The degree-diameter problem for sparse graph classes

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Abstract

The degree-diameter problem asks for the maximum number of vertices in a graph with maximum degree Δ and diameter k. For fixed k, the answer is $\Theta(\Delta^k)$. We consider the degree-diameter problem for particular classes of sparse graphs, and establish the following results. For graphs of bounded average degree the answer is $\Theta(\Delta^{k-1})$, and for graphs of bounded arboricity the answer is $\Theta(\Delta^{\lfloor k/2 \rfloor})$, in both cases for fixed k. For graphs of given treewidth, we determine the the maximum number of vertices up to a constant factor. Other precise bounds are given for graphs embeddable on a given surface and apex-minor-free graphs.

Keywords: degree-diameter problem; treewidth; arboricity; sparse graph; surface graph; apex-minor-free graph

1 Introduction

Let $N(\Delta, k)$ be the maximum number of vertices in a graph with maximum degree at most Δ and diameter at most k. Determining $N(\Delta, k)$ is called the *degree-diameter* problem and is widely studied, especially motivated by questions in network design; see [22] for a survey. Obviously, $N(\Delta, k)$ is at most the number of vertices at distance at most k from a fixed vertex. For $\Delta \geqslant 3$ (which we implicitly assume), it follows that

$$N(\Delta, k) \leqslant M(\Delta, k) := 1 + \Delta \sum_{i=0}^{k-1} (\Delta - 1)^i = \frac{\Delta(\Delta - 1)^k - 2}{\Delta - 2} .$$

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This inequality is called the Moore bound [22, p. 8]. The best lower bound is

$$N(\Delta, k) \geqslant f(k)\,\Delta^k \ ,$$

for some function f. For example, the de Bruijn graph shows that $N(\Delta, k) \geqslant \left(\frac{\Delta}{2}\right)^k$; see Lemma 1. Canale and Gómez [3] established the best known asymptotic bound of $N(\Delta, k) \geqslant \left(\frac{\Delta}{1.59}\right)^k$ for sufficiently large Δ.

For a class of graphs $\mathcal{G}$, let $N(\Delta, k, \mathcal{G})$ be the maximum number of vertices in a graph in $\mathcal{G}$ with maximum degree at most Δ and diameter at most k. We consider $N(\Delta, k, \mathcal{G})$ for some particular classes $\mathcal{G}$ of sparse graphs, focusing on the case of small diameter k, and large maximum degree Δ. We prove lower and upper bounds on $N(\Delta, k, \mathcal{G})$ of the form

$$f(k)\,\Delta^{g(k)} \tag{1}$$

for some functions f and g. Since k is assumed to be small compared to Δ, the most important term in such a bound is $g(k)$. Thus our focus is on $g(k)$ with $f(k)$ a secondary concern.

We first state two straightforward examples, namely bipartite graphs and trees. The maximum number of vertices in a bipartite graph with maximum degree Δ and diameter k is $f(k)\,\Delta^{k-1}$ for some function f; see references [22, Section 2.4.4] and [1, 5]. And for trees, it is easily seen that the maximum number of vertices is within a constant factor of $(\Delta - 1)^{\lfloor k/2 \rfloor}$, which is a big improvement over the unrestricted bound of Δ^k. Some of the results in this paper can be thought of as generalisations of this observation.

The following table summarises our current knowledge, where original results are in bold.

graph class	diameter k	max. number of vertices
general		$f(k)\,\Delta^k$
3-colourable	$k \geqslant 2$	$f(k)\,\Delta^k$
triangle-free 3-colourable	$k \geqslant 4$	$f(k)\,\Delta^k$
bipartite		$f(k)\,\Delta^{k-1}$
average degree d		$f(k)\,d\Delta^{k-1}$
arboricity b		$f(k,b)\,\Delta^{\lfloor k/2 \rfloor}$
treewidth t	odd k	$ct\,(\Delta - 1)^{(k-1)/2}$
treewidth t	even k	$c\sqrt{t}\,(\Delta - 1)^{k/2}$
Euler genus g	odd k	$\leqslant c(g+1)k\,(\Delta - 1)^{(k-1)/2}$
Euler genus g	even k	$\leqslant c\sqrt{(g+1)k}\,(\Delta - 1)^{k/2}$
trees		$c\,\Delta^{\lfloor k/2 \rfloor}$

First consider the class of graphs with average degree d. In this case, we prove that the maximum number of vertices is $f(k)\,d\Delta^{k-1}$ for some function f (see Section 3). This shows that by assuming bounded average degree we obtain a modest improvement over the standard bound of $(\Delta - 1)^k$. A much more substantial improvement is obtained by considering arboricity.

The *arboricity* of a graph G is the minimum number of spanning forests whose union is G. Nash-Williams [24] proved that the arboricity of G equals

$$\max_{H\subseteq G} \left\lceil \tfrac{|E(H)|}{|V(H)|-1} \right\rceil \ , \tag{2}$$

where the maximum is taken over all subgraphs H of G. For example, it follows from Euler's formula that every planar graph has arboricity at most 3, and every graph with Euler genus g has arboricity at most $O(\sqrt{g})$. More generally, every graph that excludes a fixed minor has bounded arboricity. Note that $\delta \leqslant d \leqslant 2b$ for every graph with minimum degree δ, average degree d, and arboricity b. Arboricity is a more refined measure than average degree, in the sense that a graph has bounded arboricity if and only if every subgraph has bounded average degree.

We prove that for a graph with arboricity b the maximum number of vertices is $f(b,k)\,\Delta^{\lfloor k/2 \rfloor}$ for some function f (see Section 4). Thus by moving from bounded average degree to bounded arboricity the $g(k)$ term discussed above is reduced from $k-1$ to $\lfloor \frac{k}{2} \rfloor$. This result generalises the above-mentioned bound for trees, which have arboricity 1. The dependence on b in f can be reduced by making more restrictive assumptions about the graph.

Treewidth is a parameter that measures how tree-like a given graph is. The *treewidth* of a graph G can be defined to be the minimum integer t such that G is a spanning subgraph of a chordal[1] graph with no $(t+2)$-clique. For example, trees are exactly the connected graphs with treewidth 1. See [2, 26] for background on treewidth. Since the arboricity of a graph is at most its treewidth, bounded treewidth is indeed a more restrictive assumption than bounded arboricity. We prove that the maximum number of vertices in a graph with treewidth t is within a constant factor of $t(\Delta-1)^{(k-1)/2}$ if k is odd, and of $\sqrt{t}(\Delta-1)^{k/2}$ if k is even (and Δ is large). These results immediately imply the best known bounds for graphs of given Euler genus[2], and new bounds for apex-minor-free graphs. All these results are presented in Section 5.

Our results in Section 6 are of a different nature. There, we describe (non-sparse) graph classes for which the maximum number of vertices is not much different from the unrestricted case. In particular, we prove that for $k \geqslant 2$, there are 3-colourable graphs with $f(k)\,\Delta^k$ vertices, and for for $k \geqslant 4$, there are triangle-free 3-colourable graphs with $f(k)\,\Delta^k$ vertices. These results are in contrast to the bipartite case, in which $f(k)\,\Delta^{k-1}$ is the answer.

All undefined terminology and notation is in reference [9].

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[1] A graph is *chordal* if every induced cycle is a triangle.

[2] A *surface* is a non-null compact connected 2-manifold without boundary. Every surface is homeomorphic to the sphere with h handles or the sphere with c cross-caps. The sphere with h handles has *Euler genus* $2h$, and the sphere with c cross-caps has *Euler genus* c. The *Euler genus* of a graph G is the minimum Euler genus of a surface in which G embeds. See the monograph by Mohar and Thomassen [23] for background on graphs embedded in surfaces.

2 Basic Constructions

This section gives some graph constructions that will later be used for proving lower bounds on $N(\Delta, k, \mathcal{G})$. A *digraph* is a directed graph possibly with loops and possibly with arcs in opposite directions between two vertices. A digraph is *r-inout-regular* if each vertex has indegree r and outdegree r (where a loop at v counts in the indegree and the outdegree of v). A digraph has *strong diameter* k if for all (not necessarily distinct) vertices v and w there is a directed walk from v to w of length exactly k.

de Bruijn [4] and Good [16] independently introduced what is now known as the de Bruijn digraph $\overrightarrow{B}(r,k)$. Fiol et al. [14], and Zhang and Lin [32] showed that $\overrightarrow{B}(r,k)$ can be constructed recursively as a line digraph, as we now explain. If G is a digraph with arc set $A(G)$, then the *line digraph* $L(G)$ has vertex set $A(G)$, where (uv, vw) is an arc of $L(G)$ for all distinct arcs $uv, vw \in A(G)$. Let $\overrightarrow{B}(r,1)$ be the r-vertex digraph in which every arc is present (including loops). Now recursively define $\overrightarrow{B}(r,k) := L(\overrightarrow{B}(r,k-1))$. Then $\overrightarrow{B}(r,k)$ has r^k vertices, is r-inout-regular, and has strong diameter k; see [14, Sec. IV]. Define the de Bruijn graph $B(r,k)$ to be the undirected graph that underlies $\overrightarrow{B}(r,k)$ (ignoring loops, and replacing bidirectional arcs by a single edge).

Lemma 1. *For all integers $r \geqslant 1$ and $k \geqslant 1$ the de Bruijn graph $B(r,k)$ has r^k vertices, maximum degree at most $2r$, and diameter k. Moreover, for $k \geqslant 2$, there are sets $B_1, \dots, B_{r^{k-1}}$ of vertices in $B(r,k)$, each containing $2r-2$ or $2r$ vertices, such that each vertex of $B(r,k)$ is in exactly two of the B_i, and the endpoints of each edge of $B(r,k)$ are in some B_i.*

Proof. Clearly $B(r,k)$ has r^k vertices, has maximum degree at most $2r$, and has (undirected) diameter k (since loops can be ignored in shortest paths). It remains to prove the final claim of the lemma, where $k \geqslant 2$. For each vertex v of $\overrightarrow{B}(r,k-1)$, let B_v be the set of vertices of $B(r,k)$ that correspond to non-loop arcs incident with v in $\overrightarrow{B}(r,k-1)$. Thus $|B_v|$ equals $2r-2$ or $2r$ depending on whether there is a loop at v in $\overrightarrow{B}(r,k-1)$. Each vertex of $B(r,k)$ corresponding to an arc vw of $\overrightarrow{B}(r,k-1)$ is in exactly two of these sets, namely B_v and B_w. The endpoints of each edge of $B(r,k)$ corresponding to a path uv, vw of $\overrightarrow{B}(r,k-1)$ are both in B_v. These r^{k-1} sets, one for each vertex of $\overrightarrow{B}(r,k-1)$, define the desired sets in $B(r,k)$. $\square$

The next two lemmas will be useful later.

Lemma 2. *For every integer $q > 1$ there is a $(2q-2)$-regular graph L with $\binom{q+1}{2}$ vertices, containing cliques $L_1, \dots, L_{q+1}$ each of order q, such that each vertex in L is in exactly two of the L_i, and $L_i \cap L_j \neq \emptyset$ for all $i, j \in [1, q+1]$.*

Proof. Let L be the line graph of the complete graph K_{q+1}. That is, $V(L) := \{\{i,j\} : 1 \leqslant i,j \leqslant q+1,\, i \neq j\}$, where $L_i := \{\{i,j\} : 1 \leqslant j \leqslant q+1,\, i \neq j\}$ is a clique for each $i \in [1, q+1]$. The claimed properties are immediate. $\square$

Lemma 3. *For all integers $p \geqslant 1$ and $q \geqslant 1$ and $m \leqslant (q+1)p$ there is a bipartite graph T with bipartition C, D, such that C consists of m vertices each with degree q, and D consists of $\binom{q+1}{2}$ vertices each with degree at most $2p$, and every pair of vertices in C have a common neighbour in D.*

Proof. By Lemma 2, there is a set $D = V(L)$ of size $\binom{q+1}{2}$, containing subsets $D_1, \dots, D_{q+1}$ each of size q, such that each element of D is in exactly two of the D_i, and $D_i \cap D_j \neq \emptyset$ for all $i, j \in [1, q+1]$.

Let T be the graph with vertex set $C \cup D$, where C is defined as follows. For each $i \in [1, q+1]$ add a set C_i of p vertices to C, each adjacent to every vertex in D_i. Since $|D_i| = q$, each vertex in C has degree q. Since each element of D is in exactly two of the D_i, each vertex in D has degree $2p$.

Consider two vertices $v, w \in C$. Say $v \in C_i$ and $w \in C_j$. Let x be a vertex in $D_i \cap D_j$. Then x is a common neighbour of v and w in G.

We have proved that T has the desired properties in the case that $m = (q+1)p$. Finally, delete $(q+1)p - m$ vertices from C, and the obtained graph has the desired properties. □

3 Average Degree

This section presents bounds on the maximum number of vertices in a graph with given average degree. For fixed diameter, the upper and lower bounds are within a constant factor. We have the following rough upper bound for graphs of given minimum degree.

Proposition 4. *Every graph with minimum degree δ, maximum degree Δ and diameter k has at most $2\delta(\Delta-1)^{k-1}+1$ vertices.*

Proof. Let v be a vertex of degree δ. For $0 \leqslant i \leqslant k$, let n_i be the number of vertices at distance i from v. Thus $n_0 = 1$ and $n_i \leqslant \delta(\Delta-1)^{i-1}$ for all $i \geqslant 1$. In total, $n = \sum_{i=0}^{k} n_i \leqslant 1 + \sum_{i=1}^{k} \delta(\Delta-1)^{i-1} = 1 + \delta\,\frac{(\Delta-1)^k-1}{\Delta-2} \leqslant 1 + 2\delta(\Delta-1)^{k-1}$. □

Since minimum degree is at most average degree, we have the following corollary.

Corollary 5. *Every graph with average degree d, maximum degree Δ and diameter k has at most $2d(\Delta-1)^{k-1}+1$ vertices.*

The following is the main result of this section; it says that Corollary 5 is within a constant factor of optimal for fixed k.

Proposition 6. *For all integers $d \geqslant 4$ and $k \geqslant 3$ and $\Delta \geqslant 2d$ there is a graph with average degree at most d, maximum degree at most Δ, diameter at most k, and at least $\frac{d}{8}\lfloor\frac{\Delta}{4}\rfloor^{k-1}$ vertices.*

Proof. Let $r := \lfloor\frac{\Delta}{4}\rfloor$. Let $q := \lfloor\frac{d}{4}\rfloor \geqslant 2$. Let $p := \lfloor\frac{\Delta}{2}\rfloor - r - q + 1$. Note that $d \geqslant 4q$ and $4p \geqslant \Delta - 4q \geqslant \frac{\Delta}{2}$.

Let $B := B(r, k-2)$ be the graph from Lemma 1 with maximum degree at most $2r$, diameter $k-2$, and r^{k-2} vertices.

Let L be the $(2q-2)$-regular graph from Lemma 2 with $\binom{q+1}{2}$ vertices, containing cliques $L_1, \dots, L_{q+1}$ each of order q, such that each vertex in L is in exactly two of the L_i, and $L_i \cap L_j \neq \emptyset$ for all $i, j \in [1, q+1]$.

Let H be the cartesian product graph $L \,\square\, B$. Note that H has $\binom{q+1}{2} r^{k-2}$ vertices and has maximum degree at most $2q - 2 + 2r$. For $i \in [1, q+1]$ and $v \in V(B)$, let $X_{i,v}$ be the clique $\{(x, v) : x \in L_i\}$ in H. Since each vertex in L is in exactly two of the L_i, each vertex in H is in exactly two of the $X_{i,v}$.

Let G be the graph obtained from H as follows: for $i \in [1, q+1]$ and $v \in V(B)$, add an independent set $Y_{i,v}$ of p vertices to G completely adjacent to $X_{i,v}$; that is, every vertex in $Y_{i,v}$ is adjacent to every vertex in $X_{i,v}$. We now prove that G has the claimed properties.

The number of vertices in G is

$$|V(G)| \geqslant \sum_{i,v} |Y_{i,v}| = (q+1) r^{k-2} p \geqslant \tfrac{d}{4} \lfloor \tfrac{\Delta}{4} \rfloor^{k-2} \tfrac{\Delta}{8} \geqslant \tfrac{d}{8} \lfloor \tfrac{\Delta}{4} \rfloor^{k-1} \ .$$

To determine the diameter of G, let α and β be vertices in G. Say $\alpha \in X_{i,v} \cup Y_{i,v}$ and $\beta \in X_{j,w} \cup Y_{j,w}$. Let x be a vertex in $L_i \cap L_j$. Let $v = y_1, \dots, y_\ell = w$ be a path of length at most $k-2$ in B. Then $\alpha, (x, y_1), (x, y_2), \dots, (x, y_\ell), \beta$ is path of length at most k in G. Hence G has diameter at most k.

Consider the maximum degree of G. Each vertex in some set $Y_{i,v}$ has degree $|X_{i,v}| = |L_i| = q \leqslant \Delta$. Each vertex in some set $X_{i,v}$ has degree $2q - 2 + 2r + 2p \leqslant \Delta$. Thus G has maximum degree at most Δ.

It remains to prove that the average degree of G is at most d. There are $|V(H)| = \binom{q+1}{2} r^{k-2}$ vertices of degree at most Δ, and there are $(q+1) r^{k-2} p$ vertices of degree q. Thus the average degree is at most

$$\frac{\binom{q+1}{2} r^{k-2} \cdot \Delta \ + \ (q+1) r^{k-2} pq}{\binom{q+1}{2} r^{k-2} \ + \ (q+1) r^{k-2} p} = \frac{\frac{q}{2}\Delta \ + \ pq}{\frac{q}{2} \ + \ p}$$

Hence it suffices to prove that $q\Delta + 2pq \leqslant (q + 2p)d$. Since $\Delta \geqslant 2d$ and $d \geqslant 4q$,

$$d\Delta = \tfrac{d\Delta}{2} + \tfrac{d\Delta}{2} \geqslant \tfrac{d\Delta}{2} + d^2 \geqslant 2q\Delta + 4dq \ .$$

That is, $2d\Delta - 2q\Delta - 8qd \geqslant d\Delta - 4qd$. Since $4p \geqslant \Delta - 4q$ and $8q^2 \geqslant 0$,

$$8p(d-q) \geqslant 2(\Delta - 4q)(d-q) = 2d\Delta - 2q\Delta - 8qd + 8q^2 \geqslant d\Delta - 4qd \geqslant 4q\Delta - 4qd \ .$$

That is, $4pd + 2qd \geqslant 2q\Delta + 4pq$, as desired. Hence the average degree of G is at most d. □

Note that for particular values of k and Δ, other graphs can be used instead of the de Bruijn graph in the proof of Proposition 6 to improve the constants in our results; we omit all these details.

4 Arboricity

This section proves that the maximum number of vertices in a graph with arboricity b is $f(b,k)\cdot\Delta^{\lfloor k/2\rfloor}$ for some function f. Reasonably tight lower and upper bounds on f are established. First we prove the upper bound.

Theorem 7. *For every graph G with arboricity b, diameter k, and maximum degree Δ,*

$$|V(G)| \leqslant 4k(2b)^k\Delta^{\lfloor k/2\rfloor}+1 \ .$$

Proof. Let $G_1,\dots,G_b$ be spanning forests of G whose union is G. Orient the edges of each component of each G_i towards a root vertex. The choice of the root is arbitrary. Thus each vertex v of G has outdegree at most 1 in each G_i; therefore v has outdegree at most b in G.

Consider an unordered pair of vertices $\{v,w\}$. Let P be a shortest vw-path in G. Say P has ℓ edges. Then $\ell\leqslant k$. An edge of P oriented in the direction from v to w is called *forward*. If at least $\lceil\frac{\ell}{2}\rceil$ of the edges in P are forward, then charge the pair $\{v,w\}$ to v, otherwise charge $\{v,w\}$ to w.

Consider a vertex v. If some pair $\{v,w\}$ is charged to v then there is path of length ℓ from v to w with exactly i forward arcs, for some i and ℓ with $\lceil\frac{\ell}{2}\rceil\leqslant i\leqslant\ell\leqslant k$. Since each vertex has outdegree at most b, the number of such paths is at most $\binom{\ell}{i}b^i\Delta^{\ell-i}$. Hence the number of pairs charged to v is at most

$$\begin{aligned}\sum_{\ell=1}^{k}\sum_{i=\lceil\ell/2\rceil}^{\ell}\binom{\ell}{i}b^i\Delta^{\ell-i} &\leqslant k\sum_{i=\lceil k/2\rceil}^{k}\binom{k}{i}b^i\Delta^{k-i}\\ &= k\sum_{i=0}^{\lfloor k/2\rfloor}\binom{k}{k-i}b^{k-i}\Delta^i\\ &\leqslant k\,2^kb^k\sum_{i=0}^{\lfloor k/2\rfloor}\Delta^i\\ &\leqslant 2k(2b)^k\Delta^{\lfloor k/2\rfloor} \ .\end{aligned}$$

Hence, the total number of pairs, $\binom{n}{2}$, is at most $2k(2b)^k\Delta^{\lfloor k/2\rfloor}n$. The result follows. $\square$

We now show that the upper bound in Theorem 7 is close to being best possible (for fixed k).

Theorem 8. *For all even integers $b\geqslant 2$ and $k\geqslant 4$ and $\Delta\geqslant b$, such that $\Delta\equiv 2 \pmod 4$ or $b\equiv 0 \pmod 4$, there is a graph G with arboricity at most b, maximum degree at most Δ, diameter at most k, and at least $\frac{8}{b^2}(\frac{b\Delta}{8})^{k/2}$ vertices.*

Proof. Let $q:=\frac{\Delta}{2}$ and $p:=\frac{b}{2}$ and $\ell:=\frac{k}{2}-1$. Then q, p and ℓ are positive integers. Let $r:=\frac{(q+1)p}{2}$. Then r is a positive integer (since $\Delta\equiv 2 \pmod 4$ or $b\equiv 0 \pmod 4$).

Let B be the de Bruijn graph $B(r,\ell)$. By Lemma 1, B has diameter ℓ and r^ℓ vertices. Moreover, there are sets $B_1,\dots,B_{r^{\ell-1}}$ of vertices in B, each containing $2r-2$ or $2r$ vertices, such that each vertex of B is in exactly two of the B_i, and the endpoints of each edge in B are in some B_i. Let $r_i := |B_i|$. Thus $r_i \leqslant 2r = (q+1)p$.

By Lemma 3, for each $i \in [1, r^{\ell-1}]$ there is a bipartite graph T_i with bipartition B_i, D_i, such that B_i consists of r_i vertices each with degree q, and D_i consists of $\binom{q+1}{2}$ vertices each with degree at most $2p \leqslant b$, and each pair of vertices in B_i have a common neighbour in D_i.

Let G be the bipartite graph with bipartition $V(B) \cup D$, where $D := \cup_i D_i$ and the induced subgraph $G[B_i, D_i]$ is T_i. In G, each vertex in $V(B)$ has degree $2q \leqslant \Delta$, and each vertex in D has degree at most $b \leqslant \Delta$. Thus G has maximum degree Δ. Assign each edge in G one of b colours, such that two edges receive distinct colours whenever they have an endpoint in D in common. Each colour class induces a subgraph in which each component is a star. Hence G has arboricity at most b. Observe that

$$|V(G)| \geqslant |D| = r^{\ell-1}\tbinom{q+1}{2} \geqslant (\tfrac{b\Delta}{8})^{\ell-1}\,\tfrac{\Delta^2}{8} = (\tfrac{b\Delta}{8})^{k/2-2}\,\tfrac{\Delta^2}{8} = \tfrac{8}{b^2}(\tfrac{b\Delta}{8})^{k/2}\ .$$

It remains to prove that G has diameter at most k. Consider two vertices v and w in G. If $v \in D_i$ then let v' be a neighbour of v in B_i. If $v \in B_i$ then let v' be v. If $w \in D_j$ then let w' be a neighbour of w in B_j. If $w \in B_j$ then let w' be w. In B, there is a $v'w'$-path P of length at most ℓ. For each edge xy in P, both x and y are in some set B_a (see Lemma 1). Since x and y have a common neighbour in T_a (by Lemma 3), we can replace xy in P by a 2-edge path in T_a, to obtain a $v'w'$-path in G of length at most 2ℓ. Possibly adding the edges vv' or ww' gives a vw-path in G of length at most $2\ell+2=k$. Hence G has diameter at most k. □

Consider the case of diameter 2 graphs with arboricity b. Every such graph has average degree less than $2b$, and thus has at most $4b\Delta$ vertices by Corollary 5. We now show that this upper bound is within a constant factor of optimal. (This result is not covered by Theorem 8 which assumes $k \geqslant 4$.)

Proposition 9. *For all integers $b \geqslant 1$ and even $\Delta \geqslant 4b$ there is a graph with diameter 2, arboricity at most b, maximum degree Δ, and at least $\frac{b\Delta}{4}$ vertices.*

Proof. By Lemma 2, there is a $(2b-2)$-regular graph X with $\binom{b+1}{2}$ vertices, containing cliques $X_1,\dots,X_{b+1}$ each of order b, such that each vertex in X is in exactly two of the X_i, and $X_i \cap X_j \neq \emptyset$ for all $i,j \in [1,q+1]$.

Initialise a graph G equal to X. For $i \in [1, b+1]$, add an independent set Y_i of $p := \frac{\Delta}{2} - b + 1$ vertices to G completely adjacent to X_i.

Consider two vertices v and w in G. Say $v \in X_i \cup Y_i$ and $w \in X_j \cup Y_j$. Let x be the vertex in $X_i \cap X_j$. If $v = x$ or $x = w$ then vw is an edge in G, otherwise vxw is a path in G. Thus G has diameter 2.

Vertices in each X_i have degree $2b-2+2p = \Delta$ and vertices in each Y_i have degree $b \leqslant \Delta$. Hence G has maximum degree Δ. The number of vertices in G is more than $(b+1)p = (b+1)(\frac{\Delta}{2} - b + 1) \geqslant \frac{b\Delta}{4}$.

To calculate the arboricity of G, consider a subgraph H of G. Let $x_i := |X_i \cap V(H)|$ and $y_i := |Y_i \cap V(H)|$. Since $x_i \leqslant |X_i| = b$ and $b \geqslant 2$,

$$\sum_i \tbinom{x_i}{2} = \sum_i \tfrac{x_i(x_i-1)}{2} \leqslant \sum_i \tfrac{b(x_i-1)}{2} = \sum_i \tfrac{bx_i - b}{2} < (\sum_i \tfrac{bx_i}{2}) - b \ .$$

Since $x_i y_i \leqslant |X_i| y_i = b y_i$,

$$\sum_i \tbinom{x_i}{2} + x_i y_i \leqslant (\sum_i \tfrac{bx_i}{2} + b y_i) - b = b\big((\sum_i \tfrac{x_i}{2} + y_i) - 1\big) \ .$$

Observe that $|E(H)| \leqslant \sum_i \binom{x_i}{2} + x_i y_i$ and $|V(H)| \geqslant \sum_i \frac{x_i}{2} + y_i$ (since each vertex in X is in exactly two of the X_i). Thus $|E(H)| \leqslant b(|V(H)| - 1)$, and G has arboricity at most b by (2). □

We conclude this section with an open problem about the degree-diameter problem for graphs containing no K_t-minor. Every such graph has arboricity at most $ct\sqrt{\log t}$, for some constant $c > 0$; see [21, 28, 29]. Thus Theorem 7 implies that for every K_t-minor-free graph G with diameter k and maximum degree $\Delta \gg t$,

$$|V(G)| \leqslant 4k(ct\sqrt{\log t})^k \Delta^{\lfloor k/2 \rfloor}.$$

Improving the $f(t,k)$ term in this $f(t,k)\,\Delta^{\lfloor k/2 \rfloor}$ bound is a challenging open problem.

5 Separators and Treewidth

This section studies a separator-based approach for proving upper bounds in the degree-diameter problem. A *separation* of order s in an n-vertex graph G is a partition (A, S, B) of $V(G)$, such that $|A| \leqslant \frac{2}{3}n$ and $|B| \leqslant \frac{2}{3}n$ and $|S| \leqslant s$ and there is no edge between A and B. Fellows et al. [12] first used separators to prove upper bounds in the degree–diameter problem. In particular, they implicitly proved that every graph that has a separation of order s has $3s\,M(\Delta, \lfloor \frac{k}{2} \rfloor)$ vertices. The following lemma improves the dependence on s in this result when k is even. We include the proof by Fellows et al. [12] for completeness.

Lemma 10. *Let G be a graph with maximum degree at most Δ, and diameter at most k. Assume (A, S, B) is a separation of order s in G. Then*

$$|V(G)| \leqslant \begin{cases} 3s\,M(\Delta, \frac{k-1}{2}) & \textit{if } k \textit{ is odd} \\ \frac{3}{2}\sqrt{s}\,\Delta(\Delta-1)^{k/2-1} \ + \ 3s\,M(\Delta, \frac{k}{2} - 1) & \textit{if } k \textit{ is even} \ . \end{cases}$$

Proof. Let $n := |V(G)|$. Note that $|A| \geqslant n - |B| - s \geqslant \frac{n}{3} - s$. By symmetry, $|B| \geqslant \frac{n}{3} - s$. We use this fact repeatedly.

For $v \in A \cup B$, let $\text{dist}(v, S) := \min\{\text{dist}(v, x) : x \in S\}$. If $\text{dist}(v, S) \geqslant \lfloor k/2 \rfloor + 1$ for some $v \in A$ and $\text{dist}(w, S) \geqslant \lfloor k/2 \rfloor + 1$ for some $w \in B$, then $\text{dist}(v, w) \geqslant 2\lfloor k/2 \rfloor + 2 \geqslant k + 1$, which is a contradiction. Hence, without loss of generality, $\text{dist}(v, S) \leqslant \lfloor k/2 \rfloor$ for each $v \in A$. By the Moore bound, for each vertex $x \in S$, there are at most $M(\Delta, \lfloor k/2 \rfloor) - 1$ vertices in A at distance at most $\lfloor k/2 \rfloor$ from x. Each vertex in A is thus counted. Hence

$$\frac{n}{3} - s \leqslant |A| \leqslant s\, M(\Delta, \lfloor k/2 \rfloor) - s \ ,$$

implying $n \leqslant 3s\, M(\Delta, \lfloor k/2 \rfloor)$. This proves the result of Fellows et al. [12] mentioned above, and proves the case of odd k in the theorem.

Now assume that $k = 2\ell$ is even. Suppose on the contrary that

$$\frac{n}{3} > \frac{\sqrt{s}}{2}\, \Delta(\Delta - 1)^{\ell-1} + s\, M(\Delta, \ell - 1) \ .$$

First consider the case in which some vertex in A is at distance at least $\ell + 1$ from S. Thus every vertex in B is at distance at most $\ell - 1$ from S. By the Moore bound,

$$s\, M(\Delta, \ell - 1) - s \geqslant |B| \geqslant \frac{n}{3} - s > \frac{\sqrt{s}}{2}\, M(\Delta, \ell) + s\, M(\Delta, \ell - 1) - s \ ,$$

which is a contradiction. Now assume that every vertex in A is at distance at most ℓ from S. By symmetry, every vertex in B is at distance at most ℓ from S.

Let A' and B' be the subsets of A and B respectively at distance exactly ℓ from S. By the Moore bound, $|A - A'| \leqslant s\, M(\Delta, \ell - 1) - s$. Hence

$$|A'| = |A| - |A - A'| \geqslant \frac{n}{3} - s - s\, M(\Delta, \ell - 1) + s > \frac{\sqrt{s}}{2}\, \Delta(\Delta - 1)^{\ell-1} \ .$$

By symmetry, $|B'| > \frac{\sqrt{s}}{2}\, \Delta(\Delta - 1)^{\ell-1}$.

Let $P := \{(x, y) : x \in A', y \in B'\}$. For each pair $(x, y) \in P$, some vertex v in S is at distance ℓ from both x and y. Charge (x, y) to v. We now bound the number of pairs in P charged to each vertex $v \in S$. Say v has degree a in A and degree b in B. Thus $a + b \leqslant \Delta$. There are at most $a(\Delta - 1)^{\ell-1}$ vertices at distance exactly ℓ from v in A, and there at most $b(\Delta - 1)^{\ell-1}$ vertices at distance exactly ℓ from v in B. Thus the number of pairs charged to v is at most

$$ab(\Delta - 1)^{2\ell-2} \leqslant \tfrac{1}{4}(a + b)^2(\Delta - 1)^{2\ell-2} \leqslant \tfrac{1}{4}\Delta^2(\Delta - 1)^{2\ell-2} \ .$$

Hence

$$\frac{s}{4}\, \Delta^2(\Delta - 1)^{2\ell-2} = \left(\frac{\sqrt{s}}{2}\, \Delta(\Delta - 1)^{\ell-1}\right)^2 < |A'| \cdot |B'| = |P| \leqslant \frac{s}{4}\Delta^2(\Delta - 1)^{2\ell-2} \ .$$

This contradiction proves that $n \leqslant \frac{3}{2}\sqrt{s}\, \Delta(\Delta - 1)^{\ell-1} + 3s\, M(\Delta, \ell - 1)$. □

Lemma 10 can be written in the following convenient form.

Lemma 11. *For all $\epsilon > 0$ there is a constant c_ϵ such that for every graph G with maximum degree Δ, diameter k, and a separation of order s,*

$$|V(G)| \leqslant \begin{cases} (3 + \epsilon)s(\Delta - 1)^{(k-1)/2} & \text{if } k \text{ is odd and } \Delta \geqslant c_\epsilon \\ (\frac{3}{2} + \epsilon)\sqrt{s}\, (\Delta - 1)^{k/2} & \text{if } k \text{ is even and } \Delta \geqslant c_\epsilon\sqrt{s} \ . \end{cases}$$

Proof. First consider the the odd k case. For $\Delta \geqslant \frac{6}{\epsilon}+2$ we have $3(\frac{\Delta}{\Delta-2}) \leqslant 3+\epsilon$. Thus, by Lemma 10 and the Moore bound,

$$|V(G)| \ \leqslant\ 3s\,(\tfrac{\Delta}{\Delta-2})(\Delta-1)^{(k-1)/2} \ \leqslant\ (3+\epsilon)s(\Delta-1)^{(k-1)/2}\ .$$

Now consider the even k case. For $\Delta \geqslant \frac{3}{\epsilon}+1$ we have $\frac{3}{2}\Delta \leqslant (\frac{3}{2}+\frac{\epsilon}{2})\,(\Delta-1)$. And for $\Delta \geqslant \frac{9}{\epsilon}\sqrt{s}+2$ we have $3\sqrt{s} \ \leqslant\ \frac{\epsilon}{3}(\Delta-2) \leqslant \frac{\epsilon}{2}(\frac{\Delta-1}{\Delta})(\Delta-2)$, implying $3s\,(\frac{\Delta}{\Delta-2}) \leqslant \frac{\epsilon}{2}\sqrt{s}(\Delta-1)$. Hence, by Lemma 10 and the Moore bound,

$$\begin{aligned}|V(G)| \ &\leqslant\ \tfrac{3}{2}\sqrt{s}\,\Delta(\Delta-1)^{k/2-1} + 3s\,(\tfrac{\Delta}{\Delta-2})(\Delta-1)^{k/2-1}\\ &\leqslant\ (\tfrac{3}{2}+\tfrac{\epsilon}{2})\sqrt{s}\,(\Delta-1)^{k/2} + \tfrac{\epsilon}{2}\sqrt{s}(\Delta-1)^{k/2}\\ &\leqslant\ (\tfrac{3}{2}+\epsilon)\sqrt{s}\,(\Delta-1)^{k/2}\ .\end{aligned}$$

This completes the proof of the lemma. □

Treewidth is a key topic when studying separators. In particular, every graph with treewidth t has a separation of order $t+1$, and in fact, a converse result holds [26]. Thus Lemma 11 implies:

Theorem 12. *For all $\epsilon > 0$ there is a constant c_ϵ such that for every graph G with maximum degree Δ, treewidth t, and diameter k,*

$$|V(G)| \leqslant \begin{cases}(3+\epsilon)(t+1)(\Delta-1)^{(k-1)/2} & \text{if } k \text{ is odd and } \Delta \geqslant c_\epsilon\\ (\frac{3}{2}+\epsilon)\sqrt{t+1}\,(\Delta-1)^{k/2} & \text{if } k \text{ is even and } \Delta \geqslant c_\epsilon\sqrt{t+1}\ .\end{cases}$$

Note that Theorem 12 in the case of odd k can also be concluded from a result by Gavoille et al. [15, Theorem 3.2]. Our original contribution is for the even k case. We now show that both upper bounds in Theorem 12 are within a constant factor of optimal.

Proposition 13. *For all integers $k \geqslant 1$ and $t \geqslant 2$ and Δ there is a graph G with maximum degree Δ, diameter k, treewidth at most t, and*

$$|V(G)| \geqslant \begin{cases}\frac{1}{2}(t+1)(\Delta-1)^{(k-1)/2} & \text{if } k \text{ is odd and } \Delta \geqslant 2t-2\\ \frac{1}{2}\sqrt{t+1}\,(\Delta-1)^{k/2} & \text{if } k \text{ is even and } \Delta \geqslant 4\sqrt{2t}\ .\end{cases}$$

Proof. First consider the case of odd k. Let T be the rooted tree such that the root vertex has degree $\Delta-t$, every non-root non-leaf vertex has degree Δ, and the distance between the root and each leaf equals $\frac{k-1}{2}$. Since $t \geqslant 2$ and $\frac{\Delta-t}{\Delta-2} \geqslant \frac{1}{2}$,

$$\begin{aligned}|V(T)| \ =\ 1+(\Delta-t)\sum_{i=0}^{(k-3)/2}(\Delta-1)^i \ &=\ \frac{t-2+(\Delta-t)(\Delta-1)^{(k-1)/2}}{\Delta-2}\\ &\geqslant\ \frac{1}{2}(\Delta-1)^{(k-1)/2}\ .\end{aligned}$$

Take $t+1$ disjoint copies of T, and add a clique on their roots. This graph is chordal with maximum clique size $t+1$. Thus it has treewidth t. The maximum degree is Δ and the number of vertices is at least $\frac{1}{2}(t+1)(\Delta-1)^{(k-1)/2}$.

Now consider the case of even k. Let q be the maximum integer such that $\binom{q+1}{2} \leqslant t+1$. Thus $2 \leqslant q \leqslant \sqrt{2t} \leqslant \frac{\Delta}{4}$ and $q+1 \geqslant \sqrt{t+1}$. Let T be the tree, rooted at r, such that r has degree $\Delta - q$, every non-leaf non-root vertex has degree Δ, and the distance between r and each leaf is $\frac{k}{2}-1$. Since $q \geqslant 2$ and $\frac{\Delta-q}{\Delta-2} \geqslant \frac{1}{2}$,

$$|V(T)| \;=\; 1+(\Delta-q)\sum_{i=0}^{k/2-2}(\Delta-1)^i \;=\; \frac{q-2+(\Delta-q)(\Delta-1)^{k/2-1}}{\Delta-2} \geqslant \frac{1}{2}(\Delta-1)^{k/2-1}\;.$$

By Lemma 2, there is a $(2q-2)$-regular graph L with $\binom{q+1}{2}$ vertices, containing cliques $L_1,\dots,L_{q+1}$ each of order q, such that each vertex in L is in exactly two of the L_i, and $L_i \cap L_j \neq \emptyset$ for all $i,j \in [1,q+1]$. Let G be the graph obtained from L as follows. For each $i \in [1,q+1]$, add $\Delta - 2(q-1)$ disjoint copies of T (called i-copies), where every vertex in L_i is adjacent to the roots of the i-copies of T, as illustrated in Figure 1. It is easily verified that G has maximum degree Δ. Consider a vertex v in some i-copy of T or in L_i, and a vertex w in some j-copy of T or in L_j. Let x be in $L_i \cap L_j$. Then $\operatorname{dist}(v,x) \leqslant \frac{k}{2}$ and $\operatorname{dist}(w,x) \leqslant \frac{k}{2}$, implying $\operatorname{dist}(v,w) \leqslant k$. Hence G has diameter at most k. Let G' be the supergraph of G obtained by adding a clique on $V(L)$. Thus G' is chordal with maximum clique size $\binom{q+1}{2} \leqslant t+1$. Hence G has treewidth at most t. The number of vertices in G is at least $(q+1)(\Delta-2q+2)|V(T)| \geqslant \sqrt{t+1}\cdot\frac{\Delta}{2}\cdot(\Delta-1)^{k/2-1}$. □

We now consider the degree-diameter problem for graphs with given Euler genus. Note that the case of planar graphs has been widely studied [12, 13, 19, 25, 30, 31]. Šiagiová and Simanjuntak [27] proved that for every graph G with Euler genus g,

$$|V(G)| \leqslant c(g+1)k\,(\Delta-1)^{\lfloor k/2 \rfloor}\;, \tag{3}$$

for some absolute constant c. Eppstein [11] proved that every graph with Euler genus g and diameter k has treewidth at most $c(g+1)k$ for some absolute constant c, and Dujmovic et al. [10] proved the explicit bound of $(2g+3)k$. Theorem 12 thus implies the upper bound in (3) and improves upon it when k is even:

Theorem 14. *For all $\epsilon > 0$ there is a constant c_ϵ such that for every graph G with Euler genus g, maximum degree Δ and diameter k,*

$$|V(G)| \leqslant \begin{cases} (3+\epsilon)((2g+3)k+1)(\Delta-1)^{(k-1)/2} & \text{for odd } k \text{ and } \Delta \geqslant c_\epsilon \\ (\frac{3}{2}+\epsilon)\sqrt{(2g+3)k+1}\,(\Delta-1)^{k/2} & \text{for even } k \text{ and } \Delta \geqslant c_\epsilon\sqrt{(2g+3)k+1}. \end{cases}$$

In our companion paper [25] we further investigate the degree-diameter problem for graphs on surfaces, providing an improved upper bound and a new lower bound.

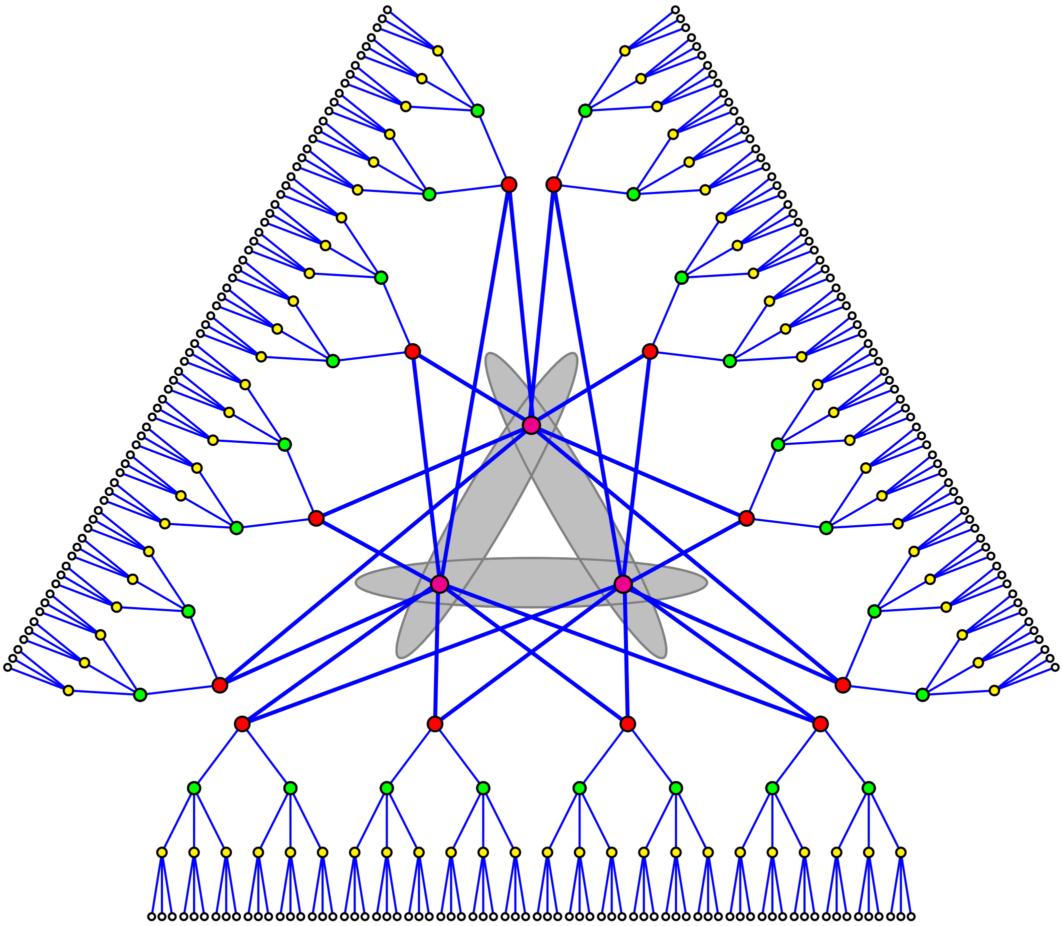

Figure 1: Construction in Proposition 13 for even k. Here $\Delta = 4$ and $k = 8$ and $t = 2$.

To obtain an upper bound of the form $n \leqslant f(k)\,\Delta^{\lfloor k/2 \rfloor}$ using the separator-based approach, one needs a separation of order bounded by a function of the graph's diameter. In some sense, the graphs that have a separation of bounded order are precisely the graphs with bounded treewidth. See Reed's survey [26] for a precise statement here. Thus the separator-based method only works for graphs whose treewidth is bounded by a function of their diameter. The minor-closed graph classes with this property are precisely those that exclude a fixed apex graph as a minor [11]. Here a graph H is *apex* if $H - v$ is planar for some vertex v of H. For example, K_5 and $K_{3,3}$ are apex. Eppstein [11] proved that for some apex graph H and some function f (depending on H), the treewidth of every H-minor-free graph G is at most $f(\operatorname{diam}(G))$. This is called the *diameter-treewidth* or *bounded local treewidth* property; also see [6, 7, 10, 17]. Demaine and Hajiaghayi [8] strengthened Eppstein's result by showing that one can take $f(k) = ck$ for some constant $c = c(H)$. Thus the next result follows from Theorem 12.

Theorem 15. *For every fixed apex graph H there is a constant $c = c(H)$, such that for every H-minor-free graph G with diameter k,*

$$|V(G)| \leqslant \begin{cases} ck\,(\Delta-1)^{(k-1)/2} & \text{if } k \text{ is odd} \\ c\sqrt{k}\,(\Delta-1)^{k/2} & \text{if } k \text{ is even and } \Delta \geqslant c\sqrt{k} \end{cases}.$$

As discussed above, for minor-closed classes, Theorem 15 is the strongest possible result that can be obtained using the separator-based method.

6 3-Colourable and Triangle-Free Graphs

As mentioned in the introduction, it is well known that the maximum number of vertices in a bipartite graph is $f(k)\,\Delta^{k-1}$. We now show that this bound does not hold for the more general class of 3-colourable graphs. In fact, we construct 3-colourable graphs where the number of vertices is within a constant factor of the Moore bound. First note that Kawai and Shibata [20] (building on the work of Harner and Entringer [18]) proved that for large $k \gtrsim \log r$, the de Bruijn graph $B(r,k)$, which roughly has $\left(\frac{\Delta}{2}\right)^k$ vertices, is 3-colourable. The constructions below have the advantage of not assuming that k is large.

In what follows a *pseudograph* is an undirected graph possibly with loops. A loop at a vertex v counts for 1 in the degree of v. A pseudograph H is k-*good* if for all (not necessarily distinct) vertices v and w there is a vw-walk of length exactly k in H.

Given pseudographs H_1 and H_2, the *direct product* graph $H_1 \times H_2$ has vertex set $V(H_1) \times V(H_2)$, where $(v,x)(w,y) \in E(H_1 \times H_2)$ if and only if $vw \in E(H_1)$ and $xy \in E(H_2)$.

Lemma 16. *Let H_1 and H_2 be k-good pseudographs with maximum degree Δ_1 and Δ_2 respectively. Then $H_1 \times H_2$ has $|V(H_1)| \cdot |V(H_2)|$ vertices, maximum degree $\Delta_1\Delta_2$, and diameter at most k. Moreover, if H_2 is loopless and c-colourable, then $H_1 \times H_2$ is c-colourable.*

Proof. Clearly $H_1 \times H_2$ has $|V(H_1)| \cdot |V(H_2)|$ vertices and maximum degree $\Delta_1\,\Delta_2$. Let (v,x) and (w,y) be distinct vertices of G. To prove that G has diameter at most k, we construct a $(v,x)(w,y)$-walk of length at most k in G. Since H_1 is k-good, there is a walk $v = v_0, v_1, \dots, v_k = w$ of length k in H_1. Since H_2 is k-good, there is a walk $x = x_0, x_1, \dots, x_k = y$ of length k in H_2. Thus $(v,x) = (v_0,x_0), (v_1,x_1), \dots, (v_k,x_k) = (w,y)$ is a walk of length k between (v,x) and (w,y) in $H_1 \times H_2$. Hence $H_1 \times H_2$ has diameter at most k. Finally, colouring each vertex (v,x) of $H_1 \times H_2$ by the colour assigned to x in a c-colouring of H_2 gives a c-colouring of $H_1 \times H_2$. □

Lemma 17. *K_3 is k-good for all $k \geqslant 2$.*

Proof. Let $v, w \in V(K_3) = \{0,1,2\}$. If there is a vw-walk of length $k-2$, then there is a vw-walk of length k (just repeat one edge twice). Thus the claim follows from the $k=2$ and $k=3$ cases. Without loss of generality, $v=0$. For $k=2$, one of 010, 021 and 012 is a vw-walk of length 2. For $k=3$, one of 0120, 0121 and 0102 is a vw-walk of length 3. □

To obtain results for triangle-free graphs we use the following:

Lemma 18. *C_5 is k-good for all $k \geqslant 4$.*

Proof. Say $V(C_5) = \{0, 1, 2, 3, 4\}$ and $E(C_5) = \{01, 12, 23, 34, 40\}$. Let $v, w \in V(C_5)$. If there is a vw-walk of length $k-2$, then there is a vw-walk of length k (just repeat one edge twice). Thus the claim follows from the $k = 4$ and $k = 5$ cases. Without loss of generality, $v = 0$. For $k = 4$, one of 01010, 04321, 01212, 04343 and 01234 is a vw-walk of length 4. For $k = 5$, one of 012340, 040101, 043232, 012323 and 010404 is a vw-walk of length 5. □

Lemmas 16 and 17 imply:

Lemma 19. *Let H be a k-good pseudograph with maximum degree Δ for some $k \geqslant 2$. Then $H \times K_3$ is a 3-colourable graph with $3|V(H)|$ vertices, maximum degree 2Δ, and diameter at most k.*

Lemma 20. *Let H be a k-good pseudograph with maximum degree Δ for some $k \geqslant 4$. Then $H \times C_5$ is a 3-colourable triangle-free graph with $5|V(H)|$ vertices, maximum degree 2Δ, and diameter at most k.*

Proof. For any graph G (without loops), if $H \times G$ contains a triangle $(a, u)(b, v)(c, w)$, then uvw is a triangle in G (even if H has loops). Since C_5 is triangle-free, $H \times C_5$ is triangle-free. Thus Lemmas 16 and 18 imply the claim. □

For particular values of Δ and k, various constructions for the degree-diameter problem can be used in the following lemma to give large 3-colourable and triangle-free graphs.

Proposition 21. *Let H be a graph with maximum degree Δ and diameter $k \geqslant 2$. Then there is a 3-colourable graph with $3|V(H)|$ vertices, maximum degree $2\Delta+2$ and diameter at most k. Moreover, if $k \geqslant 4$ then there is a 3-colourable triangle-free graph with $5|V(H)|$ vertices, maximum degree $2\Delta + 2$, and diameter at most k.*

Proof. Let H' be the pseudograph obtained from H by adding a loop at each vertex. Thus H' is k-good and has maximum degree $\Delta + 1$. Lemmas 19 and 20 imply that $H' \times K_3$ and $H' \times C_5$ satisfy the claims. □

This result implies that for fixed $k \geqslant 2$ and $\Delta \gg k$, the maximum number of vertices in a 3-colourable graph is within a constant factor of the unrestricted case. And the same conclusion holds for $k \geqslant 4$ for 3-colourable triangle-free graphs.

We now give a concrete example:

Theorem 22. *For all integers $\Delta \geqslant 4$ and $k \geqslant 2$, there is a 3-colourable graph with $3\lfloor \frac{\Delta}{4} \rfloor^k$ vertices, maximum degree at most Δ, and diameter at most k. Moreover, if $k \geqslant 4$ then there is a 3-colourable triangle-free graph with $5\lfloor \frac{\Delta}{4} \rfloor^k$ vertices, maximum degree at most Δ, and diameter at most k.*

Proof. Let $r := \lfloor \frac{\Delta}{4} \rfloor$. Let H be the undirected pseudograph underlying the de Bruijn digraph $\overrightarrow{B}(r,k)$ including any loops. Lemma 1 shows that H has r^k vertices, maximum degree at most $2r$, and is k-good. Lemma 19 shows that $H \times K_3$ satisfies the first claim. Lemma 20 implies that $H \times C_5$ satisfies the second claim. □

We now give ad-hoc constructions of triangle-free graphs with diameter 2 and 3. These lower bounds are within a constant factor of the Moore bound. Let $\mathbb{Z}_p$ be the cyclic group with p elements. For $a, b \in \mathbb{Z}_p$, let $\text{dist}(a,b) := \min\{a-b, b-a\}$. Here, as always, addition and subtraction are in the group, so $\text{dist}(a,b) \geqslant 0$.

Proposition 23. *For all $\Delta \geqslant 20$ there is a triangle-free graph with diameter 2, maximum degree at most Δ, and at least $(2\lfloor \frac{\Delta+4}{8} \rfloor + 2)^2$ vertices.*

Proof. Let $p := 2\lfloor \frac{\Delta+4}{8} \rfloor + 2$. Thus $p \geqslant 8$ is even. Let G be a graph with vertex set $\mathbb{Z}_p^2$. Thus $|V(G)| = (2\lfloor \frac{\Delta+4}{8} \rfloor + 2)^2$. Let (v_1, v_2) denote a vertex v in G. For distinct vertices v and w, define the *vw-vector* to be (a,b), where $a \leqslant b$ and $\{a,b\} = \{\text{dist}(v_1,w_1), \text{dist}(v_2,w_2)\}$. Then $vw \in E(G)$ if and only if $a = 1$ and $b \neq 2$. Observe that G is $4(p-3)$-regular, and $4(p-3) \leqslant \Delta$.

We now show that the distance between distinct vertices v, w in G is at most 2. Consider the following cases for the vw-vector (a,b), where without loss of generality, $(a,b) = (\text{dist}(v_1,w_1), \text{dist}(v_2,w_2))$:

Case $(0, \geqslant 1)$: Since $p \geqslant 8$, there exists $y \in \mathbb{Z}_p$ such that $\text{dist}(v_2,y) \notin \{0,2\}$ and $\text{dist}(w_2,y) \notin \{0,2\}$. Then $(v_1+1, y) = (w_1+1, y)$ is a common neighbour of v and w.

Case $(1,2)$: Since $p \geqslant 8$, there exists $x \in \mathbb{Z}_p$ such that $\text{dist}(v_1,x) \notin \{0,2\}$ and $\text{dist}(w_1,x) \notin \{0,2\}$. Since $\text{dist}(v_2,w_2) = 2$ there exists $y \in \mathbb{Z}_p$ such that $\text{dist}(v_2,y) = \text{dist}(w_2,y) = 1$. Then (x,y) is a common neighbour of v and w.

Case $(1, \neq 2)$: Then v and w are adjacent.

Case $(\geqslant 2, \geqslant 2)$: Since $p \geqslant 8$, there exists $x \in \mathbb{Z}_p$ such that $\text{dist}(w_1,x) = 1$ and $\text{dist}(v_1,x) \notin \{0,2\}$. Similarly, there exists $y \in \mathbb{Z}_p$ such that $\text{dist}(w_2,y) \notin \{0,2\}$ and $\text{dist}(y,v_1) = 1$. Then (x,y) is a common neighbour of v and w.

Suppose on the contrary that G contains a triangle T. For each edge uv of T, we have $\text{dist}(u_i,v_i) = 1$ for some $i \in [1,2]$. In this case, say uv is *type* i. Since there are three pairs of vertices in T and only two types, two pairs of vertices in T have the same type. Say $T = uvw$. Without loss of generality, uv and vw are both type 1. That is, $\text{dist}(u_1,v_1) = 1$ and $\text{dist}(v_1,w_1) = 1$. Thus $\text{dist}(u_1,w_1) \in \{0,2\}$, in which case $uw \notin E(G)$. This contradiction shows that G contains no triangle. □

Proposition 24. *For all $\Delta \geqslant 42$ there is a triangle-free graph with diameter 3, maximum degree at most Δ, and at least $(2\lfloor \frac{\Delta+6}{12} \rfloor + 4)^3$ vertices.*

Proof. Let $p := 2\lfloor \frac{\Delta+6}{12} \rfloor + 4$. Thus $p \geqslant 12$ is even. Let H be the graph with vertex set $\mathbb{Z}_p$, where $ab \in E(H)$ whenever $\text{dist}(a,b) \geqslant 3$. Observe that every pair of vertices in H have a common neighbour (since $p \geqslant 12$).

Define a graph G with vertex set $V(G) := \mathbb{Z}_p^3$. Thus $|V(G)| = p^3$. Let (v_1, v_2, v_3) denote a vertex v in G. For distinct vertices v and w, define the *vw-vector* to be (a,b,c), where $a \leqslant b \leqslant c$ and $\{a,b,c\} = \{\text{dist}(v_1,w_1), \text{dist}(v_2,w_2), \text{dist}(v_3,w_3)\}$. Then $vw \in E(G)$ if and only if $a = 0$ and $b = 1$ and $c \geqslant 3$. Observe that G is $6(p-5)$-regular, and $6(p-5) \leqslant \Delta$.

We now show that the distance between distinct vertices v, w in G is at most 3. Consider the following cases for the vw-vector, where without loss of generality, $(a,b,c) = (\text{dist}(v_1,w_1), \text{dist}(v_2,w_2), \text{dist}(v_3,w_3))$:

Case $(0,0,\geqslant 1)$: Let u be a common neighbour of v_3 and w_3 in H. Then $(v_1+1, v_2, u) = (w_1+1, w_2, u)$ is a common neighbour of v and w.

Case $(0,1,1)$: Then $(v_1+3, w_2, v_3) = (w_1+3, w_2, v_3)$ is a common neighbour of v and w.

Case $(0,1,2)$: Let y be a common neighbour of v_2 and w_2 in H. Since $\text{dist}(v_3,w_3) = 2$, there is an element z such that $\text{dist}(v_3,z) = \text{dist}(w_3,z) = 1$. Then $(v_1,y,z) = (w_1,y,z)$ is a common neighbour of v and w.

Case $(0,1,\geqslant 3)$: Then v and w are adjacent.

Case $(0,\geqslant 2,\geqslant 2)$: Since $\text{dist}(v_2,w_2) \geqslant 2$, there is an element $y \in \{w_2-1, w_2+1\}$ such that $\text{dist}(v_2,y) \geqslant 3$. Similarly, $\text{dist}(w_3,z) \geqslant 3$ for some $z \in \{v_3-1, v_3+1\}$. Then (v_1,y,z) is a common neighbour of v and w.

Case $(1,\geqslant 1,\geqslant 1)$: Since $v_2 \neq w_2$, there is an element $u \in \{w_2, w_2+2, w_2-2\}$ such that $\text{dist}(v_2,u) \geqslant 3$. Let u' be such that $\text{dist}(u,u') = \text{dist}(w_2,u') = 1$. Let z be a common neighbour of v_3 and z_3 in H. Then

$$(v_1,v_2,v_3)(w_1,u,v_3)(w_1,u',z)(w_1,w_2,w_3)$$

is a vw-path of length 3.

Case $(\geqslant 2,\geqslant 2,\geqslant 2)$: Since $p \geqslant 6$ and $\text{dist}(v_1,w_1) \geqslant 2$, there exists $x \in \mathbb{Z}_p$ such that $\text{dist}(w_1,x) = 1$ and $\text{dist}(v_1,x) \geqslant 3$). Similarly, there exists $y,z \in \mathbb{Z}_p$ such that $\text{dist}(w_2,y) = 1$ and $\text{dist}(v_2,y) \geqslant 3$, and $\text{dist}(v_3,z) = 1$ and $\text{dist}(w_3,z) \geqslant 3$. Then

$$(v_1,v_2,v_3)(x,v_2,z)(w_1,y,z)(w_1,w_2,w_3)$$

is a vw-path of length 3.

Thus G has diameter at most 3.

Suppose on the contrary that G contains a triangle T. For each edge uv of T, we have $u_i = v_i$ for exactly one value of $i \in [1,3]$. In this case, say uv is *type* i. First suppose that at least two of the edges in T are the same type. Then all three edges in T are the same type. Without loss of generality, $u_1 = v_1 = w_1$. Then the subgraph of G induced by $\{u,v,w\}$ (ignoring the first coordinate) is a subgraph of the graph in Proposition 23, which is triangle-free. Now assume that all three edges in T have distinct types. Without loss of generality, $u_1 = v_1$ and $u_2 = w_2$ and $v_3 = w_3$. Since $uv \in E(G)$, without loss of generality, $\text{dist}(u_2,v_2) = 1$ and $\text{dist}(u_3,v_3) \geqslant 3$. Thus $\text{dist}(v_2,w_2) = 1$ and $\text{dist}(u_3,w_3) \geqslant 3$. Since $vw \in E(G)$ and $u_1 = v_1$, we have $\text{dist}(u_1,w_1) = \text{dist}(v_1,w_1) \geqslant 3$. We have shown that $\text{dist}(u_1,w_1) \geqslant 3$ and $u_2 = w_2$ and $\text{dist}(u_3,w_3) \geqslant 3$. Thus the uw-vector is $(0,3,3)$, implying $uw \notin E(G)$. This contradiction shows that G is triangle-free. □

Finally, note that the graphs in Propositions 23 and 24 have bounded chromatic number. In Proposition 23, colour each vertex v by $(v_1 \bmod 2, v_2 \bmod 2)$. For each edge vw, we have $\text{dist}(v_i, w_i) = 1$ for some i. Since p is even, $v_i \not\equiv w_i \pmod 2$. Thus, this is a valid 4-colouring. In Proposition 24, colouring each vertex v by $(v_1 \bmod 2, v_2 \bmod 2, v_3 \bmod 2)$ gives an 8-colouring.

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