New Lower Bounds for 28 Classical Ramsey Numbers

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Abstract

We establish new lower bounds for 28 classical two and three color Ramsey numbers, and describe the heuristic search procedures used. Several of the new three color bounds are derived from the two color constructions; specifically, we were able to use $(5, k)$-colorings to obtain new $(3, 3, k)$-colorings, and $(7, k)$-colorings to obtain new $(3, 4, k)$-colorings. Some of the other new constructions in the paper are derived from two well known colorings: the Paley coloring of K_{101} and the cubic coloring of K_{127}.

1 Introduction

The classical Ramsey number $R(k_1, k_2, \ldots, k_r)$ is the smallest integer n such that in any r-coloring of the edges of the complete graph K_n there is a monochromatic copy of K_{k_c} for some $1 \leqslant c \leqslant r$. In this note, we examine colorings for $r = 2$ and $r = 3$, describe some new variations on computer construction methods and are able to improve the lower bounds for 28 classical Ramsey numbers. When discussing the procedures for improving lower bounds, we use the term *bad subgraphs* to mean monochromatic complete graphs of order k_c in any of the colors $1 \leqslant c \leqslant r$. The reader is referred to Radziszowski's survey on Small Ramsey Numbers [6] for basic terminology related to the problem. The survey contains a comprehensive summary of the current state of the art.

In this report the emphasis is on establishing new lower bounds for a range of Ramsey numbers of the form $R(k_1, k_2)$. Our primary focus is on cases where $4 \leqslant k_1 \leqslant 6$ and where current lower bounds are between 100 and 200, though several bounds outside this range are also presented. Some of these colorings are then used to obtain lower bounds for three-color classical Ramsey numbers. The new values for two- and three-color numbers are listed in Tables 1 and 2, respectively.

The graphs were constructed by three variations on a basic method. These three variations involve searching through colorings such that the adjacency (coloring) matrix is partitioned into (1) cyclic orbits, (2) square sub-blocks with cyclic orbits, or (3) single edge orbits. In the first case we get circle colorings, which have been used many times in this context [6]. In the second case we have colorings such that the adjacency matrix can be partitioned into square circulant submatrices (of the same size); and in the third case we have arbitrary coloring matrices.

Given a particular Ramsey number, $R(k_1, k_2)$ and a value of n, the goal is to find a good (k_1, k_2)-coloring of K_n. We begin by searching for circle colorings. If that search succeeds, then we might increase n, and repeat. If it doesn't succeed and if n is not prime, then we expand our search space by considering all integer factorizations $n = md$, and search for circulant block colorings where the adjacency matrix is partitioned into an $m \times m$ array of $d \times d$ circulant blocks.

If neither of these searches succeeds, but if either of them produces colorings with a relatively small number of bad subgraphs, then the coloring is used as a starting configuration and a search which recolors single edges is applied. The decision as to whether a given number of bad subgraphs is small enough to merit further investigation is made empirically.

In a further variation on this method, a few of the new colorings presented here were obtained by using one of two well known circulant colorings to create initial colorings, and then applying the single edge recoloring procedure. These two special colorings are the Paley (or quadratic) coloring of K_{101} and the cubic coloring of K_{127}. Recall that the former coloring was used to establish the current lower bound for $R(6, 6)$ [4], while the latter was used to establish lower bounds for both $R(4, 12)$ [8] and $R(4, 4, 4)$ [3]. The cubic residue graph of order 127 may be of further interest, since it was conjectured by the first author to be a Folkman graph i.e., a K_4-free graph such that in any two-coloring of the edges, there is a monochromatic K_3 (see, for example, Conjecture 4.4 in [7] which contains a fascinating discussion of this and related problems).

2 New Lower Bounds

As indicated above, we present two tables, one each for two-color and three-color lower bounds. A few of the cases listed were obtained using *ad hoc* methods, which are outlined below. For the other cases, we stayed fairly close to the general approach described above. In the column labeled *method* we indicate how we obtained the initial coloring that gave, or led to, the indicated lower bound. Since these were computer searches, and the success of such searches depends to some extent on the computer time available, we attempted to assign approximately the same time to each problem.

In Table 2, new lower bounds for 3-color Ramsey numbers are shown. Some of these

Ramsey Number	Old Bound	New Bound	Method
R(4, 8)	58	59	circulant blocks
R(4,11)	98	102	circulant blocks
R(4,13)	133	138	cubic(127)
R(4,14)	141	147	cubic(127)
R(4,15)	153	155	cubic(127)
R(4,16)	164	166	cubic(127)
R(5,10)	144	149	circulant-minus
R(5,11)	171	174	circulant
R(5,12)	191	194	circulant
R(5,13)	213	218	circulant
R(5,14)	239	242	circulant
R(5,15)	265	269	circulant
R(5,16)	290	293	circulant
R(6, 7)	113	115	Paley(101)
R(6, 8)	132	134	Paley(101)
R(6, 9)	169	175	circulant
R(6,10)	179	185	circulant
R(7, 9)	241	242	circulant

Table 1: New lower bounds for 2-color classical Ramsey numbers.

colorings were found by using one of our $(5,k)$-colorings, and *splitting* the K_5-free color graph into two K_3-free color graphs, therby obtaining a $(3,3,k)$-coloring. In two other cases we used one of our $(7,k)$-colorings to obtain a $(3,4,k)$-coloring. Here we split the K_7-free color graph into a K_3-free graph and a K_4-free graph.

All of the colorings described in this paper are available at

www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p11/data

and

http://cs.indstate.edu/ge/RAMSEY/ExTa.

3 Search Methods

Our basic method begins by considering colorings where the adjacency matrix is either a circulant or can be partitioned into square circulant submatrices of the same size. For a graph of order n, let m be a positive integer such that $m \mid n$. We can partition the coloring matrix A into $m \times m$ square blocks, as follows

$$A = \begin{bmatrix} C_1 & C_2 & \cdots & C_m \\ C_2^t & C_{m+1} & \cdots & C_{2m-1} \\ \vdots & \vdots & \ddots & \vdots \\ C_m^t & C_{2m-1}^t & \cdots & C_{\binom{m+1}{2}} \end{bmatrix}, \tag{1}$$

Ramsey Number	Old Bound	New Bound	Method
R(3,3,6)	60	61	circulant blocks
R(3,3,10)	147	150	circulant
R(3,3,11)	162	174	splitting
R(3,3,12)	185	194	splitting
R(3,3,13)	212	217	splitting
R(3,3,14)	233	242	splitting
R(3,3,15)		269	splitting
R(3,3,16)		291	splitting
R(3,4,7)	145	152	splitting
R(3,4,9)	229	242	splitting

Table 2: New lower bounds for 3-color classical Ramsey numbers.

and constrain each of the blocks C_i to be a $d \times d$ circulant matrix, with $d = n/m$. Note that the diagonal blocks must be symmetric circulant matrices.

For $m = 1$, the entire coloring matrix of K_n is a symmetric circulant. For $m > 1$, the off-diagonal matrices can be either symmetric or asymmetric circulant blocks. The former case significantly reduces the size of the search space, which makes things faster, but may eliminate some good colorings. When off-diagonal blocks are asymmetric, the number of required color choices is

$$m\lfloor d/2 \rfloor + \binom{m}{2} d. \tag{2}$$

Whereas, if they are required to be symmetric, the number of color choices is

$$m\lfloor d/2 \rfloor + \binom{m}{2}(\lfloor d/2 \rfloor + 1). \tag{3}$$

Let V denote the vector of color assignments that determines a coloring of K_n where the adjacency matrix is partitioned into circulant blocks, as in (1). When $m = 1$ (and we have a circle coloring), the coloring can be completely specified by a vector $V = (a_1, \ldots, a_{\lfloor n/2 \rfloor})$. For $m > 1$, the length of the coloring vector V is either (2) or (3). In the case where we consider recoloring single edges, the coloring vector V has length $\binom{n}{2}$.

In the algorithms described below, we use the notation $V_{i,c}$ to indicate the vector obtained from a coloring vector V by assigning color c to component i of vector V.

For a given coloring vector V, let $f_c(V)$ denote the number of monochromatic complete graphs of order k_c in color c. During the search, the goal is to minimize f_c for each $1 \leqslant c \leqslant r$; but attempting to minimize a simple count of bad subgraphs does not appear to be the best strategy. So we adjust our objective function by defining the *score* of a coloring vector V as a weighted sum

$$score(V) = \sum_{c=1}^{r} w_c \cdot f_c(V)$$

where w is a scalar array of *weights* adjusted during the course of the algorithm. The array w is useful for two reasons. The first reason applies to off-diagonal Ramsey numbers and deals with the relative difficulty of eliminating badgraphs in the two colors. Consider the case of an off diagonal Ramsey number, $R(s,t)$, where $s < t$. One will find that it is easier to eliminate the monochromatic K_t subgraphs than to eliminate the monochromatic K_s subgraphs. Local search procedures will tend to generate colorings with a large number of bad K_s subgraphs and few (or no) bad K_t subgraphs. This tendency increases as $|s-t|$ increases. To combat this problem, one can use weights such that $w_s > w_t$. A good rule of thumb is to begin the search procedure with weights that satisfy

$$\frac{t}{s} \leqslant \frac{w_s}{w_t} \leqslant \left(\frac{t}{s}\right)^2.$$

A second reason for using weights is that they help avoid getting stuck in local minima. As the search proceeds, we adjust the weights so they are proportional to a moving average of the bad subgraph counts. For the problems in this paper, we updated w for each color c by

$$w_c = \frac{Kw_c + f_c}{(K+1)\sum_{i=1}^{r} f_i}$$

where K is a positive constant. The choice of K seems important. When K is too small, the weights will oscillate wildly, and so will the number of bad subgraphs. On the other hand, if K is too large, the weights will change too slowly, and the algorithm is susceptible to getting trapped in local minima. Values of K between 10 and 100 seem to be effective for the problems considered in this paper.

In addition, the weights can be changed randomly. The relative size of the random contribution to the weights is analogous to the temperature variable in simulated annealing [5].

In the discussions below, we have occasion to mention the CPU time required to obtain a certain bound. All such times are approximate times needed for a single CPU core, running at 3.0 gHz (an average). Several bounds obtained here required months of CPU time. In these cases, the work was performed using 288 CPU cores at the Department of Mathematics and Computer Science at the Indiana State University. The majority of these CPUs are third generation Intel *i5*'s. With this setup, a month of CPU time can be covered in less than three hours.

3.1 Method 1

Our basic search method combines steepest descent with tabu search [1]. An outline of the search procedure is displayed in the Algorithm 1. We emphasize that the coloring vector V can be any one of the three types mentioned above: circulant, block-circulant, or individual edges.

The procedure begins with a random coloring vector (which determines the adjacency coloring matrix) and maintains a record of the last L colorings. During each iteration, the procedure considers each of the possible recoloring vectors $V_{i,c}$ not in the tabu list. A recoloring vector that produces the minimum score is chosen and applied.

If L is large relative to n, it is possible for the algorithm to enter a state in which no recoloring $V_{i,c}$ can be used without revisiting a state on the tabu list. If this (rare) event occurs, we restart with a new random coloring. When coloring graphs of order n, where $150 < n < 200$ (which accounts for several of the cases considered here), setting $L = 1000$ seemed to produce good results.

Algorithm 1: Steepest descent with tabu search

```
Input: Coloring vector V, number of colors r, size of a tabu list L
begin
    Set elements of V to random colors;
    H ← {};
    repeat
        W ← {};
        for 1 ⩽ i ⩽ |V| do
            for 1 ⩽ c ⩽ r do
                if V_{i,c} ∉ H then
                    W = W ∪ V_{i,c};
                    Compute score(V_{i,c});
        M ← {V ∈ W | score(V) is a minimum};
        if M ≠ ∅ then
            Randomly choose new V ∈ M;
            H ← H ∪ V;
            if |H| > L then
                Remove an element from H in FILO manner;
            Adjust weights;
    until score(V) = 0 or M = ∅;
    return V;
```

3.2 Method 2

In addition to the weight based method given in Algorithm 1, we also used a procedure based on simulated annealing [5]. This method is not suitable for circulants, but performed well for several block-circulant colorings, and is a good alternative when the recoloring of the individual edges is considered. This local improvement search procedure is outlined in Algorithm 2. At the beginning of the procedure, we find the set of all edges contained in bad subgraphs of order k_c in any of the colors $1 \leqslant c \leqslant r$. This edge set is denoted by I and can be created during the bad subgraph counting procedure, using very little additional CPU time. In the inner loop of Algorithm 2, we apply simulated annealing on edges from the set I. The initial temperature T_0 and the cooling rate depends on n (the order of the graph) and the initial *score*. The inner loop is limited by j_{max} iterations, after which we update the set I. Typically we would set $j_{max} = |V| / 4$. In most cases when T reached

0, the coloring can be improved further, so we restarted the procedure using the newly obtained coloring vector V. The process is repeated until the further improvements can not be made.

To perform a circulant block search, several modification are required. Initially V is set to a random coloring vector. Next, we set $I = \{1, \ldots, |V|\}$. The optimal choice of the initial temperature T_0 and cooling rate varies greatly for different n, m and r, and we attempt to determine them empirically. In practice, if $score(V)$ is decreasing more slowly than expected, we restart the entire procedure using the same initial parameters.

Algorithm 2: Search procedure based on simulated annealing

```
Input: Coloring vector V, number of colors r, initial temperature T_0, number of
       iterations of the inner loop j_max
begin
    T ← T_0;
    s ← ∞;
    repeat
        I ← {};
        for 1 ⩽ i ⩽ |V| do
            c ← V[i];
            if f_c(V_{i,c+1}) < f_c(V) then
                I ← I ∪ {i};
        for 1 ⩽ j ⩽ j_max do
            Randomly choose i ∈ I;
            Randomly choose c ∈ {1, ..., r}, c ≠ V[i];
            s* ← score(V_{i,c});
            if s* − s ⩽ 0 or e^{(s−s*)/T} > random(0, 1) then
                V ← V_{i,c};
                s ← s*;
        Reduce T;
        Adjust weights;
    until s = 0 or T = 0;
    return V;
```

3.3 Special Constructions

Several lower bounds presented in this paper required a combination of our basic search methods. A brief summary of how these colorings were obtained is given below.

3.3.1 $R(5, 10)$

The lower bound for $R(5, 10)$ (labeled *circulant-minus* in Table 1) required two steps. First, we began with a circulant coloring of K_{149} that was nearly a good coloring. This particular coloring had 149 monochromatic K_5's and no monochromatic K_{10}'s. After we had applied a local search procedure, one problem vertex was deleted, and a local search for a good coloring succeeded.

3.3.2 $R(4, 8)$

To improve the lower bound on $R(4, 8)$, we examined colorings of K_n for $60 \leqslant n \leqslant 64$, and for all m such that $m \mid n$ and $2 \leqslant m \leqslant 10$. The circulant block search was performed using the simulated annealing procedure. We then performed local searchs on those graphs we found with fewer than 250 bad subgraphs. For $m = 4$ and $m = 5$, graphs on 60 vertices were found and used to extract a $(4, 8)$-coloring on 58 vertices. In the first case, the block construction contained 60 monochromatic K_4's and 45 monochromatic K_8's. In the second, it contained 63 monochromatic K_4's and no monochromatic K_8's. In both cases, the best coloring was reached after removing two vertices. We note that these graphs on 60 vertices were not those with the minimum possible number of bad subgraphs (among block circulant colorings). Many colorings were found with fewer than 100 bad subgraphs, and some with fewer than 50, but we were not be able to modify these to obtain good colorings.

To obtain a new $(4, 8)-$coloring on K_{58} vertices, several hours of CPU time were required. We spent over a month of CPU time trying to improve this solution. We found several close graphs on K_{59} and K_{60}. On K_{59}, we were able to find a coloring with only 3 monochromatic K_4's and no monochromatic K_8's.

3.3.3 $R(4, 11)$

Perhaps our most complicated search was for the case of $R(4, 11)$. This search required four steps, beginning with a $(4, 8)$-coloring, extending it to a $(4, 9)$-coloring, then to a $(4, 10)$-coloring, and finally to a $(4, 11)$-coloring. We started from a circulant block construction for $n = 60$ and $m = 4$, the same one we had used for $R(4, 8)$. We extended this graph by appending circulant 15×15-blocks to the 60×60 coloring, and without changing the coloring on the first 60 vertices. A promising $(4, 9)$-coloring on 75 vertices was found. Next we added another layer of 15×15-blocks and obtained an even more promising $(4, 10)$-coloring with $n = 90$ and $m = 6$. Then we extended this coloring to a (nearly good) $(4, 11)$-coloring on 105 vertices. After repeatedly reaching local minima, we became convinced that we were not going to find a good coloring while maintaining the circulant block structure. At that point we removed four vertices and applied a local search procedure recoloring individual edges, and obtained a good coloring on 101 vertices, thus establishing the new lower bound for $R(4, 11)$. The promising coloring on 105 vertices required several hours of CPU time to locate. The final local search required approximately a week of CPU time.

3.3.4 $R(3, 3, 6)$

A graph that we find interesting was uncovered when applying a circulant block search to the three color Ramsey number $R(3, 3, 6)$. The best result was obtained for $n = 60$

and $m = 6$, and no additional corrections were required. The coloring has significant symmetry. The graph in the first color has $2^{23} \cdot 5 = 41943040$ automorphisms; the color two graph 80 automorphisms; and the color three graph has 40 automorphisms.

This coloring was discovered using the procedure similar to the one described in Algorithm 1, and the search took only a few minutes of CPU time. It is interesting that several other search procedures we tried were not able to find a good coloring on K_{60}.

3.4 Using Special Colorings

In certain cases, when two Ramsey numbers $R(s,t)$ and $R(s,t+k)$ are close, we tried to obtain a new lower bound for $R(s,t+k)$ by using the coloring that established the lower bound for $R(s,t)$ as a starting point for the $(s,t+k)$ search. Copies of (s,t)-coloring were used as starting blocks in a block circulant coloring that we hoped to manipulate into an $(s,t+k)$-coloring. If the order of the desired coloring was not an integer multiple of the order of the good (s,t) coloring, we deleted rows and columns until we had a matrix of the desired size. If the resulting graph had only a few bad subgraphs, we tried to improve it either by using a block circulant search or a single edge recoloring procedure.

3.4.1 Colorings derived from Paley coloring of K_{101}

To obtain the new bounds for $R(6,7)$ and $R(6,8)$ we used the Paley coloring of K_{101}, which was the graph used to establish the longstanding lower bound for $R(6,6)$ [4]. In both cases, the initial block coloring created from 101×101 circulant blocks was reduced to a matrix of the desired size, and the resulting coloring contained a relatively large high number of bad subgraphs. By applying the block circulant search procedure, we reduced the number of bad subgraphs significantly. To obtain good colorings, several days of CPU time were required. Additionally, several months of CPU time were spent trying to further improve the bounds. During this final stage, both bounds were improved by one.

Let us note that a similar result can be obtained for $R(6,7)$ by using only the single edge recoloring procedure. but the search required more CPU time. For $R(6,8)$, it was necessary to make improvements in the block circulant structure first, and then apply single edge recoloring.

3.4.2 Colorings derived from a cubic coloring of K_{127}

The new lower bounds for $R(4,t)$ for $13 \leqslant t \leqslant 16$ were obtained by using the cubic coloring of K_{127}. The cubic coloring was used to establish the lower bound for R(4,12) [8]. The previous bound for $R(4,13)$ was only 5 greater than that for $R(4,12)$, and in the case of $R(4,14)$ the difference was only 13. It was expected that the search would succeed at least for $t = 13$, but surprisingly, we quickly obtained the other good colorings.

The initial block coloring created from copies of the cubic residue graph on K_{127} contained only few bad subgraphs. However, performing a local search on those colorings was particularly slow. In all four cases, we feel that further improvement is possible. But better bounds would likely require years of CPU time using this method.

3.5 Splitting

This section describes a simple method that allowed us to reach most of the lower bounds given in Table 2. The idea is to use known $(R(s,t)-u,k)$-colorings to create (s,t,k)-colorings, for some $u \geqslant 1$. We noticed that some of the bounds for Ramsey numbers of the form $R(3,3,k)$ were close to the corresponding bounds for numbers of the form $R(5,k)$. A similar observation was made with regard to $R(3,4,k)$ and $R(7,k)$. In the first case, 5 is just one less than the Ramsey number $R(3,3)$, and in the second case 7 is two less than $R(3,4)$. So it appeared conceivable that we could use known $(5,k)$-colorings to obtain new $(3,3,k)$-colorings, and similarly use $(7,k)$-colorings to obtain new $(3,4,k)$-colorings. In the first case, we have to split the first color into two K_3-free graphs. In the second case, we have to split the first color into a K_3-free graph and a K_4-free graph.

As a starting point we used 2-color circulant colorings obtained by the method described in Section 3.1. First, we attempt to split the color one graph into two circulants. If good coloring can not be achieved, we continue the process by recoloring individual edges. At this point, we still do not modify the third color. In most of the cases, we found good colorings almost instantly.

Of course there is no guarantee that this procedure will work in general. In some cases, we were not able to find a good coloring using a given circulant, so we generated other non-isomorphic circulant colorings and tried each of them. Usually after few attempts the procedure succeeded. In several cases, we were unable to find good circulant colorings. But if the number of bad subgraphs was small enough, we ran the single edge recoloring procedure again, this time allowing changes to the third color. We were thereby able to find, for example, a good $(3,3,12)$-coloring on 193 vertices. But the process took a few hours of CPU time. Note that the lower bounds we found for $R(3,3,13)$ and $R(3,3,16)$ are slightly less than those found for $R(5,13)$ and $R(5,16)$, reflective of the fact that we could not complete the splitting procedure for our best $(5,13)$ and $(5,16)$ colorings, but did succeed using smaller colorings.

4 Conclusions

One of the motivations of our work was to apply a fixed amount of resources, both computer and human, to a range of Ramsey problems, and learn to what extent the lower bounds could be improved. We focused on classical two-color Ramsey numbers where the best known lower bound was between 100 and 200. We did this for two reasons. First, neither of us had ever applied serious effort to problems in this range, and second, we felt these would be problems where progress could be made. For problems where the current lower bound is less than 100, we felt that it was unlikely that progress would be made by simple methods (the case of $R(4,8)$ turned out to be an exception). For problems where the current lower bound is significantly greater than 200, we felt that the required computer time would usually be too great.

In each of the cases considered, we applied all of the methods described here, and few others. Other methods tried, but not mentioned here include the use of general (i.e., non-cyclic) Cayley colorings. It is remarkable that for every problem considered here, circulant colorings do better (produce good colorings for a larger value of n) than Cayley colorings from other groups. One obvious explanation for this is that cyclic groups have elements of larger average order than other groups, but it is still somewhat surprising.

Finally we considered the case of $R(8,8)$, where the current lower bound seems much too small, as one may observe from the Figure (the data is taken from [6]). Despite signifcant computer time spent searching for circulant colorings in the 282 to 286 range, we did not find anything new. It is tempting to conclude that if there is a circulant coloring that improves the lower bound for $R(8,8)$, it is quite a bit larger than the current best coloring. In fact, even the related case of $R(7,8)$ is surprisingly difficult, especially considering that the current bound of 217 is only 12 more than the $R(7,7)$ bound. We applied a variety of methods to this problem with no success.

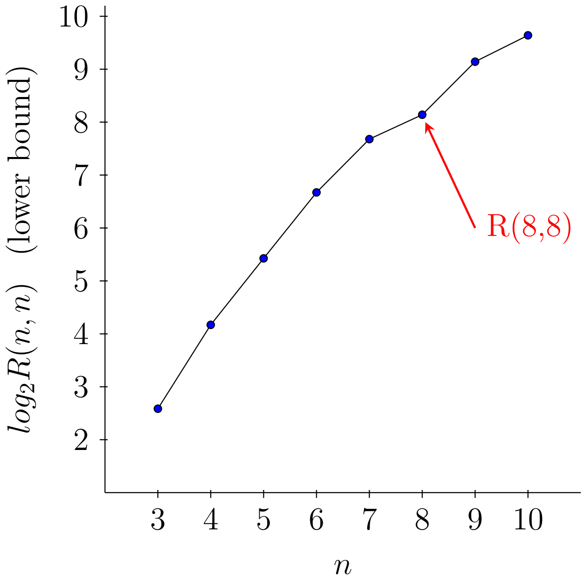

Figure 1: Log plot of current lower bounds for $R(n,n)$

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