1, k-1)$. Then the number of alternating paths in $B(u,v)$ beginning with the edge $u \to u_k$ is the Poupard number $P_n(k)$. In other words, the number of alternating sequences of length $2n+1$ beginning with $(2n-k+1, k-1)$ is $P_n(k)$.*

Proof. Assume that $F_n(k)$ is the number of alternating paths in $B(u,v)$ with first edge $u \to u_k$ labeled by $(2n-k+1, k-1)$ for $1 \leqslant k \leqslant 2n+1$. It is easy to check that $F_1(1)=0, F_1(2)=1, F_1(3)=0$. We claim that for $1 \leqslant k \leqslant n+1$,

$$F_n(k) = \sum_{i=1}^{k-1} \sum_{j=i}^{2n-i} F_{n-1}(j), \tag{13}$$

and for $n+1 < k \leqslant 2n+1$,

$$F_n(k) = F_n(2n-k+2). \tag{14}$$

We prove (13) first. Let

$$x = (u \to u_k \to u_k^i \to u_k^{ij} \to \cdots \to v)$$

be a path in $B(u,v)$. By Proposition 9, we can assume that the edges $u \to u_k, u_k \to u_k^i$ and $u_k^i \to u_k^{ij}$ are labeled by $(2n-k+1, k-1), (2n-i, i-1)$ and $(2n-j-1, j-1)$ respectively. Since the path x is valid and $w(x) = \mathbf{baba}\cdots\mathbf{ba}$, by Theorem 15, we see that

$$k-1 > i-1,\ 2n-k+1 > i-1, \tag{15}$$

and

$$2n-j-1 \geqslant i-1,\ 2n-i > 2n-j-1. \tag{16}$$

Suppose that

$$x'' = (u_k^i \to u_k^{ij} \to \cdots \to v)$$

is an alternating path in $B(u_k^i, v)$. By induction, the number of alternating paths in $B(u_k^i, v)$ with first edge $u_k^i \to u_k^{ij}$ is $F_{n-1}(j)$.

If $1 \leqslant k \leqslant n+1$, then we have $2n-k+1 \geqslant k-1$. By (15), we see that $i \leqslant k-1$. By (16), we have $i \leqslant j \leqslant 2n-i$. That is to say, only the paths beginning with $u_k \to u_k^i$ $(1 \leqslant i \leqslant k-1)$ and $u_k^i \to u_k^{ij}$ $(i \leqslant j \leqslant 2n-i)$ will contribute to $F_n(k)$. Therefore, the equation (13) holds.

If $n+1 < k \leqslant 2n+1$, i.e., $2n-k+1 < k-1$, then by (15) we have $i \leqslant 2n-k+1$ and by (16), we have $i \leqslant j \leqslant 2n-i$. Let $k' = 2n-k+2$, then $1 \leqslant k' \leqslant n$ and so

$$F_n(k) = \sum_{i=1}^{2n-k+1} \sum_{j=i}^{2n-i} F_{n-1}(j) = \sum_{i=1}^{k'-1} \sum_{j=i}^{2n-i} F_{n-1}(j) = F_n(k') = F_n(2n-k+2).$$

Thus the equation (14) holds.

Now we show that $F_n(k)$ $(1 \leqslant k \leqslant n+1)$ satisfies the relation (12). By (13), we have

$$\begin{aligned}
&2F_n(k-1) - F_n(k-2) - 2F_{n-1}(k-2) \\
&= 2\sum_{i=1}^{k-2}\sum_{j=i}^{2n-i} F_{n-1}(j) - \sum_{i=1}^{k-3}\sum_{j=i}^{2n-i} F_{n-1}(j) - 2F_{n-1}(k-2) \\
&= \sum_{i=1}^{k-2}\sum_{j=i}^{2n-i} F_{n-1}(j) + \sum_{j=k-2}^{2n-k+2} F_{n-1}(j) - 2F_{n-1}(k-2) \\
&= \sum_{i=1}^{k-2}\sum_{j=i}^{2n-i} F_{n-1}(j) + \sum_{j=k-1}^{2n-k+1} F_{n-1}(j) \\
&= \sum_{i=1}^{k-1}\sum_{j=i}^{2n-i} F_{n-1}(j) = F_n(k),
\end{aligned}$$

where the third equation follows from the fact $F_{n-1}(k-2) = F_{n-1}(2n-k+2)$, which holds by induction. This completes the proof. □

Corollary 21. *The Poupard numbers $P_n(k)$ can be defined recursively as follows. For $1 \leqslant k \leqslant n+1$,*

$$P_n(k) = \sum_{i=1}^{k-1}\sum_{j=i}^{2n-i} P_{n-1}(j),$$

where $P_1(1) = 0, P_1(2) = 1, P_1(3) = 0$ and $P_n(1) = 0$ for all $n \geqslant 1$, and $P_n(k) = P_n(2n-k+2)$ for $n+1 < k \leqslant 2n+1$.

The following corollary was also obtained by Mahajan [13] algebraically.

Corollary 22. *The coefficient of $\mathbf{d}^n$ is the reduced tangent number t_n.*

To conclude, we remark that since the **cd**-index of B_n depends only on the poset structure of B_n and $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d}) = \psi_{u,v}(\mathbf{c},\mathbf{d})$ when $BG(u,v)$ is isomorphic to B_n, the complete **cd**-index of $[u,v]$ is combinatorial invariant. However, there lacks of a direct proof of this fact in the viewpoint of the complete **cd**-index. In this paper, we provide such a proof by labeling the edges of $[u,v]$ by pairs of nonnegative integers, and show that this labeling is independent of the specific interval $[u,v]$ as long as its Bruhat graph is isomorphic to B_n. Then we can compute the **ab**-polynomial $\widetilde{\phi}_{u,v}(\mathbf{a},\mathbf{b})$ of $[u,v]$ according to this labeling (Proposition 12), and get rid of the specific Coxeter group.

It would be interesting to find a direct correspondence between the set of up-down permutations on $[n]$ beginning with k $(1 \leqslant k \leqslant n)$ and the set of valid sequences of length n beginning with $(n-k, k-1)$.

References

[1] L. J. Billera. Flag enumeration in polytopes, Eulerian partially ordered sets and Coxeter groups. *Proceedings of the International Congress of Mathematicians*, vol. IV, pages 2389–2415. Hindustan Book Agency, New Delhi, 2010.

[2] L. J. Billera and F. Brenti. Quasisymmetric functions and Kazhdan-Lusztig polynomials. *Isr. J. Math.*, 184:317–348, 2011.

[3] A. Björner and F. Brenti. Combinatorics of Coxeter groups. *Graduate Texts in Mathematics*, Vol. 231, Springer-Verlag, New York, 2005.

[4] M. J. Dyer. Hecke algebras and shellings of Bruhat intervals. *Compos. Math.*, 89:91–115, 1993.

[5] M. J. Dyer. Proof of Cellini's conjecture on self-avoiding paths in Coxeter groups. *Compos. Math.*, 148:548–554, 2012.

[6] R. C. Entringer. A combinatorial interpretation of the Euler and Bernoulli numbers. *Nieuw. Arch. Wisk.*, 14:241–246, 1966.

[7] N. J. Y. Fan and Liao He. On the non-negativity of the complete cd-index. *Discrete Math.*, to appear.

[8] D. Foata and Guo-Niu Han. The doubloon polynomial triangle. *Ramanujan J.*, 23:107–126, 2010.

[9] G. Hetyei. On the cd-variation polynomials of André and simsun permutations. *Discrete Comput. Geom.*, 16(3):259–275, 1996.

[10] J. E. Humphreys. Reflection groups and Coxeter groups. *Cambridge Studies in Advanced Mathematics*, Cambridge Univ. Press, Cambridge, 1990.

[11] K. Karu. On the complete cd-index of a Bruhat interval. *J. Algebraic Comb.*, 38:527–541, 2013.

[12] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53:165–184, 1979.

[13] S. Mahajan. The cd-index of the Boolean lattice. 2002. arXiv:math/0211390

[14] C. Poupard. Deux propriétés des arbres binaires ordonnés stricts. *Eur. J. Comb.*, 10:369–374, 1989.

[15] M. Purtill. André permutations, lexicographic shellability and the cd-index of a convex polytope. *Trans. Amer. Math. Soc.*, 338:77–104, 1993.

[16] R. P. Stanley. A survey of Eulerian posets. In *"Polytopes: Abstract, Convex, and Computational"*, T. Bisztriczky, P. McMullen, R. Schneider, A. I. Weiss, eds., NATO ASI Series C, Vol. 440, Kluwer Academic Publishers, 1994.

[17] S. Sundaram. The homology representation of the symmetric group on Cohen-Macaulay subposets of the partition lattice. *Adv. Math.*, 104:225–296, 1994.

[18] S. Sundaram. The homology of partitions with an even number of blocks. *J. Algebraic Comb.*, 4:69–92, 1995.