On exponential growth of degrees

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Abstract

A short proof to a recent theorem of Giambruno and Mishchenko is given in this note.

1 The theorem

The following theorem was recently proved by Giambruno and Mishchenko.

Theorem 1. *[1, Theorem 1] For every $0 < \alpha < 1$, there exist $\beta > 1$ and $n_0 \in \mathbb{N}$, such that for every partition λ of $n \geqslant n_0$ with $\max\{\lambda_1, \lambda'_1\} < \alpha n$*

$$f^\lambda \geqslant \beta^n.$$

The proof of Giambruno and Mishchenko is rather complicated and applies a clever order on the cells of the Young diagram. It should be noted that Theorem 1 is an immediate consequence of Rasala's lower bounds on minimal degrees [2, Theorems F and H]. The proof of Rasala is very different and not less complicated; it relies heavily on his theory of degree polynomials. In this short note we suggest a short and relatively simple proof to Theorem 1.

First, note that the following weak version is an immediate consequence of the hook-length formula.

Lemma 2. *The theorem holds for every $0 < \alpha < \frac{1}{2e}$.*

Proof. Under the assumption, for every $(i, j) \in [\lambda]$

$$h_{i,j} \leqslant h_{1,1} \leqslant \lambda_1 + \lambda'_1 \leqslant 2\alpha n.$$

Hence, by the hook-length formula together with Stirling's formula, for sufficiently large n

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in[\lambda]} h_{i,j}} \geqslant \frac{n!}{(2\alpha n)^n} \geqslant \frac{(\frac{n}{e})^n}{(2\alpha n)^n} = \beta^n,$$

where, by assumption, $\beta := \frac{1}{2e\alpha} > 1$. □

2 Two lemmas

Lemma 3. *For every $\lambda \vdash n$*

$$\prod_{\substack{(i,j)\in[\lambda] \\ 1<i}} h_{ij} \leqslant (n-\lambda_1)!$$

Proof. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) \vdash n$ let $\bar{\lambda} := (\lambda_2, \ldots, \lambda_t) \vdash n - \lambda_1$. Then

$$1 \leqslant f^{\bar{\lambda}} = \frac{(n-\lambda_1)!}{\prod_{\substack{(i,j)\in[\lambda] \\ 1<i}} h_{ij}}.$$

□

Lemma 4. *For every $\lambda \vdash n$ and $1 \leqslant k \leqslant \lambda_1$,*

$$\prod_{(1,j)\in[\lambda]} h_{1j} \leqslant \binom{n}{k} \left(\lambda_1 + \left\lfloor \frac{n-\lambda_1}{k} \right\rfloor\right)!.$$

Proof. Obviously, $h_{1,1} > h_{1,2} > \cdots > h_{1,\lambda_1}$. Since $h_{1,1} \leqslant n$ it follows that

$$\prod_{\substack{(1,j)\in[\lambda] \\ j\leqslant k}} h_{1j} \leqslant (n)_k$$

and

$$\prod_{\substack{(1,j)\in[\lambda] \\ k<j}} h_{1j} \leqslant (h_{1,k})_{\lambda_1 - k}.$$

To complete the proof, notice that, by definition, $\sum_{i=1}^{k} \bar{\lambda}'_i \leqslant n - \lambda_1$. Hence $\bar{\lambda}'_k \leqslant \lfloor \frac{n-\lambda_1}{k} \rfloor$ and thus

$$h_{1,k} = \lambda_1 - k + \lambda'_k = \lambda_1 - k + 1 + \bar{\lambda}'_k \leqslant \lambda_1 + \bar{\lambda}'_k \leqslant \lambda_1 + \left\lfloor \frac{n-\lambda_1}{k} \right\rfloor.$$

We conclude that

$$\prod_{\substack{(1,j)\in[\lambda] \\ k<j}} h_{1j} \leqslant (h_{1,k})_{\lambda_1-k} \leqslant \left(\lambda_1 + \left\lfloor \frac{n-\lambda_1}{k} \right\rfloor\right)_{\lambda_1-k} = \frac{(\lambda_1 + \lfloor \frac{n-\lambda_1}{k} \rfloor)!}{(\lfloor \frac{n-\lambda_1}{k} \rfloor + k)!} \leqslant \frac{(\lambda_1 + \lfloor \frac{n-\lambda_1}{k} \rfloor)!}{k!}.$$

Thus

$$\prod_{(1,j)\in[\lambda]} h_{1j} = \prod_{\substack{(1,j)\in[\lambda]\\ j\leqslant k}} h_{1j} \prod_{\substack{(1,j)\in[\lambda]\\ k<j}} h_{1j} \leqslant (n)_k \frac{\lambda_1 + \lfloor \frac{n-\lambda_1}{k} \rfloor !}{k!} = \binom{n}{k} \left(\lambda_1 + \left\lfloor \frac{n-\lambda_1}{k} \right\rfloor\right)!.$$

□

3 Proof of Theorem 1

For the sake of simplicity the floor notation is omitted in this section.

By Lemmas 3 and 4,

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in[\lambda]} h_{ij}} = \frac{n!}{\prod_{(1,j)\in[\lambda]} h_{1j} \prod_{\substack{(i,j)\in[\lambda]\\ 1<i}} h_{ij}} \geqslant \frac{n!}{(n-\lambda_1)!\binom{n}{k}(\lambda_1 + \frac{n-\lambda_1}{k})!}$$

$$= \frac{(n-k)!k!}{(n-\lambda_1)!(\lambda_1 + \frac{n-\lambda_1}{k})!}.$$

Denote $\gamma_n := \frac{\lambda_1}{n}$. By Lemma 2, we may assume that $\frac{1}{2e} < \gamma_n < \alpha$. Choose $k = \epsilon n$ for a constant $\epsilon = \epsilon(\alpha)$ to be defined later. Let $c_n := \frac{1-\gamma_n}{\epsilon}$. Thus, by definition, $\frac{1-\alpha}{\epsilon} < c_n < \frac{2e-1}{2e\epsilon}$. By the Stirling's formula, the lower bound in the RHS asymptotically equals to

$$\frac{((1-\epsilon)n)!(\epsilon n)!}{((1-\gamma_n)n)!(\gamma_n n + \frac{1-\gamma_n}{\epsilon})!} \sim \sqrt{\frac{\epsilon(1-\epsilon)}{(1-\gamma_n)(\gamma_n + \frac{c_n}{n})}} \cdot \frac{(1-\epsilon)^{(1-\epsilon)n}\epsilon^{\epsilon n}}{(1-\gamma_n)^{(1-\gamma_n)n}(\gamma_n + \frac{c_n}{n})^{(\gamma_n + \frac{c_n}{n})n}} \cdot \left(\frac{e}{n}\right)^{c_n}.$$

Hence, for sufficiently large n

$$\lim_{n\to\infty} \inf (f^\lambda)^{1/n}$$

$$\geqslant \lim_{n\to\infty} \inf \left(\sqrt{\frac{\epsilon(1-\epsilon)}{(1-\gamma_n)(\gamma_n + \frac{c_n}{n})}} \cdot \frac{(1-\epsilon)^{(1-\epsilon)n}\epsilon^{\epsilon n}}{(1-\gamma_n)^{(1-\gamma_n)n}(\gamma_n + \frac{c_n}{n})^{(\gamma_n + \frac{c_n}{n})n}} \cdot \left(\frac{e}{n}\right)^{c_n} \right)^{1/n}$$

$$= \lim_{n\to\infty} \inf \frac{\epsilon^\epsilon (1-\epsilon)^{1-\epsilon}}{\gamma_n^{\gamma_n}(1-\gamma_n)^{1-\gamma_n}}.$$

The function $f(x) := x^x(1-x)^{1-x}$ is differentiable in the open interval $(0,1)$, symmetric around its minimum at $x = \frac{1}{2}$, decreasing in $(0, \frac{1}{2}]$, increasing in $[\frac{1}{2}, 1)$, strictly less than 1 in this interval and tends to 1 at the boundaries. Thus, for every $0 < 1-\beta \leqslant x \leqslant \beta < 1$, $f(x) \leqslant f(\beta) = f(1-\beta)$.

Now, if $1 - \frac{1}{2e} > \alpha$ then $\frac{1}{2e} < \gamma_n < \alpha < 1 - \frac{1}{2e}$; hence $f(\gamma_n) < f(1-\frac{1}{2e}) = f(\frac{1}{2e})$. If $1 - \frac{1}{2e} \leqslant \alpha$ then $1 - \alpha \leqslant \frac{1}{2e} < \gamma_n < \alpha$; hence $f(\gamma_n) < f(\alpha)$. It follows that

$$\lim_{n\to\infty} \inf (f^\lambda)^{1/n} \geqslant \lim_{n\to\infty} \inf \frac{f(\epsilon)}{f(\gamma_n)} > \begin{cases} \frac{f(\epsilon)}{f(\alpha)}, & 1 - \frac{1}{2e} \leqslant \alpha; \\ \frac{f(\epsilon)}{f(\frac{1}{2e})}, & \text{otherwise.} \end{cases}$$

Choosing $\epsilon := \delta \min\{1-\alpha, \frac{1}{2e}\}$ for some $0 < \delta < 1$ we conclude that

$$\lim_{n\to\infty} \inf (f^\lambda)^{1/n} > \begin{cases} \frac{f(\delta(1-\alpha))}{f(\alpha)} = \frac{f(\delta(1-\alpha))}{f(1-\alpha)}, & 1-\alpha \leqslant \frac{1}{2e}; \\ \frac{f(\frac{\delta}{2e})}{f(\frac{1}{2e})}, & \text{otherwise.} \end{cases}$$

Since the function f is strictly decreasing in $(0, \frac{1}{2e}]$, the lower bound is greater than 1. The proof is complete.

□

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References

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