Cross-Intersecting Erdős-Ko-Rado Sets in Finite Classical Polar Spaces

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Abstract

A cross-intersecting Erdős-Ko-Rado set of generators of a finite classical polar space is a pair (Y, Z) of sets of generators such that all $y \in Y$ and $z \in Z$ intersect in at least a point. We provide upper bounds on $|Y| \cdot |Z|$ and classify the cross-intersecting Erdős-Ko-Rado sets of maximum size with respect to $|Y| \cdot |Z|$ for all polar spaces except some Hermitian polar spaces.

Keywords: Erdős-Ko-Rado Theorem; Polar Space; Association Scheme; Cross-intersecting Family

1 Introduction

Erdős-Ko-Rado sets (*EKR sets*) were introduced by Erdős, Ko, and Rado [6] as a family of k-element subsets of $\{1, \ldots, n\}$ such that the elements of the family pairwise intersect non-trivially. In particular, Erdős, Ko, and Rado partially classified all such Y of maximum size.

Theorem 1 (Theorem of Erdős, Ko, and Rado)**.** *Let be $n \geqslant 2k$. Let Y be an EKR set of k-element subsets of $\{1, \ldots, n\}$. Then*

$$|Y| \leqslant \binom{n-1}{k-1}$$

with equality for $n > 2k$ if and only if Y is set of all k-elemental sets containing a fixed element.

Stronger versions of this theorem were later proven by several authors including the famous works by Wilson [23], and by Ahlswede and Khachatrian [1].

This theorem for EKR sets was generalized to many structures, including subspaces of projective spaces [3, 7, 12] and generators (maximal totally isotropic, respectively, singular subspaces) of polar spaces [3, 4, 18, 19]. In polar spaces the problem is partially open, since the maximum size of EKR sets of generators of $\mathsf{H}(2d-1, q^2)$, $d > 3$ odd, is still unknown. To the knowledge of the author the best known upper bound is given in [13].

There exists the following modification of the original problem which generated a lot of interest: a *cross-intersecting EKR set* is a pair (Y, Z) of sets of subsets with k elements of $\{1, \dots, n\}$ such that all $y \in Y$ and $z \in Z$ intersect non-trivially. If one wants to generalize the theorem of Erdős, Ko, and Rado to this structure, then the following question arises: how do we measure the size of (Y, Z)? There are at least two natural choices. Either one goes for an upper bound for $|Y| + |Z|$ or one tries to find an upper bound for $|Y| \cdot |Z|$. In the set case the first project was pursued in [9], while the second one was completed in [15]. Results for vectors spaces are due to Tokushige [21]. Again this problem can be generalized to polar spaces, where a *cross-intersecting EKR set of generators* is a pair (Y, Z) of sets of generators such that all $y \in Y$ and $z \in Z$ intersect in at least a point. In this setting this paper is only concerned with an upper bound for $|Y| \cdot |Z|$ and a classification of all cross-intersecting EKR sets reaching this bound.

One additional motivation for this problem is the following: as mentioned before the problem of EKR sets of maximum size in $\mathsf{H}(2d-1, q^2)$ is still open for $d > 3$ odd. Let p be a point of $\mathsf{H}(2d-1, q^2)$ and let X be an EKR set of $\mathsf{H}(2d-1, q^2)$. Furthermore, let Y be the set of generators of X on p and Z the set of generators of X not on p. Now in the quotient geometry of p isomorphic to $\mathsf{H}(2d-3, q^2)$ the projection of the generators of Y and Z onto the quotient geometry is a cross-intersecting EKR set. So both problems are related.

One last thing to point out is that this work does not provide tight upper bounds for cross-intersecting EKR sets in $\mathsf{H}(2d-1, q^2)$ for all $d > 2$. The problem is very similar to the open problem of the maximum size of EKR sets in $\mathsf{H}(9, q^2)$. Therefore, it could be reasonable to first solve the problem of the maximum size of cross-intersecting EKR sets in $\mathsf{H}(7, q^2)$ and then generalize the technique to EKR sets in $\mathsf{H}(9, q^2)$.

2 Projective Spaces & Polar Spaces

We refer to [10] for details on *projective spaces*. A projective space $\mathrm{PG}(n-1, q)$ of *projective dimension* $n-1$ (respectively *vector space dimension* n) over the field with q elements has exactly

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \prod_{i=1}^{k} \frac{q^{n-i+1} - 1}{q^i - 1}$$

subspaces of (vector space) dimension k.

Remark 2. All the used eigenvalue formulas are more convenient if we use vector space dimensions and not projective dimensions. Consequently, the word *dimension* will always refer to the vector space dimension of a subspace.

We call 1-dimensional subspaces *points*, 2-dimensional subspaces *lines*, 3-dimensional subspaces *planes*, and $(n-1)$-dimensional subspaces of an n-dimensional space *hyperplanes*. We denote the number of points in $\mathrm{PG}(n-1,q)$ by

$$[n]_q := \begin{bmatrix} n \\ 1 \end{bmatrix}_q.$$

So we have

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{[n-i+1]_q}{[i]_q}.$$

We shall write $\begin{bmatrix} n \\ k \end{bmatrix}$ instead of $\begin{bmatrix} n \\ k \end{bmatrix}_q$ whenever the choice for q is clear. We will often use the following analog of the recursive definition of binomial coefficients.

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = \begin{bmatrix} n \\ k+1 \end{bmatrix} + q^{(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} \tag{1}$$

A *polar space of rank* d is an incidence geometry with subspaces of dimension from 0 to d defined by a non-degenerate reflexive sesquilinear form or a non-degenerate quadratic form. The *finite classical polar spaces* are $\mathsf{Q}^+(2d-1,q)$, $\mathsf{Q}(2d,q)$, $\mathsf{Q}^-(2d+1,q)$, $\mathsf{W}(2d-1,q)$, where q is a prime power, $\mathsf{H}(2d-1,q)$, and $\mathsf{H}(2d,q)$, where q is the square of a prime power. We refer to [11] for details. Denote the maximal totally isotropic, respectively, singular subspaces of (vector space) dimension d as *generators*. We denote the tangent space of a totally isotropic subspace s of a polar space by $s^\perp$. Each subspace of (vector space) dimension $d-1$ of a polar space is incident with exactly q^e+1 generators, where $e=0$ for $\mathsf{Q}^+(2d-1,q)$, $e=1/2$ for $\mathsf{H}(2d-1,q)$, $e=1$ for $\mathsf{Q}(2d,q)$ and $\mathsf{W}(2d-1,q)$, $e=3/2$ for $\mathsf{H}(2d,q)$, and $e=2$ for $\mathsf{Q}^-(2d+1,q)$. It is well known that a polar space possesses exactly

$$\prod_{i=0}^{d-1}(q^{i+e}+1)$$

generators and $(q^{d+e-1}+1)[d]$ points.

3 The Association Scheme of a Polar Space

We need some basic properties of an association scheme of generators on a dual polar space of rank d and type e. A complete introduction to association schemes can be found in [2, Ch. 2].

Definition 3. Let X be a finite set. A d-class association scheme is a pair $(X, \mathcal{R})$, where $\mathcal{R} = \{R_0, \dots, R_d\}$ is a set of symmetric binary relations on X with the following properties:

1. $\{R_0, \dots, R_d\}$ is a partition of $X \times X$.
2. R_0 is the identity relation.
3. There are numbers p_{ij}^k such that for $x, y \in X$ with xR_ky there are exactly p_{ij}^k elements z with xR_iz and zR_jy.

The number $n_i := p_{ii}^0$ is called the i-valency of R_i. The total number of elements of X is

$$n := |X| = \sum_{i=0}^{d} n_i.$$

The relations R_i are described by their adjacency matrices $A_i \in \mathbb{C}^{n \times n}$ defined by

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } xR_iy \\ 0 & \text{otherwise.} \end{cases}$$

There exist (e.g. in [2, p. 45]) idempotent Hermitian matrices $E_j \in \mathbb{C}^{n \times n}$ (hence they are positive semidefinite) with the properties

$$\sum_{j=0}^{d} E_j = \mathbf{I}, \qquad E_0 = n^{-1}\mathbf{J},$$

$$A_j = \sum_{i=0}^{d} P_{ij} E_i, \qquad E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i,$$

where $P = (P_{ij}) \in \mathbb{C}^{(d+1)\times(d+1)}$ and $Q = (Q_{ij}) \in \mathbb{C}^{(d+1)\times(d+1)}$ are the so-called eigenmatrices of the association scheme. Notice that this implies that the matrices A_j have $d+1$ common eigenspaces V_i, where V_i is the eigenspace of A_j with eigenvalue P_{ij}. We will use the fact that the matrices A_j have common eigenspaces without further notice.

The generators of a polar space define an association scheme if we say that two generators a and b are in relation R_i if and only if $\operatorname{codim}(a \cap b) = i$. Hence a cross-intersecting EKR set (Y, Z) is a set of vertices such that there are no edges between Y and Z in the (distance-regular) graph associated with A_d. This scheme is *cometric*, so there exists a natural ordering of its E_j's and its associated eigenspaces V_j [2, Sec. 2.7, Sec. 9.4]. The matrix P can be found in the literature (for example in [20] or [22, Theorem 4.3.6]). In particular, the eigenvalues of A_d are

$$(-1)^r q^{\binom{d-r}{2} + \binom{r}{2} + e(d-r)} \tag{2}$$

for $r \in \{0, 1, \dots, d\}$. Here $V_0 = \langle \mathbf{j} \rangle$, where $\mathbf{j}$ is the all-ones vector. Furthermore, notice that all eigenspaces V_i of an association scheme are pairwise orthogonal (see [2, Ch. 2]).

4 An Algebraic Bound

We shall apply a technique that was, to the knowledge of the author, first used by Haemers in [8]. The author learned about this technique from a paper by Tokushige [21], where he uses a variant of the result based on the work of Ellis, Friedgut, and Pipel [5]. In this section we use the names d and E_i slightly differently compared to the previous section on association schemes. We do so for two reasons:

- For all the cases, which we consider, the d and E_i of the previous section is the same as in this section.
- We want to provide a complete version of Lemma 7, which is best given for regular graphs.

Let G be a graph with n vertices $\{1, \ldots, n\}$. A matrix $A = (a_{xy}) \in \mathbb{C}^{n \times n}$ is called *extended weight adjacency matrix* of G if A is symmetric, and

1. $a_{xy} \leqslant 0$ if x and y are non-adjacent,
2. $a_{xx} = 0$,
3. the all-ones vector $\mathbf{j}$ is an eigenvector of A.
4. $a_{xy} > 0$ for one (x, y) with x and y adjacent.

Let $V_0, V_1, \ldots, V_d$ be eigenspaces or subspaces of eigenspaces of A such that

1. $V_0 = \langle \mathbf{j} \rangle$,
2. $\mathbb{C}^n = V_0 \perp V_1 \perp \ldots \perp V_d$.

Let λ_i be the eigenvalue corresponding to V_i. Denote the eigenvalue of $\mathbf{j}$ by k. Denote the smallest eigenvalue with eigenvectors in $V_1 \perp \ldots \perp V_d$ by λ_-. Denote the intersection of the eigenspace of λ_- with $V_1 \perp \ldots \perp V_d$ by V_-. Denote the largest eigenvalue with eigenvectors in $V_1 \perp \ldots \perp V_d$ by λ_+. Denote the intersection of the eigenspace of λ_+ with $V_1 \perp \ldots \perp V_d$ by V_+. Denote $\max\{-\lambda_-, \lambda_+\}$ by λ_b. We say that λ_b is the *second largest absolute eigenvalue* of A. Notice that this actually does not mean that λ_b is smaller than k. The last condition ($a_{xy} > 0$) on extended weight adjacency matrices is merely technical and simplifies some of the stated results as it guarantees $\lambda_- < 0 < \lambda_+$.

A *characteristic vector* χ_Y of a subset $Y \subseteq \{1, \ldots, n\}$ is defined by

$$\chi_i = \begin{cases} 1 & \text{if } i \in Y \\ 0 & \text{if } i \notin Y \end{cases}.$$

Let E_0 be the projection operator onto $\langle \mathbf{j} \rangle$, i.e. $E_0 = \mathbf{J}/n$, where $\mathbf{J}$ is the *all-ones matrix*. More generally, let E_i be the projection operator onto the eigenspace of λ_i for $0 < i \leqslant d$. Let I_- be the subset of $\{1, \ldots, d\}$ with $\lambda_i = \lambda_-$ and let I_+ be the subset of

$\{1, \ldots, d\}$ with $\lambda_i = \lambda_+$. Then $V_- = \perp_{i \in I_-} V_i$ and $V_+ = \perp_{i \in I_+} V_i$, so we can define the projections E_-, receptively, E_+ onto V_-, receptively, V_+ by

$$E_- = \sum_{i \in I_-} E_i, \qquad \text{respectively,} \qquad E_+ = \sum_{i \in I_+} E_i.$$

We can write any vector of $\chi \in V$ as

$$\chi = \frac{\chi^\intercal \mathbf{j}}{n} \mathbf{j} + \sum_{i=1}^{d} E_i \chi.$$

As the λ_i are the eigenvalues of A, we can write

$$A = \frac{k}{n} \mathbf{J} + \sum_{i=1}^{d} \lambda_i E_i.$$

An *independent set* is a set of pairwise non-adjacent vertices of a graph. Ellis, Friedgut, and Pipel [5] used the following result, a generalization of the Hoffman bound for independent sets in graphs. In this paper we say that a *cross-intersecting independent set* is a pair (Y, Z) such that there are no edges between Y and Z. In particular cross-intersecting sets are cross-intersecting independent sets for the appropriate graphs.

Lemma 4. *Let A be an extended weight adjacency matrix of a regular graph G with n vertices. Let k be the eigenvalue of the all-ones vector $\mathbf{j}$, and let λ_b the second largest absolute eigenvalue of A. Let (Y, Z) be a cross-intersecting independent set. If $k + \lambda_b > 0$, then*

$$\sqrt{|Y| \cdot |Z|} \leqslant \frac{\lambda_b}{k + \lambda_b} n.$$

Remark 5. In [5] Lemma 4 is restricted to *weight adjacency matrices* or *pseudo adjacency matrices* where a_{ij} is zero if i and j are non-adjacent and $a_{ij} > 0$ if i and j are adjacent. Our more general definition does not change anything and will turn out to be more convenient in Section 6 where we shall not bother to calculate the exact minimum of the Hoffman bound.

Tokushige reformulates Lemma 4 in a more detailed way in [21, Lemma 2]. Unfortunately, his reformulation does not point out some details which are necessary for the special case handled in this paper. Furthermore, our notation differs significantly, so we shall prove Lemma 4 again as well as our version of Tokoshige's result, Lemma 7. First recall the inequality between the geometric and the arithmetic mean.

Lemma 6 (Inequality of Arithmetic and Geometric Means)**.** *Let $0 \leqslant \alpha, \beta \leqslant 1$. Then we have*

$$\sqrt{1 - \alpha} \sqrt{1 - \beta} \leqslant 1 - \sqrt{\alpha \beta}$$

with equality if and only if $\alpha = \beta$.

Our variant of Tokoshige's lemma is as follows.

Lemma 7. *If equality holds in Lemma 4, then $|Y| = |Z| = \alpha n$ for some $\alpha \in \mathbb{R}$. Furthermore, one of the following cases occurs.*

(a) We have $\lambda_+ = \lambda_b > -\lambda_-$, $\chi_Y = \alpha\mathbf{j} + v_+$, and $\chi_Z = \alpha\mathbf{j} - v_+$ for some vector $v_+ \in V_+$.

(b) We have $\lambda_+ < \lambda_b = -\lambda_-$, $\chi_Y = \alpha\mathbf{j} + v_-$, and $\chi_Z = \alpha\mathbf{j} + v_-$ for some vector $v_- \in V_-$. In this case $Y = Z$, and Y is an independent set of maximum size.

(c) We have $\lambda_+ = \lambda_b = -\lambda_-$, $\chi_Y = \alpha\mathbf{j} + v_- + v_+$, and $\chi_Z = \alpha\mathbf{j} + v_- - v_+$ for some vectors $v_- \in V_-$ and $v_+ \in V_+$.

Proof. Let $\chi = \chi_Y$ be the characteristic vector of Y. Let $\psi = \chi_Z$ be the characteristic vector of Z. We suppose that (Y, Z) is a cross-intersecting independent set. Since A is an extended weight adjacency matrix, we have $\chi^\intercal A\psi \leqslant 0$

Put $y = |Y|$ and $z = |Z|$. Suppose without loss of generality $y \geqslant z > 0$. Recall the definitions of E_i, E_+ and E_- at the beginning of the chapter on page 5. Write χ and ψ as

$$\chi = \frac{y}{n}\mathbf{j} + \sum_{i=1}^{d} E_i\chi,$$

$$\psi = \frac{z}{n}\mathbf{j} + \sum_{i=1}^{d} E_i\psi.$$

Recall that

$$A = \frac{k}{n}\mathbf{J} + \sum_{i=1}^{d} \lambda_i E_i.$$

Let E_+ the projection operator onto V_+ and E_- the projection operator onto V_-. We have

$$\begin{aligned}
0 &\geqslant \chi^\intercal A\psi \\
&\geqslant \frac{k}{n}\chi^\intercal \mathbf{J}\psi - \sum_{i=1}^{d} \lambda_i \chi^\intercal E_i\psi && (3)\\
&\geqslant \frac{kyz}{n} - \sum_{i=1}^{d} |\lambda_i \chi^\intercal E_i\psi| && (4)\\
&\geqslant \frac{kyz}{n} - \lambda_b \sum_{i=1}^{d} |(E_i\chi)^\intercal E_i\psi| && (5)\\
&\geqslant \frac{kyz}{n} - \lambda_b \sum_{i=1}^{d} \sqrt{(E_i\chi)^\intercal E_i\chi}\sqrt{(E_i\psi)^\intercal E_i\psi} && (6)
\end{aligned}$$

$$\geqslant \frac{kyz}{n} - \lambda_b \sqrt{\sum_{i=1}^{d} (E_i\chi)^\intercal (E_i\chi)} \cdot \sqrt{\sum_{i=1}^{d} (E_i\psi)^\intercal (E_i\psi)} \tag{7}$$

Notice that we apply the Cauchy-Schwartz inequality twice when going from (5) to (7). Furthermore χ and ψ are 0-1-vectors, so

$$y = \chi^\intercal \chi = \frac{y^2}{n} + \sum_{i=1}^{d} (E_i\chi)^\intercal (E_i\chi), \tag{8}$$

and

$$z = \frac{z^2}{n} + \sum_{i=1}^{d} (E_i\psi)^\intercal (E_i\psi). \tag{9}$$

If we put this back into our previous inequality (7), then we have by Lemma 6

$$\begin{aligned} 0 &\geqslant \frac{kyz}{n} - \lambda_b \sqrt{yz} \sqrt{1 - \frac{y}{n}} \sqrt{1 - \frac{z}{n}} && (10) \\ &\geqslant \frac{kyz}{n} - \lambda_b \sqrt{yz} (1 - \sqrt{yz}/n). && (11) \end{aligned}$$

Rearranging yields Lemma 4.

If this bound is tight, then we have equality from (3) to (11). Equality between (4) and (5) shows that we have $(E_i\chi)^\intercal E_i\psi = 0$ for all i such that $|\lambda_i| \neq \lambda_b$. Equality between (5) and (6) shows that there exists a factor α_i with $E_i\chi = \alpha_i E_i\psi$ for all $i \in \{1, \dots, d\}$. Equality between (6) and (7) shows that there exist constants β with $\sqrt{(E_i\chi)^\intercal E_i\chi} = \beta\sqrt{(E_i\psi)^\intercal E_i\psi}$ for all $i \in \{1, \dots, d\}$. Hence, either $E_i\chi = E_i\psi = 0$ or $\beta = |\alpha_i|$. These two observations show that there $E_i\chi = E_i\psi = 0$ if $|\lambda_i| \neq \lambda_b$. Hence,

$$\begin{aligned} \chi &= \frac{y}{n}\mathbf{j} + E_-\chi + E_+\chi \\ \psi &= \frac{z}{n}\mathbf{j} + E_-\psi + E_+\psi. \end{aligned}$$

Equality between (10) and (11) shows $y = z$ (see Lemma 6). By

$$\sum_{i=1}^{d} (E_i\psi)^\intercal (E_i\psi) = z - \frac{z^2}{n} = y - \frac{y^2}{n} = \sum_{i=1}^{d} (E_i\chi)^\intercal (E_i\chi) = \beta \sum_{i=1}^{d} (E_i\psi)^\intercal (E_i\psi)$$

we derive $\beta = 1$. Then equality between (3) and (4) shows that $\alpha_i = 1$ if $\lambda_i < 0$ and that $\alpha_i = -1$ if $\lambda_i > 0$. Hence, $E_-\chi = E_-\psi$ and $E_+\chi = -E_+\psi$. This proves Lemma 7. $\square$

5 Cross-intersecting EKR Sets of Maximum Size

In this section we shall calculate tight upper bounds for all polar spaces except $\mathsf{H}(2d-1, q^2)$, and classify all examples in case of equality. For all polar spaces except $\mathsf{H}(2d-1, q^2)$ we can imitate the approach of Pepe, Storme, and Vanhove [18]. Recall from Section 3 that we have a natural ordering of the eigenspaces $V_0(= \langle \mathbf{j} \rangle), V_1, \dots, V_d$ of the association scheme which we defined on generators of a polar space of rank d.

Lemma 8. *Let $\mathcal{P}$ be a polar space over $\mathbb{F}_q$ with parameter e and let A_d be the adjacency matrix matrix of the disjointness graph of generators of $\mathcal{P}$. Let $V_0, V_1, \dots, V_d$ the common eigenspaces of the corresponding association scheme. Then $k = q^{\binom{d}{2}+de}$. Additionally, we have the following.*

- *If $\mathcal{P} \in \{\mathsf{Q}^+(2d-1, q), \mathsf{H}(2d-1, q)\}$, then $\lambda_b = q^{\binom{d}{2}}$. Moreover, if d is even, then $\lambda_b = \lambda_+ > -\lambda_-$ and $V_+ = V_d$; if d is odd, then $\lambda_b = -\lambda_- > \lambda_+$ and $V_- = V_d$.*
- *If $\mathcal{P} \in \{\mathsf{Q}(2d, q), \mathsf{W}(2d-1, q)\}$, then $\lambda_b = q^{\binom{d}{2}}$. Moreover, if d is even, then $\lambda_b = -\lambda_- = \lambda_+$, $V_- = V_1$ and $V_+ = V_d$; if d is odd, then $\lambda_b = -\lambda_- > \lambda_+$ and $V_- = V_1 \perp V_d$.*
- *If $\mathcal{P} \in \{\mathsf{H}(2d, q), \mathsf{Q}^-(2d+1, q)\}$, then $\lambda_b = q^{\binom{d-1}{2}+de}$. Moreover, $\lambda_b = -\lambda_- > \lambda_+$ and $V_- = V_1$.*

Proof. The eigenvalues of A_d are given in (2) as

$$(-1)^r q^{\binom{d-r}{2}+\binom{r}{2}+e(d-r)}.$$

For $r = 0$ this is the eigenvalue that belongs to the all-ones vector $\mathbf{j}$, so with k defined as in Lemma 4 we have

$$k = q^{\binom{d}{2}+de}.$$

For $e = 0$ note that the absolute eigenvalues for $r = 0$ and $r = d$ are equal. Therefore, the eigenspace belonging to k has dimension at least 2 which make k also the second largest absolute eigenvalue. Hence, we have the following for the different polar spaces. For $e \in \{0, 1/2\}$ (i.e. $\mathcal{P} \in \{\mathsf{Q}^+(2d-1, q), \mathsf{H}(2d-1, q)\}$) the second largest absolute eigenvalue occurs if and only if $r = d$, for $e = 1$ (i.e. $\mathcal{P} \in \{\mathsf{Q}(2d, q), \mathsf{W}(2d-1, q)\}$) the second largest absolute eigenvalue occurs if and only if $r \in \{1, d\}$, for $e \in \{3/2, 2\}$ (i.e. $\mathcal{P} \in \{\mathsf{H}(2d, q), \mathsf{Q}^-(2d+1, q)\}$) the second largest absolute eigenvalue occurs if and only if $r = 1$. □

Recall that a cross-intersecting EKR set (Y, Z) is a set of vertices of the disjointness graph such that there are no edges between Y and Z. Hence, using Lemma 7 and the classification of EKR sets of generators given in [18] we get the following result.

Corollary 9. *Let (Y, Z) be a cross-intersecting EKR set of maximum size of a finite classical polar space $\mathcal{P}$ not isomorphic to $\mathsf{Q}(2d, q)$ with d even, $\mathsf{W}(2d-1, q)$ with d even, $\mathsf{Q}^+(2d-1, q)$ with d even, or $\mathsf{H}(2d-1, q^2)$, where $|Y| \cdot |Z|$ reaches the bound in Lemma 4. Then $Y = Z$, and Y is an EKR set of maximum size.*

Proof. By Lemma 7 and Lemma 8, all cross-intersecting EKR, which reach the bound, are EKR sets of maximum size. These EKR sets exist as shown in [18]. □

The following cases besides $\mathsf{H}(2d-1, q^2)$ remain.

Theorem 10. *Let (Y, Z) be a cross-intersecting EKR set of generators of maximum size of a polar space $\mathcal{P}$, with $\mathcal{P}$ isomorphic to $\mathsf{Q}(2d, q)$, d even, $\mathsf{W}(2d-1, q)$, d even, or $\mathsf{Q}^+(2d-1, q)$, d even. Let n be the number of generators of $\mathcal{P}$. Then we have the following:*

- *If $\mathcal{P} = \mathsf{Q}^+(2d-1, q)$, then $\sqrt{|Y| \cdot |Z|}$ is at most $n/2$, and if this bound is reached, then there is a $v_+ \in V_d$ such that $\chi_Y = \mathbf{j}/2 + v_+$ and $\chi_Z = \mathbf{j}/2 - v_+$.*
- *If $\mathcal{P} \in \{\mathsf{Q}(2d, q), \mathsf{W}(2d-1, q)\}$, then $\sqrt{|Y| \cdot |Z|}$ is at most the number of generators on a fixed point, and if this bound is reached, then there are $v_- \in V_1$ and $v_+ \in V_d$ such that $\chi_Y = \alpha\mathbf{j} + v_- + v_+$ and $\chi_Z = \alpha\mathbf{j} + v_- - v_+$, with $\alpha = \frac{1}{q^d+1}$.*

Proof. Apply Lemma 7 and Lemma 8. □

Similar to [18] we shall continue by classifying the more complicated cases.

5.1 The Hyperbolic Quadric of Even Rank

The generators of $\mathsf{Q}^+(2d-1, q)$ can be partitioned into two sets X_1 and X_2 of generators (commonly known as *latins* and *greeks*) with $|X_1| = |X_2| = n/2$. For $x_1 \in X_1$ and $x_2 \in X_2$ the codimension of the intersection of $x_1 \cap x_2$ is odd. For $x_1, x_2 \in X_1$ the codimension of the intersection of $x_1 \cap x_2$ is even. This implies for d even that (X_1, X_2) is a cross-intersecting EKR set of maximum size according to Theorem 10. There exist $x_1, x_2 \in X_1$ with $\dim(x_1 \cap x_2) = 0$ if d is even, so (X_1, X_1) is not a cross-intersecting EKR set.

Theorem 11. *Let (Y, Z) be a cross-intersecting EKR set of maximum size of $\mathsf{Q}^+(2d-1, q)$ with d even. Then $Y = X_i$ and $Z = X_j$ for $\{i, j\} = \{1, 2\}$.*

Proof. By Theorem 10, we have $\chi_Y = \mathbf{j}/2 + v_+$ and $\chi_Z = \mathbf{j}/2 - v_+$. As in Theorem 16 of [18] $V_0 = \langle \mathbf{j} \rangle$ is spanned by $\chi_{X_1} + \chi_{X_2}$, and V_d is spanned by $\chi_{X_1} - \chi_{X_2}$. Recall $\chi_Y + \chi_Z = \mathbf{j}$ and $\chi_Y - \chi_Z = 2v_+ \in V_d$. Hence, $\chi_Y = \chi_{X_i}$ and $\chi_Z = \chi_{X_j}$ for some $\{i, j\} = \{1, 2\}$. Hence $\{Y, Z\} = \{X_1, X_2\}$. □

5.2 The Parabolic Quadric and the Symplectic Polar Space of Even Rank

Before we classify cross-intersecting EKR sets of generators of maximum size in parabolic quadrics and symplectic polar spaces, we mention a few simple, most likely well-known lemmas. We include their proofs to make this article more self-contained.

Lemma 12. *Let $\chi \in \langle \mathbf{j} \rangle \perp V$ for some eigenspace V of an (extended weight) adjacency matrix A of a k-regular graph with n vertices associated with eigenvalue λ. Then the characteristic vector e_i of the i-th vertex satisfies*

$$e_i^\intercal A\chi = \frac{\chi^\intercal \mathbf{j}}{n}(k-\lambda) + \lambda e_i^\intercal \chi.$$

Proof. As $\chi \in \langle \mathbf{j} \rangle \perp V$, we can write $\chi = \alpha\mathbf{j} + v$ for some $v \in V$ and $\alpha = \frac{\chi^\intercal \mathbf{j}}{n}$. Then

$$\begin{aligned} e_i^\intercal A\chi &= e_i^\intercal A(\alpha\mathbf{j} + v) \\ &= e_i^\intercal(\alpha k\mathbf{j} + \lambda v) \\ &= e_i^\intercal(\alpha(k-\lambda)\mathbf{j} + \lambda\chi) \\ &= \frac{\chi^\intercal \mathbf{j}}{n}(k-\lambda) + \lambda e_i^\intercal \chi. \end{aligned}$$

□

Corollary 13. *Let $\chi, \psi \in \langle \mathbf{j} \rangle \perp V_- \perp V_+$ for some eigenspaces V_- and V_+ of a (weighted) adjacency matrix A of a k-regular graph with n vertices, where λ_- is the eigenvalue associated with V_- and λ_+ is the eigenvalue associated with V_+. If $\chi = \alpha\mathbf{j} + v_- + v_+$ and $\psi = \alpha\mathbf{j} + v_- - v_+$ for some $\alpha \in \mathbb{R}$, $v_- \in V_-$, and $v_+ \in V_+$, then*

$$\begin{aligned} e_i^\intercal A\chi &= \alpha(k-\lambda_-) + \frac{(\lambda_- + \lambda_+)}{2} e_i^\intercal \chi + \frac{(\lambda_- - \lambda_+)}{2} e_i^\intercal \psi, \\ e_i^\intercal A\psi &= \alpha(k-\lambda_-) + \frac{(\lambda_- + \lambda_+)}{2} e_i^\intercal \psi + \frac{(\lambda_- - \lambda_+)}{2} e_i^\intercal \chi. \end{aligned}$$

Proof. We have $\chi + \psi \in \langle \mathbf{j} \rangle \perp V_-$ and $\chi - \psi \in V_+$. By Lemma 12 and $\mathbf{j}^\intercal v_- = 0 = \mathbf{j}^\intercal v_+$,

$$\begin{aligned} e_i^\intercal A(\chi+\psi) &= \frac{(\chi+\psi)^\intercal \mathbf{j}}{n}(k-\lambda_-) + \lambda_- e_i^\intercal(\chi+\psi) \\ e_i^\intercal A(\chi-\psi) &= \lambda_+ e_i^\intercal(\chi-\psi). \end{aligned}$$

Now the equations $2e_i^\intercal A\chi = e_i^\intercal A(\chi+\psi) + e_i^\intercal A(\chi-\psi)$, $2e_i^\intercal A\psi = e_i^\intercal A(\chi+\psi) - e_i^\intercal A(\chi-\psi)$ and $(\chi+\psi)^\intercal \mathbf{j} = 2\alpha n$ yield the assertion. □

Recall for this section that all matrices A_i of a given association scheme have the common eigenspaces V_j.

Lemma 14. *For the adjacency matrix A_{d-s}, $0 < s < d$, of the dual polar graph of $\mathsf{Q}(2d,q)$ or $\mathsf{W}(2d-1,q)$ the eigenspace V_1 is associated with eigenvalue*

$$\lambda_{-,s} := -\begin{bmatrix} d-1 \\ s \end{bmatrix} q^{\binom{d-s}{2}} + \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{\binom{d-s+1}{2}},$$

the eigenspace V_d is associated with eigenvalue

$$\lambda_{+,s} := (-1)^{d-s}\begin{bmatrix} d \\ s \end{bmatrix} q^{\binom{d-s}{2}},$$

and the eigenspace $\langle \mathbf{j} \rangle$ is associated with eigenvalue

$$k_s := \begin{bmatrix} d \\ s \end{bmatrix} q^{\binom{d-s+1}{2}} = \left(\begin{bmatrix} d-1 \\ s \end{bmatrix} + \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{d-s} \right) q^{\binom{d-s+1}{2}}.$$

Proof. See [22, Theorem 4.3.6], where the eigenvalue of V_j for A_i is given by

$$\sum_{0,j-i\leqslant u\leqslant d-i,j} (-1)^{j+u} \begin{bmatrix} d-j \\ d-i-u \end{bmatrix} \begin{bmatrix} j \\ u \end{bmatrix} q^{(u+i-j)(u+i-j+2e-1)/2+\binom{j-u}{2}}.$$

Note that $e = 1$ in the considered cases. For $j = 0, 1, d$, and $i = d-s$, this formula yields the assertion. The last equality is an application of (1). $\square$

If a cross-intersecting EKR set (Y, Z) of $\mathsf{Q}(2d, q)$ satisfies $\chi_Y, \chi_Z \in V_0 \perp V_1$, then $Y = Z$, and Y is an EKR set as before. So only the case $\chi_Y, \chi_Z \in V_0 \perp V_1 \perp V_d$ remains. In the following denote V_1 by V_- and V_d by V_+. Furthermore, as in Lemma 7 we write

$$\chi_Y = \alpha\mathbf{j} + v_- + v_+ = \frac{|Y|}{n}\mathbf{j} + v_- + v_+ = \frac{\lambda_b}{k+\lambda_b}\mathbf{j} + v_- + v_+$$

and

$$\chi_Z = \alpha\mathbf{j} + v_- - v_+ = \frac{|Z|}{n}\mathbf{j} + v_- - v_+ = \frac{\lambda_b}{k+\lambda_b}\mathbf{j} + v_- - v_+$$

with $v_- \in V_-$ and $v_+ \in V_+$.

Proposition 15. *Let (Y, Z) be a cross-intersecting EKR set of $\mathsf{Q}(2d, q)$ or $\mathsf{W}(2d-1, q)$, d even, of maximum size such that $Y \nsubseteq Z$. Let $g \in Y \setminus Z$.*

(a) If $d-s$ is even, then g meets 0 elements of Z in dimension s.

(b) If $d-s$ is odd, then g meets 0 elements of Y in dimension s.

(c) If $d-s$ is even, then g meets

$$\begin{bmatrix} d \\ s \end{bmatrix} q^{\binom{d-s}{2}}$$

elements of Y in dimension s.

(d) If $d-s$ is odd, then g meets

$$\begin{bmatrix} d \\ s \end{bmatrix} q^{\binom{d-s}{2}}$$

elements of Z in dimension s.

Proof. We can calculate these numbers with Lemma 14 and Corollary 13 by choosing $\chi_{\{g\}}$ as e_i. For A_{d-s} the parameters are given by

$$\begin{aligned}
k_s - \lambda_{-,s} &= \left(\begin{bmatrix} d-1 \\ s \end{bmatrix} + \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{d-s} \right) q^{\binom{d-s+1}{2}} \\
&+ \begin{bmatrix} d-1 \\ s \end{bmatrix} q^{\binom{d-s}{2}} - \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{\binom{d-s+1}{2}} \\
&= q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} \left(q^{d-s} + 1 \right) + q^{\binom{d-s+1}{2}} \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} \left(q^{d-s} - 1 \right) \\
&\overset{\text{Def.}}{=} q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} \left(q^{d-s} + 1 \right) + q^{d-s} \cdot q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} (q^s - 1) \\
&= q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} \left(q^d + 1 \right),
\end{aligned}$$

for $d - s$ even

$$\begin{aligned}
\lambda_{-,s} + \lambda_{+,s} &\overset{(1)}{=} 2 \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{\binom{d-s+1}{2}}, \\
\lambda_{-,s} - \lambda_{+,s} &\overset{(1)}{=} -2 \begin{bmatrix} d-1 \\ s \end{bmatrix} q^{\binom{d-s}{2}},
\end{aligned}$$

for $d - s$ odd

$$\begin{aligned}
\lambda_{-,s} + \lambda_{+,s} &\overset{(1)}{=} -2 \begin{bmatrix} d-1 \\ s \end{bmatrix} q^{\binom{d-s}{2}}, \\
\lambda_{-,s} - \lambda_{+,s} &\overset{(1)}{=} 2 \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} q^{\binom{d-s+1}{2}}.
\end{aligned}$$

Recall that

$$\frac{(\chi_Y + \chi_Z)^\intercal \mathbf{j}}{2n} = \frac{\lambda_+}{k + \lambda_+} = \frac{q^{\binom{d}{2}}}{q^{\binom{d+1}{2}} + q^{\binom{d}{2}}} = \frac{1}{q^d + 1}.$$

Hence by Corollary 13 and $g \notin Z$ (i.e. $e_i^T \psi = 0$),

$$\begin{aligned}
e_i^\intercal A_{d-s} \chi_Y &= \frac{k_s - \lambda_{-,s}}{q^d + 1} + \frac{(\lambda_- + \lambda_+)}{2} e_i^\intercal \chi \\
&= q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} + \frac{(\lambda_- + \lambda_+)}{2} e_i^\intercal \chi.
\end{aligned}$$

If $d - s$ is even, then by (1) and $e_i^\intercal \chi_Y = 1$

$$e_i^\intercal A_{d-s} \chi_Y = q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} + q^{\binom{d-s+1}{2}} \begin{bmatrix} d-1 \\ s-1 \end{bmatrix} = q^{\binom{d-s}{2}} \begin{bmatrix} d \\ s \end{bmatrix}.$$

If $d-s$ is odd, then by $e_i^\intercal \chi_Y = 1$

$$e_i^\intercal A_{d-s}\chi_Y = q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} - q^{\binom{d-s}{2}} \begin{bmatrix} d-1 \\ s \end{bmatrix} = 0.$$

All the remaining cases are either similar or trivial. □

Corollary 16. *Let (Y, Z) be a cross-intersecting EKR set of $\mathsf{Q}(2d, q)$ or $\mathsf{W}(2d-1, q)$, d even, of maximum size such that $Y \nsubseteq Z$. Then $Y \cap Z = \emptyset$.*

Proof. Let g be as in Proposition 15. By Proposition 15, no element of Y meets g in the same dimension as an element of Z, hence $Y \cap Z = \emptyset$. □

Now we have a strong combinatorial information about cross-intersecting EKR sets (Y, Z) of maximum size which are not EKR sets. By adding some geometrical arguments this leads to a complete classification of cross-intersecting EKR sets of maximum size in these parabolic and symplectic polar spaces as we shall see in the following.

Lemma 17. *Let g_1, g_2, g_3 be pairwise different generators of a polar space of rank d such that*

$$\dim(g_1 \cap g_2 \cap g_3) = d-2 \quad \textit{and} \quad g_1 \cap g_2 \neq g_1 \cap g_3$$
$$\textit{and} \quad \dim(g_1 \cap g_2) = \dim(g_1 \cap g_3) = d-1.$$

Then $g_2 \cap g_3$ is a $(d-2)$-dimensional subspace of g_1.

Proof. We have

$$\dim(g_1 \cap g_2), \dim(g_1 \cap g_3) > \dim(g_1 \cap g_2 \cap g_3).$$

By $\dim(g_1 \cap g_2) = \dim(g_1 \cap g_3) = d-1$,

$$\dim(\langle g_1 \cap g_2, g_1 \cap g_3 \rangle) = \dim(g_1 \cap g_2) + \dim(g_1 \cap g_3) - \dim(g_1 \cap g_2 \cap g_3) = d = \dim(g_1).$$

Hence,

$$\langle g_1 \cap g_2, g_1 \cap g_3 \rangle = g_1. \tag{12}$$

Suppose contrary to the first part of the assertion that $g_2 \cap g_3 \nsubseteq g_1$. By (12), we have

$$\begin{aligned}
&\dim(\langle g_1 \cap g_2, g_1 \cap g_3, g_2 \cap g_3 \rangle) \\
=&\dim(g_2 \cap g_3) + \dim(\langle g_1 \cap g_2, g_1 \cap g_3 \rangle) - \dim(g_2 \cap g_3 \cap \langle g_1 \cap g_2, g_1 \cap g_3 \rangle) \\
=&\dim(g_2 \cap g_3) + \dim(g_1) - \dim(g_1 \cap g_2 \cap g_3) \\
>&(d-2) + d - (d-2) = d.
\end{aligned}$$

As $\langle g_1 \cap g_2, g_1 \cap g_3, g_2 \cap g_3 \rangle$ is a totally isotropic subspace of a polar space rank d, this is a contradiction. Hence, $g_2 \cap g_3 \subseteq g_1$. Furthermore, $\dim(g_2 \cap g_3) = \dim(g_1 \cap g_2 \cap g_3) = d-2$. □

Lemma 18. *Let (Y, Z) be a cross-intersecting EKR set of $\mathsf{Q}(2d, q)$ or $\mathsf{W}(2d-1, q)$, d even, of maximum size with $Y \neq Z$. Let $g, h \in Y$ disjoint. Let $\pi_1, \dots, \pi_{[d]} \subseteq g$ be the $[d]$ subspaces of dimension $d-1$ of g. Then the following holds:*

(a) Exactly $[d]$ elements $z_1, \dots, z_{[d]}$ of Z meet g in dimension $d-1$.

(b) We have $\{z_i \mid i \in \{1, \dots, [d]\}\} = \{\langle \pi_i, \pi_i^\perp \cap h \rangle \mid i \in \{1, \dots, [d]\}\}$.

(c) We have that $z_i \cap z_j$ is a $(d-2)$-dimensional subspace of g.

Proof. By Proposition 15 (d) and $Y \neq Z$, exactly $[d]$ elements of Z meet g in dimension $d-1$. This shows (a). By Proposition 15 (b), $\dim(z_i \cap z_j) < d-1$ for $i \neq j$. Hence, each hyperplane π_i of g lies in exactly one element z_i, and z_i satisfies $z_i \subseteq \pi_i^\perp$. Since (Y, Z) is a cross-intersecting EKR set, all z_i meet h in at least a point. Since $\pi_i \subseteq g$, $\dim(\pi_i) = d-1$, and $g \cap h = \emptyset$, we see that $\pi_i^\perp \cap h$ is a point. Hence,

$$\{z_i \mid i \in \{1, \dots, [d]\}\} = \{\langle \pi_i, \pi_i^\perp \cap h \rangle \mid i \in \{1, \dots, [d]\}\}.$$

This shows (b). Lemma 17 (with $g_1 = g, g_2 = z_i, g_3 = z_j$) shows (c). □

5.2.1 The Parabolic Quadric $\mathsf{Q}(2d, q)$, d even

Let s be a subspace of $\mathrm{PG}(2d, q)$. We write $Y \subseteq s$ if all elements of Y are subspaces of s, $Y \cap s$ for all elements of Y in s, and $Y \setminus s$ for all elements of Y not in s.

Lemma 19. *Let g and h be disjoint generators of $\mathsf{Q}(2d, q)$. Then $\mathcal{Q} := \langle g, h \rangle \cap \mathsf{Q}(2d, q)$ is isomorphic to $\mathsf{Q}^+(2d-1, q)$.*

Proof. The generators g and h are disjoint, hence $\mathcal{Q}$ is not degenerate. The hyperplane $\langle g, h \rangle$ obviously contains generators, hence $\mathcal{Q}$ does not have type $\mathsf{Q}^-(2(d-1)+1, q)$. Therefore, the intersection $\langle g, h \rangle \cap \mathsf{Q}(2d, q)$ is isomorphic to $\mathsf{Q}^+(2d-1, q)$. □

Lemma 20. *Let (Y, Z) be a cross-intersecting EKR set of $\mathsf{Q}(2d, q)$, d even, of maximum size such that $Y \cap Z = \emptyset$. Let $g \in Y$. Let $\tilde{Y}$ be the set of generators of Y disjoint to g.*

(a) There exists a hyperplane η of type $\mathsf{Q}^+(2d-1, q)$ such that $g, \tilde{Y} \subseteq \eta$.

(b) Let $z \in Z$. If z meets an element of $\{g\} \cup \tilde{Y}$ in a subspace of dimension $d-1$, then $z \in \eta$ and all elements of Z disjoint to z are in η.

(c) If $\tilde{g} \in Y$ and $\dim(\tilde{g} \cap h) = d-2$ for an $h \in \tilde{Y}$, then $\tilde{g}$ and the $q^{\binom{d}{2}}$ generators disjoint to $\tilde{g}$ are in η.

Proof. By Proposition 15 (c), a generator $g \in Y$ is disjoint to $q^{\binom{d}{2}}$ generators of Y. Let $h \in \tilde{Y}$. By Lemma 19, $\eta := \langle g, h \rangle$ has type $\mathsf{Q}^+(2d-1, q)$. We shall show $\tilde{Y} \subseteq \eta$.

Suppose to the contrary that there exists a generator $\tilde{h} \in \tilde{Y}$ not in η. In the notation of Lemma 18 we write

$$Z_a = \{z_i \mid i \in \{1, \dots, [d]\}\}$$

$$= \{z \in Z \mid \dim(z \cap a) = d-1\} \subseteq Z$$

for $a \in Y$.

Define the set of points

$$\mathcal{P} := \{\tilde{h} \cap z \mid z \in Z_g\}.$$

By Lemma 18 (c), $|\mathcal{P}| = |Z_g| = [d]$. Furthermore, $Z_g \subseteq \eta$ as the elements of Z_g meet h non-trivially, so $\mathcal{P} \subseteq \tilde{h} \cap \eta$. Hence,

$$[d] = |\mathcal{P}| \leqslant |\tilde{h} \cap \eta| = [d-1].$$

This is a contradiction. Thus, $\tilde{Y} \subseteq \eta$. This proves (a).

To prove (b), suppose without loss of generality that z meets g in a subspace of dimension $d-1$. Then $z \subseteq \eta$ as $z \in Z_g$ and $Z_g \subseteq \eta$. Let π be a $(d-1)$-dimensional space of h disjoint to z. Define $\tilde{z}$ as $\langle \pi, \pi^\perp \cap g\rangle$. Then $\tilde{z} \in Z_h$ by Lemma 18 (b) and $\tilde{z}$ is disjoint to z by Proposition 15 (b). Then $\tilde{z} \subseteq \eta$. Hence, $\langle z, \tilde{z}\rangle = \eta$. By (a), all elements of Z disjoint to z are in η. This shows (b).

Let $\tilde{g} \in Y$ with $\dim(h \cap \tilde{g}) = d-2$. By Lemma 18 (c), there exists a $z \in Z_h$ with $h \cap \tilde{g} \subseteq z$. By (b), $\tilde{g} \subseteq \eta$. Let $\tilde{h} \in Y$ be disjoint to $\tilde{g}$. As (Y, Z) is a cross-intersecting EKR set, z meets $\tilde{h}$ in a point. Let $\bar{z} \in Z_{\tilde{h}}$ with z disjoint to $\bar{z}$ ($\bar{z}$ exists for the same reasons as $\tilde{z}$ exists). By (b) and $\dim(h \cap z) = d-1$, $\langle z, \bar{z}\rangle = \eta$. Hence, $\tilde{h} \subseteq \eta$. By (a), all elements of Y disjoint to $\tilde{g}$ are in η. This shows (c). □

Corollary 21. *Let (Y, Z) be a cross-intersecting EKR set of $\mathsf{Q}(2d, q)$, d even, of maximum size such that $Y \cap Z = \emptyset$. Let $g_1, g_2, g_3 \in Y$. Then*

$$\dim(g_1 \cap g_2) \in \{0, d-2\} \qquad \text{and} \qquad \dim(g_2 \cap g_3) = 0$$

implies that $g_1 \subseteq \langle g_2, g_3\rangle$.

Proof. As g_2 and g_3 are disjoint, $\langle g_2, g_3\rangle$ is a hyperplane η. By Lemma 20 (a), $g_1 \subseteq \langle g_2, g_3\rangle$ if $\dim(g_1 \cap g_2) = 0$. By Lemma 20 (c), $g_1 \subseteq \langle g_2, g_3\rangle$ if $\dim(g_1 \cap g_2) = d-2$. □

Corollary 22. *Let (Y, Z) be a cross-intersecting EKR set of $\mathsf{Q}(2d, q)$, d even, of maximum size such that $Y \cap Z = \emptyset$. Suppose that there are $g, h, \tilde{g} \in Y$ with*

$$\dim(g \cap h) = d-2 \text{ and } \dim(g \cap \tilde{g}) > 0.$$

Let η be the hyperplane, which contains g and all elements of Y disjoint to g (see Proposition 20 (a)). If $\tilde{g}$ and h are disjoint, then $\tilde{g} \in \eta$ or all generators $\tilde{h} \in Y$ with $\dim(\tilde{g} \cap \tilde{h}) = d-2$ meet g non-trivially.

Proof. Suppose that $\tilde{g}$ and h are disjoint. By Proposition 15 (b), we have that $\dim(g \cap \tilde{g})$ is even. By $\dim(g \cap h) = d-2$ and $\dim(g \cap \tilde{g}) > 0$, we have that $\dim(g \cap \tilde{g}) = 2$. Let $\tilde{h} \in Y$ with $\dim(\tilde{g} \cap \tilde{h}) = d-2$. If g and $\tilde{h}$ are disjoint, then $\tilde{g} \subseteq \eta = \langle g, \tilde{h}\rangle$. Hence, if $\tilde{g} \nsubseteq \eta$, then $\dim(g \cap \tilde{h}) > 0$. □

We need the following bound.

Lemma 23. *Let $q \geqslant 2$. Let $d \geqslant 1$. Then*

$$\prod_{i=1}^{d-1}(q^i+1) \leqslant \frac{2q^d}{q^d+1}\left(q^{\binom{d}{2}} - q^{\binom{d-1}{2}} + 1\right) + q^{\binom{d-2}{2}+2(d-2)}.$$

Proof. We will prove the assertion by induction over d. It can be easily checked that the assertion is true for $d \leqslant 4$. If the assertion is true for $d \geqslant 4$, then

$$\begin{aligned}\prod_{i=1}^{d}(q^i+1) &\leqslant (q^d+1)\left(\frac{2q^d}{q^d+1}\left(q^{\binom{d}{2}} - q^{\binom{d-1}{2}} + 1\right) + q^{\binom{d-2}{2}+2(d-2)}\right)\\ &\overset{(*)}{\leqslant} \frac{2q^{d+1}}{q^{d+1}+1}\left(q^{\binom{d+1}{2}} - q^{\binom{d}{2}} + 1\right) + q^{\binom{d-1}{2}+2(d-1)}.\end{aligned}$$

The difference between the right side of $(*)$ and the left side of $(*)$ equals

$$\frac{q^d}{q^{d+1}+1}\left(2q^{\binom{d}{2}+2} - 2q^{\binom{d}{2}+1} - 3q^{\binom{d}{2}} + 2q^{\binom{d-1}{2}} - q^{\binom{d-1}{2}-2} - 2q^{d+1} + 2q - 2\right)$$

which is a positive expression for $q \geqslant 2$. □

Proposition 24. *Let (Y, Z) be a cross-intersecting EKR set of $\mathsf{Q}(2d, q)$ of maximum size such that $Y \cap Z = \emptyset$. Then there exists a hyperplane η such that $Y, Z \subseteq \eta$.*

Proof. Let $g \in Y$. In the view of Lemma 20, we find a hyperplane η that contains g, the set Y_1 of $q^{\binom{d}{2}}$ generators of Y disjoint to g, and, by Proposition 15 and Lemma 20 (c), the set Y_2 of $\begin{bmatrix}d\\2\end{bmatrix}q^{\binom{d-2}{2}}$ generators of Y which meet g in dimension $d-2$.

Suppose contrary to the assertion that there exists an element $\tilde{g} \in Y$ that is not in η. Then $\tilde{g}$ and the set Y_3 of $q^{\binom{d}{2}}$ generators of Y disjoint to $\tilde{g}$ lie in a second hyperplane $\eta' \neq \eta$ by Lemma 20. Let Y_4 be the set of $\begin{bmatrix}d\\2\end{bmatrix}q^{\binom{d-2}{2}}$ generators of Y which meet $\tilde{g}$ in dimension $d-2$.

First we show $|(Y_2 \cup Y_4) \setminus (Y_1 \cup Y_3)| \geqslant \begin{bmatrix}d\\2\end{bmatrix}q^{\binom{d-2}{2}}$. If $Y_2 \cap Y_3$ is empty, then the claim is clear. Suppose that there exists a generator $h \in Y_2 \cap Y_3$. By Corollary 22, then Y_4 is disjoint to Y_1. The claim follows.

Now we want to show that $Y_1 \cap Y_3$ is empty. Suppose that there exists a generator $h \in Y_1 \cap Y_3$. By Corollary 21 with $g_1 = \tilde{g}$, $g_2 = h$, and $g_3 = g$, we obtain the contradiction $\tilde{g} \subseteq \langle g, h\rangle = \eta$. Hence,

$$\begin{aligned}|Y| &\geqslant |Y_1| + |Y_3| + |(Y_2 \cup Y_4) \setminus (Y_1 \cup Y_3)| + 2\\ &= 2\left(q^{\binom{d}{2}} + 1\right) + \begin{bmatrix}d\\2\end{bmatrix}q^{\binom{d-2}{2}} \geqslant 2\left(q^{\binom{d}{2}} + 1\right) + q^{\binom{d-2}{2}+2(d-2)}. \qquad (13)\end{aligned}$$

According to Theorem 10,

$$|Y| = \prod_{i=1}^{d-1}(q^i+1).$$

This contradicts (13) by Lemma 23. □

Theorem 25. *Let (Y, Z) be a cross-intersecting EKR set of $\mathsf{Q}(2d, q)$, or $\mathsf{W}(2d-1, q)$, q even, of maximum size such that $Y \neq Z$. Then d even and (Y, Z) is a cross-intersecting EKR set of generators of a subgeometry $\mathcal{Q}$ isomorphic to $\mathsf{Q}^+(2d-1, q)$ (i.e. Y are latins and Z are greeks).*

Proof. First consider $\mathsf{Q}(2d, q)$. By Corollary 16, then $Y \cap Z = \emptyset$. By Proposition 24, $Y, Z \subseteq \eta$ for some hyperplane η isomorphic to $\mathsf{Q}^+(2d-1, q)$ if not $Y = Z$. Hence, (Y, Z) is a cross-intersecting set of $\mathsf{Q}^+(2d-1, q)$ of maximum size. These sets were classified in Theorem 11.

The part of the assertion for $\mathsf{W}(2d-1, q)$, q even, follows, since $\mathsf{Q}(2d, q)$ and $\mathsf{W}(2d-1, q)$ are isomorphic for q even. □

5.2.2 The Symplectic Polar Space $\mathsf{W}(2d-1, q)$, d even, q odd

Similar to [18] we use the following property of $\mathsf{W}(2d-1, q)$, d even (see [17, 1.3.6, 3.2.1, 3.3.1]):

Theorem 26. *Let ℓ_1, ℓ_2, ℓ_3 be three pairwise disjoint lines of $\mathsf{W}(3, q)$, q odd. Then the number of lines meeting ℓ_1, ℓ_2, ℓ_3 is 0 or 2.*

Theorem 27. *Let (Y, Z) be a cross-intersecting EKR set of maximum size of $\mathsf{W}(2d-1, q)$, d even, q odd. Then $Y = Z$.*

Proof. Suppose to the contrary that $Y \neq Z$. By Corollary 16, then $Y \cap Z = \emptyset$. By Proposition 15 (c), we can find two disjoint generators g and h in Y. Again by Proposition 15 (c), there are exactly $q[d][d-1]/(q+1)$ generators $Y' \subseteq Y$ which meet g in a subspace of dimension $d-2$. The generator g has $[d][d-1]/(q+1)$ subspaces of dimension $d-2$. Hence, we find a subspace $\ell \subseteq g$ of dimension $d-2$ such that ℓ is contained in q elements of Y'. Since q is odd, there are at least three elements y_1, y_2, y_3 of Y through ℓ.

Consider the quotient geometry $\mathcal{W}_3$ of ℓ isomorphic to $\mathsf{W}(3, q)$ and the projection of the elements of Y and Z onto $\mathcal{W}_3$ from ℓ. Since elements of Y do not meet each other in dimension $d-1$ by Proposition 15, y_1, y_2, y_3 are three disjoint lines in $\mathcal{W}_3$ after projection. The subspace $\ell^\perp \cap h$ has dimension 2, so we find a subspace $\tilde{\ell} \subseteq h$ of dimension $d-2$ disjoint to $\ell^\perp$. Let π_1, π_2, π_3 be subspaces of dimension $d-1$ in h with $\tilde{\ell} \subseteq \pi_1, \pi_2, \pi_3$. By Lemma 18 (b), we find $z_1, z_2, z_3 \in Z$ with $\pi_i \subseteq z_i$ for $i \in \{1, 2, 3\}$. By Lemma 18 (c), the pairwise meets of z_1, z_2, z_3 are contained in $\tilde{\ell}$. As we have that $\tilde{\ell}$ meets $\ell^\perp$ trivially, we have that z_1, z_2, z_3 are projected onto three disjoint lines on $\mathcal{W}_3$. These three lines have to meet the projections of y_1, y_2, and y_3, since (Y, Z) is a cross-intersecting EKR set. This contradicts Theorem 26. □

6 The Hermitian Polar Space $\mathsf{H}(2d-1, q^2)$

It is well-known (see for example [14]) that the linear programming bound given in [13] can be reformulated as a weighted Hoffman bound. Hence, Lemma 4 is applicable if $d > 1$. The original bound on EKR sets on $\mathsf{H}(2d-1, q^2)$ is as follows.

Theorem 28 ([13]). *Let Y be an EKR set of $\mathsf{H}(2d-1,q^2)$ with $d>2$ odd. Then*

$$|Y| \leqslant \frac{nq^{d-1} - f_1(q^{d-1}-1)\,(1-c)}{q^{2d-1}+q^{d-1}+f_1(q^{d-1}-1)c} \approx q^{d^2-2d+2},$$

where $n=\prod_{i=0}^{d-1}(q^{2i+1}+1)$, $f_1 = q^2[d]_{q^2}\frac{q^{2d-3}+1}{q+1}$ and $c = \frac{q^2-q-1+q^{-2d+3}}{q^{2d}-1}$.

The result by Luz [14] which shows that the linear programming bound is a special case of the weighted Hoffman bound[1] also holds for cross-intersecting EKR sets, but we feel that we should show this directly, since the transition from the linear programming technique used in [13] to the weighted Hoffman bound is not obvious. We shall prove a cross-intersecting result similar to [13] in the following.

Theorem 29. *Let (Y,Z) be a cross-intersecting EKR set of $\mathsf{H}(2d-1,q^2)$ with $d>2$. Then*

$$\sqrt{|Y|\cdot|Z|} \leqslant \frac{n\lambda_b}{\lambda_b-k} \approx q^{d^2-2d+2},$$

where $n=\prod_{i=0}^{d-1}(q^{2i+1}+1)$, $\lambda_b = -q^{(d-1)^2} - \alpha\left(1-f_1\frac{1-c}{n}\right)$, $k=q^{d^2}+\alpha f_1\left(c+\frac{1-c}{n}\right)$, $f_1 = q^2[d]_{q^2}\frac{q^{2d-3}+1}{q+1}$, $c=\frac{q^2-q-1+q^{-2d+3}}{q^{2d}-1}$, and

$$\alpha = \begin{cases} q^{d(d-1)} - q^{(d-1)^2} & \text{if } d \text{ odd},\\ \frac{q^{d(d-1)}-q^{(d-1)^2}}{1-2(1-c)f_1/n} & \text{if } d \text{ even}.\end{cases}$$

Proof of Theorem 29. Let $d>1$. Let A_d be the disjointness matrix as defined in Section 3. Consider the matrix A defined by

$$A = A_d - \alpha E_1 + \frac{\alpha f_1 c}{n}\mathbf{J} + \alpha f_1 \frac{1-c}{n}\mathbf{I}$$

Claim 1. Our first claim is that α is positive. For d odd, this is obvious. For d even it is sufficient to show that $1-2f_1/n$ is positive as we have $c\in[0,1]$ and $f_1>0$. By $d\geqslant 4$,

$$\begin{aligned} 1 &< \frac{(q+1)(q^2-1)}{2q^2} \\ &\leqslant \frac{\left(\prod_{i=0}^{d-3}(q^{2i+1}+1)\right)(q^{2d-1}+1)}{2q^2[d]_{q^2}} \\ &\leqslant \frac{\left(\prod_{i=0}^{d-1}(q^{2i+1}+1)\right)(q+1)}{2q^2[d]_{q^2}\,(q^{2d-3}+1)} = \frac{n}{2f_1}. \end{aligned}$$

Claim 2. Our second claim is that A is an extended weight adjacency matrix. By Section 3, it is clear that the entry (x,y) of E_1 equals $Q_{i,1}/n$ if x and y meet in codimension i. It was shown in [13, Equations (6)–(11)] that the following[2] holds:

[1] This seems to be part of the mathematical folklore for a long time, but the author is not aware of any source older than [14].

[2] Note the following about [13]:

(a) $Q_{0,1} = f_1$,

(b) $Q_{d-1,1} = f_1 c$,

(c) $Q_{s,1} \geqslant f_1 c$ if $s < d$,

(d) $Q_{d,1} < 0$.

Hence, the entry (x, y) of the matrix A is 0 if $x = y$, it is less than or equal to zero if $1 \leqslant \operatorname{codim}(x \cap y) \leqslant d - 1$, and it is larger than 1 if x and y are disjoint. This shows that A is an extended weight adjacency matrix of the disjointness graph of generators.

Claim 3. Our third claim is that the second largest absolute eigenvalue of A is

$$q^{(d-1)^2} + \alpha \left(1 - f_1 \frac{1-c}{n}\right),$$

and that

$$k = q^{d^2} + \alpha f_1 \left(c + \frac{1-c}{n}\right).$$

Recall that A is a linear combination of the A_is of the scheme, so the common eigenspaces of the A_is are the eigenspaces of A. Notice for the following that $c \in [0, 1]$ and $\alpha, f_1 > 0$. By (2), the eigenvalues of A are

$$\begin{aligned}
&q^{d^2} + \alpha f_1 \left(c + \frac{1-c}{n}\right) \text{ for } \langle \mathbf{j} \rangle,\\
&-q^{(d-1)^2} - \alpha \left(1 - f_1 \frac{1-c}{n}\right) \text{ for } V_1,\\
&(-1)^r q^{(d-r)^2 + r(r-1)} + \alpha f_1 \frac{1-c}{n} \text{ for } V_r \text{ with } 1 < r < d,\\
&(-1)^d q^{d(d-1)} + \alpha f_1 \frac{1-c}{n} \text{ for } V_d.
\end{aligned}$$

If d even, then we have that

$$q^{(d-1)^2} + \alpha \left(1 - f_1 \frac{1-c}{n}\right) = q^{d(d-1)} + \alpha f_1 \frac{1-c}{n}$$

is the second largest absolute eigenvalue. If d odd, then

$$q^{(d-1)^2} + \alpha \left(1 - f_1 \frac{1-c}{n}\right) = q^{d(d-1)} - \alpha f_1 \frac{1-c}{n}$$

1. The equations (6)–(11) in [13] do not depend on d odd.

2. The relations in [13] are numbered $-1, 0, \ldots, d$ for a Hermitian polar space of rank $d + 1$. This implies that one has to replace an index i by $i + 1$ (in particular, d by $d + 1$) if one goes from the notation of [13] to the notation of this article.

3. We use ϵ in [13] instead of e as in this article, so instead of type $\epsilon = -1/2$ we have type $e = 1/2$.

is the second largest absolute eigenvalue. This proves our claim.

Now we can apply Lemma 4 with these values. Note that k has approximately size q^{d^2+d-2}, the second largest absolute eigenvalue λ_b has approximately size $q^{d(d-1)}$, and n has approximately size q^{d^2}. Therefore,

$$\frac{n\lambda_b}{\lambda_b + k}$$

has approximately size q^{d^2-2d+2}. □

Note that the normal adjacency matrix of the disjointness graph of generators of $\mathsf{H}(2d-1, q^2)$ only yields q^{d^2-d} as an upper bound, so this improves the bound significantly.

For the sake of completeness we want to mention the cross-intersecting EKR sets for $d = 2$ as Lemma 4 only considered the case $d > 2$. We will do this after providing a general geometrical result on (maximal) cross-intersecting EKR sets, where we call a cross-intersecting EKR set (Y, Z) *maximal* if there exists no generator x such that $(Y \cup \{x\}, Z)$ or $(Y, Z \cup \{x\})$ is a cross-intersecting EKR set.

Lemma 30. *Let (Y, Z) be a maximal cross-intersecting EKR set in a finite classical polar space of rank d. If two distinct elements $y_1, y_2 \in Y$ meet in a subspace of dimension $d-1$, then all elements of Z meet this subspace in at least a point.*

Proof. Assume that there exists a generator z which meets y_1 and y_2 in points p, r not in $y_1 \cap y_2$. Then $\langle p, r, y_1 \cap y_2 \rangle$ is a totally isotropic subspace of dimension $d+1$. Contradiction. □

Theorem 31. *Let (Y, Z) be a maximal cross-intersecting EKR set of $\mathsf{H}(3, q^2)$ with $|Y| \geqslant |Z|$. Then one of the following cases occurs:*

(a) The set Y is the set of all lines of $\mathsf{H}(3, q^2)$, and $Z = \emptyset$. Here $|Y| \cdot |Z| = 0$.

(b) The set Y is the set of all lines meeting a fixed line ℓ in at least a point, and $Z = \{\ell\}$. Here $|Y| \cdot |Z| = (q^2 + 1)q + 1$.

(c) The set Y is the set of all lines on a fixed point p, and $Y = Z$. Here $|Y| \cdot |Z| = (q+1)^2$.

(d) The set Y is the set of lines meeting two disjoint lines ℓ_1, ℓ_2, and $Z = \{\ell_1, \ell_2\}$. Here $|Y| \cdot |Z| = 2(q^2 + 1)$.

(e) The set Y is the set of lines meeting three disjoint lines ℓ_1, ℓ_2, ℓ_3, and Z is the set of all $q + 1$ lines meeting the lines of Y. Here $|Y| \cdot |Z| = (q + 1)^2$.

Proof. Assume that (a) does not occur.

By Lemma 30, as soon as two elements of Y meet in a point p, then all elements of Z contain p. Hence, (b) occurs or at least 2 elements of Z meet in p. Hence, all elements of Y contain p by Lemma 30. This is case (c).

So assume that Y and vice-versa Z only consist of disjoint lines. If there are two lines $\ell_1, \ell_2 \subseteq Z$, then there are $q^2 + 1$ (disjoint) lines L meeting ℓ_1 and ℓ_2 in a point (hence

$|Y| \leqslant q^2 + 1$). If more than $q + 1$ of these lines meet ℓ_1 (hence $|Y| > q + 1$), then Z contains at most two lines, since in $\mathsf{H}(3, q^2)$ exactly $q + 1$ lines meet 3 pairwise disjoint lines in a point. This yields (d). If $|Z| \geqslant 3$, then $|Y| \leqslant q + 1$ by the previous argument. We may assume $|Y| \geqslant 3$. Then it is well-known that there are exactly $q+1$ lines meeting the $q + 1$ lines of Y. Hence, we can add these lines and then Z is maximal. This yields (e). □

The author tried to prove that the maximum cross-intersecting EKR set of $\mathsf{H}(5, q^2)$ is the unique EKR of maximum size given in [18], but aborted this attempt after he got lost in too many case distinctions. The EKR set Y of all generators meeting a fixed generator in at least a line corresponds to the largest cross-intersecting EKR set (Y, Y) known to the author and has size $q^5 + q^3 + q + 1$. The largest example known to the author for $\mathsf{H}(7, q^2)$ is the following.

Example 32. Let g be a generator of $\mathsf{H}(7, q^2)$. Let Y be the set of all generators that meet g in at least a 2-dimensional subspace. Let Z be the set of all generators that meet g in at least a 3-dimensional subspace. Then (Y, Z) is a cross-intersecting EKR set.

Proof. A generator of $\mathsf{H}(7, q^2)$ is a 4-dimensional subspace. A plane and a line of a 4-dimensional subspace meet pairwise in at least a point. Hence, (Y, Z) is a cross-intersecting EKR set. □

In this example Y has

$$1 + q + q^3 + q^4 + q^5 + q^6 + q^7 + 2q^8 + q^{10} + q^{12}$$

elements, Z has

$$1 + q + q^3 + q^5 + q^7$$

elements, so in total the cross-intersecting EKR set has size

$$\sqrt{|Y| \cdot |Z|} \approx q^{19/2}.$$

The bound given in Theorem 29 for this case is approximately q^{10}. For $\mathsf{H}(2d - 1, q^2)$, $d > 4$, the largest example known to the author is the EKR set of all generators on a fixed point. The author assumes that the largest known examples are also the largest examples.

7 Summary

We summarize our results in the following table. We only list the cases, where cross-intersecting EKR sets of maximum size are not necessarily EKR sets. The table includes the size of the largest known example if it is not known if the best known bound does not seem to be tight.

Polar Space	Maximum Size $\sqrt{\lvert Y\rvert \cdot \lvert Z\rvert}$	Largest (known) Examples	Reference
$\mathsf{Q}^+(2d-1,q)$, d even	$n/2$	Y latins, Z greeks	Th. 11
$\mathsf{Q}(2d,q)$, d even	$(q+1)\cdot\ldots\cdot(q^{d-1}+1)$	Y latins and Z greeks of a $\mathsf{Q}^+(2d+1,q)$, or $Y=Z$ EKR set	Th. 25
$\mathsf{W}(2d-1,q)$, d even, q even	$(q+1)\cdot\ldots\cdot(q^{d-1}+1)$	see $\mathsf{Q}(2d,q)$	Th. 25
$\mathsf{H}(3,q^2)$	q^3+q+1	$Z=\{\ell\}$, Y all lines meeting ℓ	Th. 31
$\mathsf{H}(5,q^2)$	$\lessapprox q^5$	largest EKR set, size $\approx q^5$	Th. 29
$\mathsf{H}(7,q^2)$	$\lessapprox q^{10}$	Example 32, size $\approx q^{19/2}$	Th. 29
$\mathsf{H}(2d-1,q^2)$, $d>4$	$\lessapprox q^{(d-1)^2+1}$	all generators on a point, size $\approx q^{(d-1)^2}$	Th. 29

8 Open Problems

As mentioned before, the case $\mathsf{H}(2d-1,q^2)$ is mostly open. The next logical step would be to solve the problem for $\mathsf{H}(5,q^2)$ and $\mathsf{H}(7,q^2)$. It might be also interesting to study the same problem for proper subspaces of generators. For classical EKR sets this was done recently by Klaus Metsch [16] using mostly geometrical arguments.

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