Mutually unbiased Bush-type Hadamard matrices and association schemes

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Abstract

It was shown by LeCompte, Martin, and Owens in 2010 that the existence of mutually unbiased Hadamard matrices and the identity matrix, which coincide with mutually unbiased bases, is equivalent to that of a Q-polynomial association scheme of class four which is both Q-antipodal and Q-bipartite. We prove that the existence of a set of mutually unbiased Bush-type Hadamard matrices is equivalent to that of an association scheme of class five. As an application of this equivalence, we obtain an upper bound of the number of mutually unbiased Bush-type Hadamard matrices of order $4n^2$ to be $2n-1$. This is in contrast to the fact that the best general upper bound for the mutually unbiased Hadamard matrices of order $4n^2$ is $2n^2$. We also discuss a relation of our scheme to some fusion schemes which are Q-antipodal and Q-bipartite Q-polynomial of class 4.

1 Introduction

A *Hadamard* matrix is a matrix H of order n with entries in $\{-1, 1\}$ and orthogonal rows in the usual inner product on $\mathbb{R}^n$. Two Hadamard matrices H and K of order n are called

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unbiased if $HK^t = \sqrt{n}L$ for some Hadamard matrix L, where K^t denotes the transpose of K. In this case, it follows that n must be a perfect square. A Hadamard matrix of order n for which the row sums and column sums are all the same, necessarily $\sqrt{n}$, is called *regular*, see [12].

Definition 1. A *Bush-type* Hadamard matrix is a block Hadamard matrix $H = [H_{ij}]$ of order $4n^2$ with block size $2n$, $H_{ii} = J_{2n}$ and $H_{ij}J_{2n} = J_{2n}H_{ij} = 0$, $i \neq j$, $1 \leqslant i \leqslant 2n$, $1 \leqslant j \leqslant 2n$ where J_ℓ is the $\ell \times \ell$ matrix of ones.

It is known that for odd values of n there is no pair of unbiased Bush-type Hadamard matrices of order $4n^2$ [2]. In contrast, for even values of n, there are unbiased Bush-type Hadamard matrices of order $4n^2$ [9]. (Note that a missing necessary assumption is needed in the proof of [9, Theorem 13]. The modified version will be presented in Section 3.) One very important property of unbiased Bush-type Hadamard matrices, as is shown in section 3, is the fact that for any two unbiased pair of Bush-type Hadamard matrices H and K of order $4n^2$, $\frac{1}{2n}HK^t$ is also a Bush-type Hadamard matrix.

It was shown by LeCompte, Martin, and Owens in [10] that the existence of mutually unbiased Hadamard matrices and the identity matrix, which coincide with mutually unbiased bases, is equivalent to that of a Q-polynomial association scheme of class four which is both Q-antipodal and Q-bipartite.

Our aim in this paper is to show that the existence of unbiased Bush-type Hadamard matrices is equivalent to the existence of a certain association scheme of class five. As an application of this equivalence, we obtain an upper bound for the number of mutually unbiased Bush-type Hadamard matrices of order $4n^2$ to be $2n-1$, whereas the best general upper bound for the mutually unbiased Hadamard matrices of order $4n^2$ is $2n^2$ [9, Theorem 2]. Also we discuss a relation of our scheme to some association schemes of class four.

2 Association schemes

A *symmetric d-class association scheme*, see [1], with vertex set X of size n and d classes is a set of nonzero symmetric $(0,1)$-matrices $A_0, \dots, A_d$ with rows and columns indexed by X, such that:

1. $A_0 = I_n$, the identity matrix of order n.
2. $\sum_{i=0}^{d} A_i = J_n$.
3. For all i, j, $A_iA_j = \sum_{k=0}^{d} p_{ij}^k A_k$ for some p_{ij}^k.

It follows from property (3) that the matrices A_i necessarily commute. The vector space spanned by the matrices A_i forms a commutative algebra, denoted by $\mathcal{A}$ and called the *Bose-Mesner algebra* or *adjacency algebra*. There exists a basis of $\mathcal{A}$ consisting of primitive idempotents, say $E_0 = (1/n)J_n, E_1, \dots, E_d$. Since $\{A_0, A_1, \dots, A_d\}$ and $\{E_0, E_1, \dots, E_d\}$

are two bases in $\mathcal{A}$, there exist change-of-bases matrices $P = (P_{ij})_{i,j=0}^d$, $Q = (Q_{ij})_{i,j=0}^d$ so that

$$A_j = \sum_{i=0}^{d} P_{ij} E_j, \quad E_i = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i.$$

Since disjoint $(0,1)$-matrices A_i's form a basis of $\mathcal{A}$, the algebra $\mathcal{A}$ is closed under the entrywise multiplication denoted by $\circ$. The *Krein parameters* q_{ij}^k are defined by $E_i \circ E_j = \frac{1}{n}\sum_{k=0}^{d} q_{ij}^k E_k$. The *Krein matrix* B_i^* is defined as $B_i^* = (q_{ij}^k)_{j,k=0}^d$.

Each of the matrices A_i can be considered as the adjacency matrix of some graph without multiedges. The scheme is *imprimitive* if, on viewing the A_i's as the adjacency matrices of graphs G_i on vertex set X, at least one of the G_i, $i \neq 0$, is disconnected. Then there exists a set $\mathcal{I}$ of indices, under a suitable ordering, such that 0 and such i are elements of $\mathcal{I}$ and $\sum_{j\in\mathcal{I}} A_j = I_p \otimes J_q$ for some p, q with $1 < p < n$. Thus the set of n vertices X are partitioned into p subsets called *fibers*, each of which has size q. The set $\mathcal{I}$ defines an equivalence relation on $\{0, 1, \ldots, d\}$ by $j \sim k$ if and only if $p_{ij}^k \neq 0$ for some $i \in \mathcal{I}$. Let $\mathcal{I}_0 = \mathcal{I}, \mathcal{I}_1, \ldots, \mathcal{I}_t$ be the equivalence classes on $\{0, 1, \ldots, d\}$ by $\sim$. Then by [1, Theorem 9.4] there exist $(0,1)$-matrices $\overline{A}_j$ $(0 \leqslant j \leqslant t)$ such that

$$\sum_{i\in\mathcal{I}_j} A_i = \overline{A}_j \otimes J_q,$$

and the matrices $\overline{A}_j$ $(0 \leqslant j \leqslant t)$ define an association scheme on the set of fibers. This is called the *quotient association scheme* with respect to $\mathcal{I}$

For fibers U and V, let $\mathcal{I}(U, V)$ denote the set of indices of adjacency matrices A_i with $(A_i)_{u,v}$ for some $u \in U$, $v \in V$.

We define a $(0,1)$-matrix A_i^{UV} by

$$(A_i^{UV})_{xy} = \begin{cases} 1 & \text{if } (A_i)_{xy} = 1, x \in U, y \in V, \\ 0 & \text{otherwise.} \end{cases}$$

We define uniformity for imprimitive association schemes following [4, 7].

Definition 2. An imprimitive association scheme is called uniform if its quotient association scheme is class 1 and there exist integers a_{ij}^k such that for all fibers U, V, W and $i \in \mathcal{I}(U, V), j \in \mathcal{I}(V, W)$, we have

$$A_i^{UV} A_j^{VW} = \sum_k a_{ij}^k A_k^{UW}.$$

3 Class 5 Association Scheme

Let $\{H_1, H_2, \ldots, H_m\}$ be a set of Mutually Unbiased Regular Hadamard (MURH) matrices of order $4n^2$ with $m \geqslant 2$. Write

$$M = \begin{bmatrix} I \\ H_1/2n \\ H_2/2n \\ \cdots \\ H_m/2n \end{bmatrix} \begin{bmatrix} I & H_1^t/2n & H_2^t/2n & \ldots & H_m^t/2n \end{bmatrix}. \tag{1}$$

be the *Gramian* of the set of matrices $\{I, \frac{1}{2n}H_1, \frac{1}{2n}H_2, \ldots, \frac{1}{2n}H_m\}$. Let $B = 2n(M - I)$. Then B is a symmetric $(0, -1, 1)$-matrix. Let

$$B = B_1 - B_2,$$

where B_1 and B_2 are disjoint $(0, 1)$-matrices. By reworking a result of Mathon, see [3], we have the following:

Lemma 3. *Let $I = I_{4n^2(m+1)}$, B_1, B_2 and $B_3 = I_{m+1} \otimes J_{4n^2} - I_{4n^2(m+1)}$. Then, I, B_1, B_2, B_3 form a 3-class association scheme.*

Proof. The intersection numbers can be read off from the following easily verified equations:

$$\begin{aligned} B_1^2 &= (2n^2 + n)mI + (n^2 + \frac{3}{2}n)(m-1)B_1 \\ &\quad + (n^2 + n/2)(m-1)B_2 + (n^2 + n)B_3, \\ B_2^2 &= (2n^2 - n)mI + (n^2 - \frac{1}{2}n)(m-1)B_1 \\ &\quad + (n^2 - \frac{3}{2}n)(m-1)B_2 + (n^2 - n)B_3, \\ B_1B_2 &= (n^2 - n/2)(m-1)B_1 \\ &\quad + (n^2 + n/2)(m-1)B_2 + n^2mB_3, \\ B_1B_3 &= (2n^2 + n - 1)B_1 + (2n^2 + n)B_2, \\ B_2B_3 &= (2n^2 - n)B_1 + (2n^2 - n - 1)B_2. \square \end{aligned}$$

We now impose a further structure on the regular Hadamard matrices H_i and assume that they are all of Bush type. First we need the following.

Lemma 4. *Let H and K be two unbiased Bush-type Hadamard matrices of order $4n^2$. Let L be a $(1, -1)$-matrix so that $HK^t = 2nL$. Then L is a Bush-type Hadamard matrix.*

Proof. Let $X = I_{2n} \otimes J_{2n}$, then L is of Bush type if and only if $LX = XL = 2nX$. We calculate LX.

$$LX = \frac{1}{2n}HK^tX = 2nX.$$

Similarly, we have $XL = 2nX$. Thus L is a Bush-type Hadamard matrix. $\square$

This enables us to add two more classes and we have the following.

Theorem 5. *Let B_1, B_2 denote the matrices defined above. Let*

- $A_0 = I_{4n^2(m+1)}$
- $A_1 = I_{m+1} \otimes I_{2n} \otimes (J_{2n} - I_{2n})$
- $A_2 = I_{m+1} \otimes (J_{2n} - I_{2n}) \otimes J_{2n}$
- $A_3 = (J_{m+1} - I_{m+1}) \otimes I_{2n} \otimes J_{2n}$
- $A_4 = B_1 - A_3$
- $A_5 = B_2$

Then, $A_0, A_1, A_2, A_3, A_4, A_5$ form a 5-class association scheme.

Proof. We work out the intersection numbers, using some of the relations in Lemma 3. Note that $A_0 + A_1$, A_3, $(A_0 + A_1 + A_2)$ are block matrices of block size $2n$, ($4n^2$, respectively), where each block is either the zero or the matrix of ones. On the other hand A_4 and A_5 are block matrices of block size $2n$, where the blocks are either the zero matrix or of row and column sum n. So, it is straightforward computation to see the following:

$$
\begin{aligned}
A_1A_1 &= (2n-1)A_0 + (2n-2)A_1.\\
A_1A_2 &= (2n-1)A_2.\\
A_1A_3 &= (2n-1)A_3.\\
A_1A_4 &= (n-1)A_4 + nA_5.\\
A_1A_5 &= nA_4 + (n-1)A_5.\\
A_2A_2 &= 2n(2n-1)A_0 + 2n(2n-1)A_1 + 2n(2n-2)A_2.\\
A_2A_3 &= 2n(A_4 + A_5).\\
A_2A_4 &= (2n-1)nA_3 + (2n-2)n(A_4 + A_5).\\
A_2A_5 &= (2n-1)nA_3 + (2n-2)n(A_4 + A_5).\\
A_3A_3 &= 2mn(A_0 + A_1) + 2n(m-1)A_3.\\
A_3A_4 &= mnA_2 + (m-1)n(A_4 + A_5).\\
A_3A_5 &= mnA_2 + (m-1)n(A_4 + A_5).
\end{aligned}
$$

Using these, the facts that $A_3 + A_4 = B_1$, $A_5(A_3 + A_4) = B_2B_1$, and the intersection numbers in Lemma 3 we have:

$$
\begin{aligned}
A_4A_5 = n^2mA_1 + m(n^2-n)A_2 + (n^2 - \frac{n}{2})(m-1)A_3\\
+ (n^2 - \frac{3n}{2})(m-1)A_4 + (n^2 - \frac{n}{2})(m-1)A_5.
\end{aligned}
$$

Finally, noting that $A_4 - A_5$ is a block matrix of block size $2n$, where the blocks are either the zero matrix or of row and column sum zero, it follows that

$$(A_4 + A_5)(A_4 - A_5) = 0,$$

so we have

$$\begin{aligned} A_4A_4 = A_5A_5 = (2n^2 - n)mI + (n^2 - n)m(A_1 + A_2)+ \\ (n^2 - \frac{n}{2})(m-1)(A_3 + A_4) + (n^2 - \frac{3n}{2})(m-1)A_5. \end{aligned}$$

□

Definition 6. Two Latin squares L_1 and L_2 of size n on symbol set $\{0, 1, 2, \ldots, n-1\}$ are called *suitable* if every superimposition of each row of L_1 on each row of L_2 results in only one element of the form (a, a). A set of Latin squares in which every distinct pair of Latin squares is mutually suitable is called *Mutually Suitable Latin Squares*, denoted MSLS.

The existence and a construction method for MUBH matrices to use mutually suitable Latin squares were given in [9, Theorem 13]. However, in order to obtain Bush-type Hadamard matrices as defined here, an additional assumption on the MSLS is needed as follows.

Proposition 7. *If there are m mutually suitable Latin squares of size $2n$ with all one entries on diagonal and a Hadamard matrix of order $2n$, then there are m mutually unbiased Bush-type Hadamard matrices of order $4n^2$.*

The construction is exactly same as [9, Theorem 13]. The resulting mutually unbiased Hadamard matrices are all of Bush-type. Indeed, each Latin square has the entries 1 on diagonal, thus the resulting Hadamard matrix has the all ones matrices on diagonal blocks.

The equivalence of MOLS and MSLS was given in [9, Lemma 9]. The assumption on MOLS corresponding to MSLS with all ones entries on diagonal is that each Latin square has $(1, 2, \cdots, n)$ as the first row. The MOLS having this property is constructed by the use of finite fields as follows. For each $\alpha \in \mathbb{F}_q \setminus \{0\}$, define L_α as (i, j)-entry equal to $\alpha i + j$, where $i, j \in \mathbb{F}_q$. By switching rows so that the first row corresponds to $0 \in \mathbb{F}_q$ and mapping $\mathbb{F}_q$ to $\{1, 2, \ldots, n\}$ such that each first row becomes $(1, 2, \ldots, n)$, we obtain the desired MOLS. Thus we have the same conclusion as [9, Corollary 15].

Remark 8. (a) Rewriting $A_4A_4 = A_5A_5$ as:

$$\begin{aligned} A_4A_4 = A_5A_5 = n^2mI + (n^2 - n)mJ + n(\frac{m+1}{2} - n)(A_3 + A_4) \\ + n(\frac{3}{2} - \frac{m}{2} - n)A_5. \end{aligned}$$

It is seen that, for $m = 2n - 1$, A_5 is the adjacency matrix of a strongly regular graph and A_4 is the adjacency matrix of a Deza Graph, see [5, 6]. There exists an example satisfying $m = 2n - 1$ for $n = 2^{k-1}$ and $k \geqslant 1$ [9, Corollary 15].

(b) The association scheme of class 5 is uniform. Any two fibers define a coherent configuration, which is a strongly regular design of the second kind, see [8]. The first and second eigenmatrices and B_5^* are as follows:

$$P=\begin{pmatrix}1 & 2n-1 & 2n(2n-1) & 2nm & n(2n-1)m & n(2n-1)m\\ 1 & -1 & 0 & 0 & nm & -nm\\ 1 & 2n-1 & -2n & 2nm & -nm & -nm\\ 1 & 2n-1 & -2n & -2n & n & n\\ 1 & -1 & 0 & 0 & -n & n\\ 1 & 2n-1 & 2n(2n-1) & -2n & -n(2n-1) & -n(2n-1)\end{pmatrix},$$

$$Q=\begin{pmatrix}1 & 2n(2n-1) & 2n-1 & (2n-1)m & 2n(2n-1)m & m\\ 1 & -2n & 2n-1 & (2n-1)m & -2nm & m\\ 1 & 0 & -1 & -m & 0 & m\\ 1 & 0 & 2n-1 & -2n+1 & 0 & -1\\ 1 & 2n & -1 & 1 & -2n & -1\\ 1 & -2n & -1 & 1 & 2n & -1\end{pmatrix},$$

$$B_5^*=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & m & m-1 & 0 & 0\\ 0 & m & 0 & 0 & m-1 & 0\\ m & 0 & 0 & 0 & 0 & m-1\end{pmatrix}.$$

Thus the association scheme certainly satisfies [4, Proposition 4.7]. Note that the scheme is not Q-polynomial since no column has all distinct entries.

Since the Krein number $q_{1,2}^1=\frac{2n-m-1}{m+1}$ must be positive, we obtain $m\leqslant 2n-1$ holds. This means that the number of MUBH matrices of order $4n^2$ is at most $2n-1$. The example attaining the upper bound is given in [9, Corollary 15].

(c) The first, second eigenmatrices and the Krein matrix B_1^* of the class 3 association scheme are as follows:

$$P=\begin{pmatrix}1 & n(2n+1)m & n(2n-1)m & 4n^2-1\\ 1 & nm & -nm & -1\\ 1 & -n & n & -1\\ 1 & -n(2n+1) & -n(2n-1) & 4n^2-1\end{pmatrix},$$

$$Q=\begin{pmatrix}1 & 4n^2-1 & (4n^2-1)m & m\\ 1 & 2n-1 & -2n+1 & 1\\ 1 & -2n-1 & 2n+1 & 1\\ 1 & -1 & -m & m\end{pmatrix},$$

$$B_1^*=\begin{pmatrix}0 & 1 & 0 & 0\\ 4n^2-1 & \frac{2(2n^2-m-1)}{m+1} & \frac{4n^2}{m+1} & 0\\ 0 & \frac{4n^2m}{m+1} & \frac{(4n^2-2)m-2}{m+1} & 4n^2-1\\ 0 & 0 & 1 & 0\end{pmatrix}.$$

This association scheme is a Q-antipodal Q-polynomial scheme of class 3. By [3, Theorem 5.8], this scheme comes from a linked systems of symmetric designs.

Next we show the converse implication as follows.

Theorem 9. *Assume that there exists an association scheme with the same eigenmatrices as in Remark 8(b). Then there exists a set of MUBH* $\{H_1, \ldots H_m\}$ *of order* $4n^2$*.*

Proof. Let $A_0, \ldots, A_5$ be the adjacency matrices of an association scheme with the same eigenmatrices as in Remark 8. Let $B_0 = A_0$, $B_1 = A_3 + A_4$, $B_2 = A_5$ and $B_3 = A_1 + A_2$. By Remark 8 B_i's form a linked system of symmetric designs. Thus we rearrange the vertices so that $B_3 = I_{m+1} \otimes J_{4n^2} - I_{4n^2(m+1)}$.

We first determine the form of A_3. Since A_1 is the adjacency matrix of an imprimitive strongly regular graph with eigenvalues $2n-1, -1$ with multiplicities $2n(m+1), 2n(2n-1)(m+1)$, A_1 is $I_{2n(m+1)} \otimes (J_{2n} - I_{2n})$ after rearranging the vertices. By $B_3 = I_{m+1} \otimes J_{4n^2} - I_{4n^2(m+1)} = A_1 + A_2$, A_2 has the desired form. Since B_3 and A_3 are disjoint and $A_2A_3 = 2n(A_4 + A_5)$, we obtain $A_3 = (J_{m+1} - I_{m+1}) \otimes I_{2n} \otimes J_{2n}$.

Letting $G = (m+1)(E_0 + E_1 + E_2)$, we have

$$\begin{aligned} G &= (m+1)(E_0 + E_1 + E_2) \\ &= \frac{1}{4n^2} \sum_{i=0}^{5} (Q_{0,i} + Q_{1,i} + Q_{2,i}) A_i \\ &= A_0 + \frac{1}{2n} A_3 + \frac{1}{2n} A_4 - \frac{1}{2n} A_5. \end{aligned}$$

Since $A_3 + A_4 + A_5 = (J_{m+1} - I_{m+1}) \otimes J_{2n} \otimes J_{2n}$, G is the following form

$$G = \begin{pmatrix} I_{2n} & \frac{1}{2n} H_{1,2} & \cdots & \frac{1}{2n} H_{1,m+1} \\ \frac{1}{2n} H_{2,1} & I_{2n} & \cdots & \frac{1}{2n} H_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n} H_{m+1,1} & \frac{1}{2n} H_{m+1,2} & \cdots & I_{2n} \end{pmatrix}$$

where $H_{i,j}$ $(i \neq j)$ is a $(1,-1)$-matrix.

We claim that $H_k := H_{k+1,1}$ $(1 \leqslant k \leqslant m)$ are mutually unbiased Bush-type Hadamard matrices. Let $\bar{A}$ denote the submatrix of entries that lie in the rows and columns on the first and $(k+1)$th blocks. We consider the principal submatrix $\bar{G}$. Since the association scheme is uniform, the restricting to indices on the first and second blocks yields an association scheme with the eigenmatrix $\bar{P} = (\bar{P}_{ij})_{i,j=0}^{5}$ obtained by putting $m = 1$.

Since $\bar{G} = (m+1)(\bar{E}_0 + \bar{E}_1 + \bar{E}_2)$ holds and $\frac{m+1}{2}\bar{E}_i$ $(i = 0, 1, 2)$ are primitive idempotents of the subscheme, we have $\bar{G}^2 = 2\bar{G}$. Expanding the left hand-side to use the form $\bar{G} = \begin{pmatrix} I_{2n} & \frac{1}{2n} H_k^t \\ \frac{1}{2n} H_k & I_{2n} \end{pmatrix}$, we obtain

$$\begin{pmatrix} I_{2n} + \frac{1}{4n^2} H_k^t H_k & \frac{1}{n} H_k^t \\ \frac{1}{n} H_k & I_{2n} + \frac{1}{4n^2} H_k H_k^t \end{pmatrix} = \begin{pmatrix} 2I_{2n} & \frac{1}{n} H_k^t \\ \frac{1}{n} H_k & 2I_{2n} \end{pmatrix}.$$

This implies that H_k is a Hadamard matrix of order $4n^2$.

Next we show H_k is of Bush-type. Now we calculate $\bar{A}_3\bar{G}$ in two ways. First we have

$$\begin{aligned}
\bar{A}_3\bar{G} &= (m+1)\bar{A}_3(\bar{E}_0+\bar{E}_1+\bar{E}_2)\\
&= (m+1)(\sum_{i=0}^{5}\bar{P}_{i3}\bar{E}_i)(\bar{E}_0+\bar{E}_1+\bar{E}_2)\\
&= (m+1)(\sum_{i=0}^{2}\bar{P}_{i3}\bar{E}_i)\\
&= 2n(m+1)(\bar{E}_0+\bar{E}_2)\\
&= 2n(m+1)(\frac{1}{4n^2(m+1)}\sum_{i=0}^{5}(\bar{Q}_{i,0}+\bar{Q}_{i,2})\bar{A}_i)\\
&= (A_0+A_1+A_3)\\
&= \begin{pmatrix} I_{2n}\otimes J_{2n} & I_{2n}\otimes J_{2n}\\ I_{2n}\otimes J_{2n} & I_{2n}\otimes J_{2n}\end{pmatrix}.
\end{aligned}$$

Second we have

$$\begin{aligned}
\bar{A}_3\bar{G} &= \begin{pmatrix} 0 & I_{2n}\otimes J_{2n}\\ I_{2n}\otimes J_{2n} & 0\end{pmatrix}\begin{pmatrix} I_{2n} & \frac{1}{2n}H_k^t\\ \frac{1}{2n}H_k & I_{2n}\end{pmatrix}\\
&= \begin{pmatrix} \frac{1}{2n}(I_{2n}\otimes J_{2n})H_k & I_{2n}\otimes J_{2n}\\ I_{2n}\otimes J_{2n} & \frac{1}{2n}(I_{2n}\otimes J_{2n})H_k^t\end{pmatrix}.
\end{aligned}$$

Comparing these two equations yields

$$(I_{2n}\otimes J_{2n})H_k = (I_{2n}\otimes J_{2n})H_k^t = 2nI_{2n}\otimes J_{2n}.$$

This implies that H_k is of Bush-type by Lemma 4.

Finally we show $H_1,\dots,H_m$ are unbiased. Let k,k' be integers such that $1\leqslant k< k'\leqslant m$. Let $\tilde{A}$ denote the submatrix of vertices that lie in the rows and columns on the first, $(k+1)$th and $(k'+1)$th blocks. We then have $\tilde{G}^2=3\tilde{G}$. Comparing the $(2,3)$-block, we obtain

$$\frac{1}{4n^2}H_kH_{k'}^t+\frac{1}{2n}I_{2n}H_{k+1,k'+1}+\frac{1}{2n}H_{k+1,k'+1}I_{2n}=\frac{3}{2n}H_{k+1,k'+1},$$

namely $\frac{1}{4n^2}H_kH_{k'}^t=\frac{1}{2n}H_{k+1,k'+1},$. Since $H_{k+1,k'+1}$ is a $(-1,1)$-matrix, H_k and $H_{k'}$ are unbiased. □

4 8 class association schemes

Linked systems of symmetric designs with specific parameters have the extended Q-bipartite double which yields an association scheme of mutually unbiased bases [11, Theorem 3.6]. Next we show an association scheme from our association schemes of class 5 has a double cover and show a relation to an association scheme of class 4 as a fusion scheme.

Theorem 10. *Let $A_0, A_1, \ldots, A_5$ be the adjacency matrices of the association scheme in Theorem 5. Define*

$$\tilde{A}_0 = \begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix}, \tilde{A}_1 = \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}, \tilde{A}_2 = \begin{pmatrix} 0 & A_1 \\ A_1 & 0 \end{pmatrix}, \tilde{A}_3 = \begin{pmatrix} A_2 & A_2 \\ A_2 & A_2 \end{pmatrix},$$
$$\tilde{A}_4 = \begin{pmatrix} A_3 & 0 \\ 0 & A_3 \end{pmatrix}, \tilde{A}_5 = \begin{pmatrix} 0 & A_3 \\ A_3 & 0 \end{pmatrix}, \tilde{A}_6 = \begin{pmatrix} A_4 & A_5 \\ A_5 & A_4 \end{pmatrix}, \tilde{A}_7 = \begin{pmatrix} A_5 & A_4 \\ A_4 & A_5 \end{pmatrix},$$
$$\tilde{A}_8 = \begin{pmatrix} 0 & A_0 \\ A_0 & 0 \end{pmatrix}.$$

Then $\tilde{A}_0, \ldots, \tilde{A}_8$ form an association scheme.

Proof. This follows from the calculation in Theorem 5. □

Remark 11. (a) The association scheme of class 8 is also uniform. The second eigenmatrix and B_8^* are as follows:

$$Q = \begin{pmatrix} 1 & 2n(2n-1) & 2n & 2n-1 & 2n(2n-1)(m+1) & (2n-1)m & 2nm & 2n(2n-1)m & m \\ 1 & -2n & 2n & 2n-1 & -2n(m+1) & (2n-1)m & 2nm & -2nm & m \\ 1 & 2n & -2n & 2n-1 & -2n(m+1) & (2n-1)m & -2nm & 2nm & m \\ 1 & 0 & 0 & -1 & 0 & -m & 0 & 0 & m \\ 1 & 0 & 2n & 2n-1 & 0 & -2n+1 & -2n & 0 & -1 \\ 1 & 0 & -2n & 2n-1 & 0 & -2n+1 & 2n & 0 & -1 \\ 1 & 2n & 0 & -1 & 0 & 1 & 0 & -2n & -1 \\ 1 & -2n & 0 & -1 & 0 & 1 & 0 & 2n & -1 \\ 1 & -2n(2n-1) & -2n & 2n-1 & 2n(2n-1)(m+1) & (2n-1)m & -2nm & -2n(2n-1)m & m \end{pmatrix},$$

$$B_8^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & m-1 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 & m-1 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 & 0 & m-1 & 0 \\ m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m-1 \end{pmatrix}.$$

Thus the association scheme certainly satisfies [4, Proposition 4.7]. Note that the scheme is not Q-polynomial since no column has all distinct entries.

(b) Letting $\tilde{B}_1 = \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3$, $\tilde{B}_2 = \tilde{A}_4 + \tilde{A}_6$ and $\tilde{B}_3 = \tilde{A}_5 + \tilde{A}_7$, $\tilde{B}_4 = \tilde{A}_8$, we obtain a fusion association scheme of class 4. The second eigenmatrix of the class 4 fusion association scheme is as follows:

$$Q = \begin{pmatrix} 1 & 4n^2 & (4n^2-1)(m+1) & 4n^2m & m \\ 1 & 0 & -m-1 & 0 & m \\ 1 & 2n & -0 & -2n & -1 \\ 1 & -2n & 0 & 2n & -1 \\ 1 & -4n^2 & (4n^2-1)(m+1) & -4n^2m & m \end{pmatrix}.$$

This association scheme is a Q-antipodal and Q-bipartite Q-polynomial scheme of class 4, see [10].

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