Sign conjugacy classes of the symmetric groups

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Abstract

A conjugacy class C of a finite group G is a sign conjugacy class if every irreducible character of G takes value 0, 1 or -1 on C. In this paper we classify the sign conjugacy classes of the symmetric groups and thereby verify a conjecture of Olsson.

Keywords: symmetric groups; characters; partitions

1 Introduction

We will begin this paper by giving the definition of sign conjugacy class for an arbitrary finite group.

Definition 1.1. *Let G be a finite group. A conjugacy class of G is a sign conjugacy class of G if every irreducible character of G takes values 0, 1 or -1 on C.*

Since we will be working with the symmetric group, we will consider partitions instead of conjugacy classes. A partition of n is a sign partition if it is the corresponding conjugacy class of S_n is a sign conjugacy class. An easy example of a sign partition of n is (n).

Definition 1.2. *Define* Sign *to be the subsets of partitions consisting of all partitions $(\gamma_1, \ldots, \gamma_r)$ for which there exists an s, $0 \leqslant s \leqslant r$, such that the following hold:*

- $\gamma_i > \gamma_{i+1} + \cdots + \gamma_r$ *for* $1 \leqslant i \leqslant s$,
- $(\gamma_{s+1}, \ldots, \gamma_r)$ *is one of the following partitions:*
  - $()$, $(1,1)$, $(3,2,1,1)$ *or* $(5,3,2,1)$,
  - $(a, a-1, 1)$ *with* $a \geqslant 2$,

– $(a, a-1, 2, 1)$ *with* $a \geqslant 4$,

– $(a, a-1, 3, 1)$ *with* $a \geqslant 5$.

The name Sign for the above set is justified by the next theorem, which classifies sign partitions.

Theorem 1.3. *A partition* γ *is a sign partition if and only if* $\gamma \in \mathrm{Sign}$.

This was first formulated by Olsson in [4] as a conjecture.

In order to prove Theorem 1.3 we will use two results from [4]. The first one of them is the following lemma (Theorem 7 of [4]).

Lemma 1.4. *A sign partition cannot have repeated parts, except possibly for the part 1, which may have multiplicity 2.*

In particular only partitions of the form $(\gamma_1, \ldots, \gamma_r)$ with either $\gamma_1 > \ldots > \gamma_r$ or $\gamma_1 > \ldots > \gamma_{r-2} > \gamma_{r-1} = \gamma_r = 1$ may be sign partitions. The next lemma can also be found in [4] (Proposition 2).

Lemma 1.5. *Let* $(\gamma_1, \ldots, \gamma_r)$ *be a partition of* n *and let* $m > n$. *Then* $(\gamma_1, \ldots, \gamma_r)$ *is a sign partition if and only if* $(m, \gamma_1, \ldots, \gamma_r)$ *is a sign partition.*

For any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ let $|\lambda| := \lambda_1 + \cdots + \lambda_k$. Also for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant \lambda_i$ let $h^\lambda_{i,j}$ denote the hook length of the node (i, j) of λ. For partitions λ, μ with $|\lambda| = n = |\mu|$ let χ^λ_μ denote the value of the irreducible character of S_n labeled by λ on the conjugacy class with cycle partition μ.

Together with the previous lemmas, the following theorem, which will be proved in Sections 2 and 3, will allow us to prove one direction of Theorem 1.3.

Theorem 1.6. *Let* $\alpha = (\alpha_1, \ldots, \alpha_h)$ *be a partition with* $h \geqslant 3$. *Assume that* $\alpha_1 > \alpha_2$, *that* $\alpha \notin \mathrm{Sign}$ *and that* $(\alpha_2, \ldots, \alpha_h) \in \mathrm{Sign}$. *Then if* $\alpha \neq (5, 4, 3, 2, 1)$ *we can find a partition* β *of* $|\alpha|$ *such that* $\chi^\beta_\alpha \notin \{0, \pm 1\}$ *and* $h^\beta_{2,1} = \alpha_1$.

The other direction of Theorem 1.3 will be proved using Lemma 1.5 and the results from Section 4, where we prove that the partitions $(\gamma_{s+1}, \ldots, \gamma_r)$ are sign partitions.

References about results on partitions and irreducible characters of S_n can be found in [1] and [3].

2 Proof of Theorem 1.6 for $\alpha_2 \leqslant \alpha_3 + \cdots + \alpha_h$

In this section we will prove Theorem 1.6 in the case where $\alpha_2 \leqslant \alpha_3 + \cdots + \alpha_h$. Since by assumption $h \geqslant 3$ and $(\alpha_2, \ldots, \alpha_h) \in \mathrm{Sign}$, we have that

$$(\alpha_2, \ldots, \alpha_h) \in \{(1,1), (3,2,1,1), (5,3,2,1)\} \cup \{(a, a-1, 1) : a \geqslant 2\} \cup \{(a, a-1, 2, 1) : a \geqslant 4\} \cup \{(a, a-1, 3, 1) : a \geqslant 5\}.$$

Also $\alpha_1 \leqslant \alpha_2 + \cdots + \alpha_h$ as $\alpha \notin \text{Sign}$ and by assumption $\alpha_1 > \alpha_2$. If

$$\begin{aligned}(\alpha_2, \ldots, \alpha_h) \in \{&(1,1), (3,2,1,1), (5,3,2,1)\} \cup \{(a, a-1, 1) : 2 \leqslant a \leqslant 4\} \\ &\cup \{(a, a-1, 2, 1) : 4 \leqslant a \leqslant 8\} \cup \{(a, a-1, 3, 1) : 5 \leqslant a \leqslant 10\}\end{aligned}$$

there are only finitely many such α and it can be checked that for each one of them Theorem 1.6 holds.

For $(\alpha_2, \ldots, \alpha_h) = (a, a-1, 1)$ with $a \geqslant 5$ let

$$\beta := \begin{cases} (2a, 2, 1^{\alpha_1 - 2}), & a+2 \leqslant \alpha_1 \leqslant 2a-2 \text{ or } \alpha_1 = 2a, \\ (a-1, a-1, a-1, 4), & \alpha_1 = a+1, \\ (2a, \alpha_1), & \alpha_1 = 2a-1. \end{cases}$$

For $(\alpha_2, \ldots, \alpha_h) = (a, a-1, 2, 1)$ with $a \geqslant 9$ let

$$\beta := \begin{cases} (2a+2, 4, 1^{\alpha_1 - 4}), & a+4 \leqslant \alpha_1 \leqslant 2a-2 \text{ or } 2a \leqslant \alpha_1 \leqslant 2a+2, \\ (2a+2, \alpha_1 - 1, 1), & \alpha_1 = a+1, \\ (2a+2, 2, 1^{\alpha_1 - 2}), & a+2 \leqslant \alpha_1 \leqslant a+3, \\ (2a+2, \alpha_1), & \alpha_1 = 2a-1. \end{cases}$$

For $(\alpha_2, \ldots, \alpha_h) = (a, a-1, 3, 1)$ with $a \geqslant 11$ let

$$\beta := \begin{cases} (2a+3, 5, 1^{\alpha_1 - 5}), & a+5 \leqslant \alpha_1 \leqslant 2a-2 \text{ or } 2a \leqslant \alpha_1 \leqslant 2a+3, \\ (2a+3, 2, 1^{\alpha_1 - 2}), & \alpha_1 = a+1 \text{ or } \alpha_1 = a+4, \\ (2a+3, \alpha_1 - 2, 1, 1), & \alpha_1 = a+2, \\ (2a+3, \alpha_1), & \alpha_1 = a+3 \text{ or } \alpha_1 = 2a-1. \end{cases}$$

It's easy to check that in each of the above cases β is a partition and that $h_{2,1}^{\beta} = \alpha_1$. In each of the above cases in can also be proved that $\chi_\alpha^\beta \notin \{0, \pm 1\}$.

Assume that $(\alpha_2, \ldots, \alpha_h) = (a, a-1, 1)$ and $a+2 \leqslant \alpha_1 \leqslant 2a-2$, that $(\alpha_2, \ldots, \alpha_h) = (a, a-1, 2, 1)$ and $a+4 \leqslant \alpha_1 \leqslant 2a-2$ or that $(\alpha_2, \ldots, \alpha_h) = (a, a-1, 3, 1)$ and $a+5 \leqslant \alpha_1 \leqslant 2a-2$. In either case $h_{1,\beta_2+1} = 2a-2 \geqslant \alpha_1$. As $h_{2,1}^\beta = \alpha_1$ it follows from the Murnaghan-Nakayama formula that

$$\chi_\alpha^\beta = (-1)^{\alpha_1 - \beta_2} \chi_{(\alpha_2, \ldots, \alpha_h)}^{(|\alpha| - \alpha_1)} + \chi_{(\alpha_2, \ldots, \alpha_h)}^{(|\alpha| - 2\alpha_1, \beta_2, 1^{\alpha_1 - \beta_2})}.$$

Since by assumption

$$\begin{aligned} h_{3,1}^{(|\alpha| - 2\alpha_1, \beta_2, 1^{\alpha_1 - \beta_2})} &= \alpha_1 - \beta_2 \geqslant a, \\ h_{1,2}^{(|\alpha| - 2\alpha_1, \beta_2, 1^{\alpha_1 - \beta_2})} &= |\alpha| - 2\alpha_1 \leqslant a-2, \end{aligned}$$

and $\alpha_2 = a$, we have that

$$\chi_\alpha^\beta = (-1)^{\alpha_1 - \beta_2} + (-1)^{\alpha_2 - 1} \chi_{(\alpha_3, \ldots, \alpha_h)}^{(|\alpha| - 2\alpha_1, \beta_2, 1^{\alpha_1 - \beta_2 - \alpha_2})}.$$

By definition of β

$$\begin{aligned} h_{1,1}^{(|\alpha|-2\alpha_1,\beta_2,1^{\alpha_1-\beta_2-a})} &= |\alpha|-2\alpha_1+\alpha_1-\beta_2-\alpha_2+1 \\ &= \alpha_3+\cdots+\alpha_h-(\alpha_4+\cdots+\alpha_h+1)+1 \\ &= \alpha_3. \end{aligned}$$

So

$$\chi_\alpha^\beta = (-1)^{\alpha_1-\beta_2}+(-1)^{\alpha_2-1+\alpha_1-\beta_2-\alpha_2+1}\chi_{(\alpha_4,\ldots,\alpha_h)}^{(\beta_2-1)} = (-1)^{\alpha_1-\beta_2}2.$$

The other cases can be computed similarly.

3 Proof of Theorem 1.6 for $\alpha_2 > \alpha_3+\cdots+\alpha_h$

In this section we will prove Theorem 1.6 for $\alpha_2 > \alpha_3+\cdots+\alpha_h$. Again, from Lemma 1.5, as $\alpha \notin \mathrm{Sign}$ but $(\alpha_2,\ldots,\alpha_h) \in \mathrm{Sign}$, we have that $\alpha_1 \leqslant \alpha_2+\cdots+\alpha_h$.

Throughout this section let k be minimal such that

$$\alpha_k+\cdots+\alpha_h < \alpha_1-\alpha_2.$$

Since $\alpha_1 \leqslant \alpha_2+\cdots+\alpha_h$, it follows that $4 \leqslant k \leqslant h+1$. Also define

$$x := \alpha_k+\cdots+\alpha_h.$$

Theorem 3.1. *Assume that the following hold:*

- $\alpha \notin \mathrm{Sign}$, $(\alpha_2,\ldots,\alpha_h) \in \mathrm{Sign}$ *and* $\alpha_1 > \alpha_2 > \alpha_3+\cdots+\alpha_h$*,*
- $k \leqslant h$*,*
- $\alpha_1-\alpha_2$ *is not a part of* α*,*
- $\alpha_{k-1} > x$*.*

Then $\beta = (|\alpha|-\alpha_1, x+1, 1^{\alpha_1-x-1})$ *is a partition,* $h_{2,1}^\beta = \alpha_1$ *and* $\chi_\alpha^\beta = (-1)^{\alpha_1-x-1}2$*.*

Proof. By definition and by assumption

$$|\alpha|-\alpha_1 = \alpha_2+\cdots+\alpha_h \geqslant \alpha_1 \geqslant x+1,$$

from which follows that β is a partition. Also clearly $h_{2,1}^\beta = \alpha_1$. We will now prove that $\chi_\alpha^\beta = (-1)^{\alpha_1-x-1}2$.

Assume first that $2\alpha_1+x > |\alpha|$. Then

$$2 = |\alpha|-\alpha_1-(\alpha_2+\cdots+\alpha_h)+2 \leqslant |\alpha|-2\alpha_1+2 \leqslant x+1$$

and so

$$h_{1,|\alpha|-2\alpha_1+2}^\beta = |\alpha|-\alpha_1+2-(|\alpha|-2\alpha_1+2) = \alpha_1.$$

It follows that

$$\chi_\alpha^\beta = (-1)^{\alpha_1 - x - 1}\chi_{(\alpha_2,\ldots,\alpha_h)}^{(|\alpha|-\alpha_1)} - \chi_{(\alpha_2,\ldots,\alpha_h)}^{\delta} = (-1)^{\alpha_1-x-1} - \chi_{(\alpha_2,\ldots,\alpha_h)}^{\delta},$$

where $\delta := (x, |\alpha| - 2\alpha_1 + 1, 1^{\alpha_1 - x - 1})$. So it is enough to prove that $\chi_{(\alpha_2,\ldots,\alpha_h)}^{\delta} = (-1)^{\alpha_1 - x}$. As $h_{1,2}^\delta \leqslant x < \alpha_{k-1} < \alpha_2$ by assumption, we have that

$$\chi_{(\alpha_2,\ldots,\alpha_h)}^{\delta} = (-1)^{\alpha_2-1}\chi_{(\alpha_3,\ldots,\alpha_h)}^{\epsilon},$$

where $\epsilon := (x, |\alpha| - 2\alpha_1 + 1, 1^{\alpha_1 - \alpha_2 - x - 1})$ (as by definition of x, $\alpha_1 - \alpha_2 > x$, so that ϵ is a partition). By minimality of k,

$$|\epsilon| < 2x + \alpha_1 - \alpha_2 - x \leqslant 2x + \alpha_{k-1}.$$

Also, as $(\alpha_2, \ldots, \alpha_h) \in \text{Sign}$ and $k - 2 \geqslant 2$,

$$\alpha_3 + \cdots + \alpha_h = |\epsilon| < 2(\alpha_k + \cdots + \alpha_h) + \alpha_{k-1} < \alpha_{k-2} + \cdots + \alpha_h$$

and then $k - 2 < 3$. Since $k \geqslant 4$ it follows that $k = 4$. As by induction $\alpha_3 > x$,

$$\chi_{(\alpha_2,\ldots,\alpha_h)}^{\delta} = (-1)^{\alpha_2-1}\chi_{(\alpha_3,\ldots,\alpha_h)}^{\epsilon} = (-1)^{\alpha_2-1+\alpha_1-\alpha_2-x-1}\chi_{(\alpha_4,\ldots,\alpha_h)}^{(x)} = (-1)^{\alpha_1-x}$$

and then the theorem holds in this case.

Assume now that $2\alpha_1 + x < |\alpha|$. Then

$$x + 1 < |\alpha| - 2\alpha_1 + 1 \leqslant |\alpha| - \alpha_1$$

and so

$$h_{1,|\alpha|-2\alpha_1+1}^\beta = |\alpha| - \alpha_1 + 1 - (|\alpha| - 2\alpha_1 + 1) = \alpha_1.$$

By definition $\alpha_2 \leqslant \alpha_1 - x - 1$ and by assumption $\alpha_2 > \alpha_3 + \cdots + \alpha_h$, so that any partition of $\alpha_2 + \cdots + \alpha_h$ has at most one hook of length α_2. So

$$\begin{aligned} \chi_\alpha^\beta &= (-1)^{\alpha_1-x-1}\chi_{(\alpha_2,\ldots,\alpha_h)}^{(|\alpha|-\alpha_1)} + \chi_{(\alpha_2,\ldots,\alpha_h)}^{(|\alpha|-2\alpha_1,x+1,1^{\alpha_1-x-1})} \\ &= (-1)^{\alpha_1-x-1} + (-1)^{\alpha_2-1}\chi_{(\alpha_3,\ldots,\alpha_h)}^{\lambda}, \end{aligned}$$

where $\lambda = (|\alpha| - 2\alpha_1, x + 1, 1^{\alpha_1 - \alpha_2 - x - 1})$. So it is enough to prove that $\chi_{(\alpha_3,\ldots,\alpha_h)}^{\lambda} = (-1)^{\alpha_1-\alpha_2-x}$.

First assume that $\alpha_{k-1} > \alpha_1 - \alpha_2$. Then

$$h_{2,1}^\lambda = \alpha_1 - \alpha_2 < \alpha_j$$

for $3 \leqslant j \leqslant k - 1$ and

$$h_{1,x+2}^\lambda = |\lambda| - x - 1 - \alpha_1 + \alpha_2 \geqslant |\lambda| - \alpha_{k-1} - \cdots - \alpha_h = \alpha_3 + \cdots + \alpha_{k-2}$$

if $x+2\leqslant\lambda_1$. If $\lambda_1=x+1$ then

$$|\lambda|=x+\alpha_1-\alpha_2+1\leqslant\alpha_{k-1}+\cdots+\alpha_h\leqslant\alpha_3+\cdots+\alpha_h=|\lambda|$$

and so in this case $k=4$. In either case

$$\begin{aligned}\chi^{\lambda}_{(\alpha_3,\ldots,\alpha_h)} &= \chi^{(\alpha_{k-1}-\alpha_1+\alpha_2+x,x+1,1^{\alpha_1-\alpha_2-x-1})}_{(\alpha_{k-1},\ldots,\alpha_h)}\\ &= (-1)^{\alpha_1-\alpha_2-x}\chi^{(x)}_{(\alpha_k,\ldots,\alpha_h)}\\ &= (-1)^{\alpha_1-\alpha_2-x}\end{aligned}$$

and so the theorem holds also in this case.

Now assume that $\alpha_{k-1}<\alpha_1-\alpha_2$. Then $k\geqslant 5$ (otherwise $\alpha_1>\alpha_2+\cdots+\alpha_h$) and

$$\alpha_{k-1}+x=\alpha_{k-1}+\cdots+\alpha_h\geqslant\alpha_1-\alpha_2$$

by definition of k. Since $\alpha_1-\alpha_2-x-1<\alpha_{k-1}$ by minimality of k and since by assumption $x<\alpha_{k-1}$ and $\alpha_1-\alpha_2$ is not a part of α, it follows similarly to the previous case that

$$\chi^{\lambda}_{(\alpha_3,\ldots,\alpha_h)}=\chi^{\mu}_{(\alpha_{k-2},\ldots,\alpha_h)},$$

where $\mu:=(\alpha_{k-2}+\alpha_{k-1}-\alpha_1+\alpha_2+x,x+1,1^{\alpha_1-\alpha_2-x-1})$. As

$$2\leqslant\alpha_{k-1}-\alpha_1+\alpha_2+x+2\leqslant x+1$$

and so

$$h^{\mu}_{1,\alpha_{k-1}-\alpha_1+\alpha_2+x+2}=\alpha_{k-2}+\alpha_{k-1}-\alpha_1+\alpha_2+x+2-(\alpha_{k-1}-\alpha_1+\alpha_2+x+2)=\alpha_{k-2}.$$

From $\alpha_1-\alpha_2$ not being a part of α and

$$x,\alpha_1-\alpha_2-x-1<\alpha_{k-1}<\alpha_{k-2}$$

it follows that

$$\chi^{\mu}_{(\alpha_{k-2},\ldots,\alpha_h)}=-\chi^{\nu}_{(\alpha_{k-1},\ldots,\alpha_h)}=(-1)^{\alpha_1-\alpha_2-x},$$

with $\nu=(x,\alpha_{k-1}-\alpha_1+\alpha_2+x+1,1^{\alpha_1-\alpha_2-x-1})$, and so the theorem holds also in this case.

At last assume that $2\alpha_1+x=|\alpha|$. Then

$$\alpha_1=|\alpha|-\alpha_1-x=\alpha_2+\cdots+\alpha_{k-1}.$$

By definition of k we then have that

$$\alpha_3+\cdots+\alpha_{k-1}=\alpha_1-\alpha_2\leqslant\alpha_{k-1}+\cdots+\alpha_h$$

and so

$$\alpha_3+\cdots+\alpha_{k-2}\leqslant\alpha_k+\cdots+\alpha_h.$$

If $k\geqslant 5$ then $k-2\geqslant 3$ and then $\alpha_{k-2}\leqslant\alpha_k+\cdots+\alpha_h$. This gives a contradiction with $(\alpha_2,\ldots,\alpha_h)\in\text{Sign}$. So $k=4$ and then $\alpha_1-\alpha_2=\alpha_3$ is a part of α, which contradicts the assumptions. □

Theorem 3.2. *Assume that the following hold:*

- $\alpha \notin \text{Sign}$, $(\alpha_2, \ldots, \alpha_h) \in \text{Sign}$ *and* $\alpha_1 > \alpha_2 > \alpha_3 + \cdots + \alpha_h$,
- $k \leqslant h$,
- $\alpha_1 - \alpha_2$ *is not a part of* α,
- $\alpha_{k-1} \leqslant x$,
- *none of the following holds:*
  - $(\alpha_{k-1}, \ldots, \alpha_h) = (3, 2, 1, 1)$ *and* $\alpha_1 = \alpha_2 + \alpha_{k-1} + \cdots + \alpha_h$,
  - $(\alpha_{k-1}, \ldots, \alpha_h) = (5, 3, 2, 1)$ *and* $\alpha_1 = \alpha_2 + \alpha_{k-1} + \cdots + \alpha_h$,
  - $(\alpha_{k-1}, \ldots, \alpha_h) = (a, a-1, 1)$ *with* $a \geqslant 2$ *and* $\alpha_1 = \alpha_2 + \alpha_{k-1} + \cdots + \alpha_h - 1$,
  - $(\alpha_{k-1}, \ldots, \alpha_h) = (a, a-1, 2, 1)$ *with* $a \geqslant 4$ *and* $\alpha_1 = \alpha_2 + \alpha_{k-1} + \cdots + \alpha_h - 3$,
  - $(\alpha_{k-1}, \ldots, \alpha_h) = (a, a-1, 3, 1)$ *with* $a \geqslant 5$ *and* $\alpha_1 = \alpha_2 + \alpha_{k-1} + \cdots + \alpha_h - 4$.

Then $\beta = (|\alpha| - \alpha_1, x+1, 1^{\alpha_1 - x - 1})$ *is a partition,* $h^{\beta}_{2,1} = \alpha_1$ *and* $\chi^{\beta}_{\alpha} = (-1)^{\alpha_1 - x - 1} 2$.

Proof. As in the previous theorem we have that $2\alpha_1 + x \neq |\alpha|$, since $\alpha_1 - \alpha_2$ is not a part of α.

Assume first that $2\alpha_1 + x > |\alpha|$. From the proof of the previous theorem ($\alpha_2 > x$ since $(\alpha_2, \ldots, \alpha_h) \in \text{Sign}$), it is enough to prove that $\chi^{\epsilon}_{(\alpha_3, \ldots, \alpha_h)} = (-1)^{\alpha_1 - \alpha_2 - x - 1}$, where $\epsilon = (x, |\alpha| - 2\alpha_1 + 1, 1^{\alpha_1 - \alpha_2 - x - 1})$. In this case it holds $k = 4$ as in the previous theorem.

Assume now that $2\alpha_1 + x < |\alpha|$. Since $\alpha_{k-1} \leqslant x < \alpha_1 - \alpha_2$ we have that $\alpha_{k-1} < \alpha_1 - \alpha_2$. As $\alpha_1 - \alpha_2$ is not a part of α it is enough, from the proof of the previous theorem, to prove that $x < \alpha_j$ for $j \leqslant k-2$ and that $\chi^{\nu}_{(\alpha_{k-1}, \ldots, \alpha_h)} = (-1)^{\alpha_1 - \alpha_2 - x - 1}$, where $\nu = (x, \alpha_{k-1} - \alpha_1 + \alpha_2 + x + 1, 1^{\alpha_1 - \alpha_2 - x - 1})$. In order to prove that $x < \alpha_j$ for $j \leqslant k-2$, it is enough to prove it for $j = k-2$. As $k \geqslant 4$, so that $k - 2 \geqslant 2$, and $(\alpha_2, \ldots, \alpha_h) \in \text{Sign}$, we have that $x = \alpha_k + \cdots + \alpha_h < \alpha_{k-2}$.

In either case it is then enough to prove that $\chi^{\lambda_y}_{(\alpha_{k-1}, \ldots, \alpha_h)} = (-1)^y$ for $\lambda_y = (x, \alpha_{k-1} - y, 1^y)$, $y = \alpha_1 - \alpha_2 - x - 1$. Notice that $0 \leqslant y \leqslant \alpha_{k-1} - 1$, since λ_y is a partition.

Clearly $h^{\lambda_y}_{2,1} = \alpha_{k-1}$. If this is the only α_{k-1}-hook of λ, then it is easy to see that $\chi^{\lambda_y}_{(\alpha_{k-1}, \ldots, \alpha_h)} = (-1)^y$. Else, due to hook lengths being decreasing along both the rows and the columns, λ_y has exactly 2 α_{k-1}-hooks and there exists $2 \leqslant j \leqslant x$ with $h^{\lambda_y}_{1,j} = \alpha_{k-1}$.

As $\alpha_{k-1} \leqslant x$ by assumption

$$\begin{aligned}(\alpha_{k-1}, \ldots, \alpha_h) \in &\{(1,1), (3,2,1,1), (5,3,2,1)\} \cup \{(a, a-1, 1) : a \geqslant 2\} \\ &\cup \{(a, a-1, 2, 1) : a \geqslant 4\} \cup \{(a, a-1, 3, 1) : a \geqslant 5\}.\end{aligned}$$

If $(\alpha_{k-1}, \ldots, \alpha_h) = (1, 1)$ then $x = 1 < 2$, so no such j exists.

If $(\alpha_{k-1}, \ldots, \alpha_h) = (3, 2, 1, 1)$ then $\lambda_y \in \{(4, 3), (4, 1, 1, 1)\}$ if such a j exists, and so $y = 0$ or $y = 2$ respectively. The second case would imply $\alpha_1 - \alpha_2 - x = 3$, which would contradict the assumption. As $\chi^{(4,3)}_{(3,2,1,1)} = 1 = (-1)^0$ the theorem holds in this case.

If $(\alpha_{k-1},\dots,\alpha_h)=(5,3,2,1)$ and there exists such a j then

$$\lambda_y\in\{(6,5),(6,4,1),(6,3,1,1),(6,1^5)\}$$

and then $y=0$, $y=1$, $y=2$ or $y=4$ respectively. In the last case $\alpha_1-\alpha_2-x=5$, which contradicts the assumption. In the other cases $\chi^{(6,5)}_{(5,3,2,1)}=1=(-1)^0$, $\chi^{(6,4,1)}_{(5,3,2,1)}=-1=(-1)^1$ and $\chi^{(6,3,1,1)}_{(5,3,2,1)}=1=(-1)^2$ and so the theorem holds also in this case.

If $(\alpha_{k-1},\dots,\alpha_h)=(a,a-1,1)$ then there exists such a j if and only if $0\leqslant y\leqslant\alpha_{k-1}-2$. If $y=\alpha_{k-1}-2$ then $\alpha_1-\alpha_2-x=\alpha_{k-1}-1$ which contradicts the assumption. In the other cases

$$\chi^{\lambda_y}_{(\alpha_{k-1},\dots,\alpha_h)}=\chi^{(a,a-y,1^y)}_{(a,a-1,1)}=(-1)^y\chi^{(a)}_{(a-1,1)}-\chi^{(a-y-1,1^{y+1})}_{(a-1,1)}=(-1)^y,$$

since $a-y-2,y+1\geqslant 1$, so that also $a-y-2,y+1<a-1$. In particular the theorem holds in this case.

If $(\alpha_{k-1},\dots,\alpha_h)=(a,a-1,2,1)$ then there exists such a j if and only if $y\neq\alpha_{k-1}-3$. For $y=\alpha_{k-1}-4$ we have that $\alpha_1-\alpha_2-x=\alpha_{k-1}-3$, which contradicts the assumptions.

For $0\leqslant y\leqslant\alpha_{k-1}-5$ then $j=4$ as $\alpha_{k-1}-y>4$, so that

$$h^{\lambda_y}_{1,4}=a+2+2-4=a.$$

So

$$\chi^{\lambda_y}_{(\alpha_{k-1},\dots,\alpha_h)}=\chi^{(a+2,a-y,1^y)}_{(a,a-1,2,1)}=(-1)^y\chi^{(a+2)}_{(a-1,2,1)}-\chi^{(a-y-1,3,1^y)}_{(a-1,2,1)}=(-1)^y-\chi^{(a-y-1,3,1^y)}_{(a-1,2,1)}$$

and

$$\chi^{(a-y-1,3,1^y)}_{(a-1,2,1)}=\begin{cases}0 & y\neq 0\\ -\chi^{(2,1)}_{(2,1)}=0 & y=0,\end{cases}$$

as

$$\begin{aligned}h^{(a-y-1,3,1^y)}_{1,1}&=a,\\ h^{(a-y-1,3,1^y)}_{2,1}&=y+3<a-1,\\ h^{(a-y-1,3,1^y)}_{1,2}&=a-y-1\leqslant a-1,\end{aligned}$$

since $0\leqslant y\leqslant\alpha_{k-1}-5=a-5$. In particular $\chi^{\lambda_y}_{(\alpha_{k-1},\dots,\alpha_h)}=(-1)^y$.

For $\alpha_{k-1}-2\leqslant y\leqslant\alpha_{k-1}-1$ then $j=3$ as $\alpha_{k-1}-y\leqslant 2$, so that

$$h^{\lambda_y}_{1,3}=a+2+1-3=a.$$

It follows that

$$\chi^{\lambda_y}_{(\alpha_{k-1},\dots,\alpha_h)}=\chi^{(a+2,a-y,1^y)}_{(a,a-1,2,1)}=(-1)^y\chi^{(a+2)}_{(a-1,2,1)}+\chi^{(2,a-y,1^y)}_{(a-1,2,1)}=(-1)^y+\chi^{(2,a-y,1^y)}_{(a-1,2,1)}.$$

As

$$\chi^{(2,a-y,1^y)}_{(a-1,2,1)}=\begin{cases}\chi^{(2,2,1^{a-2})}_{(a-1,2,1)}=0 & y=\alpha_{k-1}-2,\\ \chi^{(2,1^a)}_{(a-1,2,1)}=(-1)^{a-2}\chi^{(2,1)}_{(2,1)}=0 & y=\alpha_{k-1}-1,\end{cases}$$

as $a \geqslant 4$. In particular also in this case $\chi^{\lambda_y}_{(\alpha_{k-1},\dots,\alpha_h)} = (-1)^y$.

If $(\alpha_{k-1},\dots,\alpha_h) = (a, a-1, 3, 1)$ then there exists such a j if and only if $y \neq \alpha_{k-1} - 4$. If $y = \alpha_{k-1} - 5$ then $\alpha_1 - \alpha_2 - x = \alpha_k - 4$, in contradiction to the assumption.

For $0 \leqslant y \leqslant \alpha_{k-1} - 6$ then $j = 5$ as $\alpha_{k-1} - y > 5$ and then

$$h^{\lambda_y}_{1,5} = a + 3 + 2 - 5 = a.$$

So

$$\chi^{\lambda_y}_{(\alpha_{k-1},\dots,\alpha_h)} = \chi^{(a+3,a-y,1^y)}_{(a,a-1,3,1)} = (-1)^y \chi^{(a+3)}_{(a-1,3,1)} - \chi^{(a-y-1,4,1^y)}_{(a-1,3,1)} = (-1)^y - \chi^{(a-y-1,3,1^y)}_{(a-1,3,1)}$$

and

$$\chi^{(a-y-1,4,1^y)}_{(a-1,3,1)} = \begin{cases} 0 & y \neq 0, \\ -\chi^{(3,1)}_{(3,1)} = 0 & y = 0, \end{cases}$$

as

$$\begin{aligned} h^{(a-y-1,4,1^y)}_{1,1} &= a, \\ h^{(a-y-1,4,1^y)}_{2,1} &= y + 4 < a - 1, \\ h^{(a-y-1,4,1^y)}_{1,2} &= a - y - 1 \leqslant a - 1, \end{aligned}$$

since $0 \leqslant y \leqslant \alpha_{k-1} - 6 = a - 6$. In particular $\chi^{\lambda_y}_{(\alpha_{k-1},\dots,\alpha_h)} = (-1)^y$.

For $\alpha_{k-1} - 3 \leqslant y \leqslant \alpha_{k-1} - 1$ then $j = 4$ as $\alpha_{k-1} - y \leqslant 3$, so that

$$h^{\lambda_y}_{1,4} = a + 3 + 1 - 4 = a.$$

Then

$$\chi^{\lambda_y}_{(\alpha_{k-1},\dots,\alpha_h)} = \chi^{(a+3,a-y,1^y)}_{(a,a-1,3,1)} = (-1)^y \chi^{(a+3)}_{(a-1,3,1)} + \chi^{(3,a-y,1^y)}_{(a-1,3,1)} = (-1)^y + \chi^{(3,a-y,1^y)}_{(a-1,3,1)}.$$

As

$$\chi^{(3,a-y,1^y)}_{(a-1,3,1)} = \begin{cases} \chi^{(3,3,1^{a-3})}_{(a-1,3,1)} = 0 & y = \alpha_{k-1} - 3, \\ \chi^{(3,2,1^{a-2})}_{(a-1,3,1)} = 0 & y = \alpha_{k-1} - 2, \\ \chi^{(3,1^a)}_{(a-1,3,1)} = (-1)^{a-2} \chi^{(3,1)}_{(3,1)} = 0 & y = \alpha_{k-1} - 1, \end{cases}$$

since $a \geqslant 5$ it follows that also in this case $\chi^{\lambda_y}_{(\alpha_{k-1},\dots,\alpha_h)} = (-1)^y$. □

Theorem 3.3. *Assume that the following hold:*

- $\alpha \notin \mathrm{Sign}$, $(\alpha_2, \dots, \alpha_h) \in \mathrm{Sign}$ *and* $\alpha_1 > \alpha_2 > \alpha_3 + \cdots + \alpha_h$,
- $k \leqslant h$,
- $\alpha_1 - \alpha_2$ *is not a part of* α,
- $(\alpha_{k-1}, \dots, \alpha_h) \in \{(3, 2, 1, 1), (5, 3, 2, 1)\}$,

- $\alpha_1 = \alpha_2 + \alpha_{k-1} + \cdots + \alpha_h$.

Let c equal to 3 if $(\alpha_{k-1}, \ldots, \alpha_h) = (3, 2, 1, 1)$ or equal to 6 if $(\alpha_{k-1}, \ldots, \alpha_h) = (5, 3, 2, 1)$. Then $\beta := (|\alpha| - \alpha_1, \alpha_1 - c, 1^c)$ is a partition with $h^\beta_{2,1} = \alpha_1$ and $\chi^\beta_\alpha = (-1)^c 2$.

Proof. Since $c < \alpha_2 < \alpha_1 < \alpha_2 + \cdots + \alpha_h = |\alpha| - \alpha_1$ by assumption on α, it follows that β is a partition. Clearly $h^\beta_{2,1} = \alpha_1$.

Also, from

$$2 \leqslant \alpha_3 + \cdots + \alpha_{k-2} + 2 < \alpha_3 + \cdots + \alpha_h - c < \alpha_1 - c$$

we have that

$$\begin{aligned} h^\beta_{1,\alpha_3+\cdots+\alpha_{k-2}+2} &= |\alpha| - \alpha_1 + 2 - (\alpha_3 + \cdots + \alpha_{k-2} + 2) \\ &= \alpha_2 + \cdots + \alpha_h - \alpha_3 - \cdots - \alpha_{k-2} \\ &= \alpha_2 + \alpha_{k-1} + \cdots + \alpha_h \\ &= \alpha_1. \end{aligned}$$

If $(\alpha_{k-1}, \ldots, \alpha_h) = (3, 2, 1, 1)$ let $d = 3$. If instead $(\alpha_{k-1}, \ldots, \alpha_h) = (5, 3, 2, 1)$ let $d = 4$. Notice that $c + d = \alpha_{k-1} + \cdots + \alpha_h - 1$. Then by assumption

$$\alpha_1 - c = \alpha_2 + \alpha_{k-1} + \cdots + \alpha_h - c = \alpha_2 + d + 1.$$

It follows that

$$\chi^\beta_\alpha = (-1)^c \chi^{(|\alpha|-\alpha_1)}_{(\alpha_2,\ldots,\alpha_h)} - \chi^\delta_{(\alpha_2,\ldots,\alpha_h)} = (-1)^c - \chi^\delta_{(\alpha_2,\ldots,\alpha_h)}$$

where $\delta = (\alpha_2 + d, \alpha_3 + \cdots + \alpha_{k-2} + 1, 1^c)$.

Assume first that $k = 4$. Then $\alpha_3 + \cdots + \alpha_{k-2} = 0$ and so, as $c + 1 < \alpha_2$,

$$\chi^\delta_{(\alpha_2,\ldots,\alpha_h)} = \chi^{(d,1^{c+1})}_{(\alpha_{k-1},\ldots,\alpha_h)} = (-1)^{c-1}$$

(the last equality follows from $(\alpha_{k-1}, \ldots, \alpha_h) \in \{(3, 2, 1, 1), (5, 3, 2, 1)\}$ and from the definition of c and d) and so in this case $\chi^\beta_\alpha = (-1)^c 2$.

So assume now that $k > 4$. As $(\alpha_2, \ldots, \alpha_h) \in \text{Sign}$, it follows that $\alpha_j > \alpha_{k-1} + \cdots + \alpha_h$ for $j \leqslant k - 2$. Also

$$\delta_2 = \alpha_3 + \cdots + \alpha_{k-2} + 1 \geqslant \alpha_3 + 1 > d + 2 > 2.$$

So

$$h^\delta_{1,d+2} = \alpha_2 + d + 2 - (d + 2) = \alpha_2$$

and then as by assumption $|\delta| = \alpha_2 + \cdots + \alpha_h < 2\alpha_2$, so that δ cannot have more than 1 hook of length α_2,

$$\chi^\delta_{(\alpha_2,\ldots,\alpha_h)} = -\chi^\epsilon_{(\alpha_3,\ldots,\alpha_h)}$$

with $\epsilon = (\alpha_3 + \cdots + \alpha_{k-2}, d + 1, 1^c)$. As $h^\epsilon_{2,1} = c + d + 1 = \alpha_{k-1} + \cdots + \alpha_h < \alpha_j$ for $j \leqslant k - 2$ and then in particular also $\alpha_{k-2} \geqslant d + 1 > 2$, we have that

$$\chi^\epsilon_{(\alpha_3,\ldots,\alpha_h)} = \chi^{(\alpha_{k-2},d+1,1^c)}_{(\alpha_{k-2},\ldots,\alpha_h)} = -\chi^{(d,1^{c+1})}_{(\alpha_{k-1},\ldots,\alpha_h)} = (-1)^c.$$

In particular also in this case $\chi^\beta_\alpha = (-1)^c 2$. ☐

Theorem 3.4. *Assume that the following hold:*

- $\alpha \notin \mathrm{Sign}$, $(\alpha_2, \ldots, \alpha_h) \in \mathrm{Sign}$ *and* $\alpha_1 > \alpha_2 > \alpha_3 + \cdots + \alpha_h$,
- $k \leqslant h$,
- $\alpha_1 - \alpha_2$ *is not a part of* α,
- *one of the following holds:*
  - $(\alpha_{k-1}, \ldots, \alpha_h) = (a, a-1, 1)$ *with* $a \geqslant 2$, $\alpha_1 = \alpha_2 + \alpha_{k-1} + \cdots + \alpha_h - 1$ *and* $(\alpha_{k-2}, \ldots, \alpha_h) \notin \{(3,2,1,1), (5,3,2,1)\}$,
  - $(\alpha_{k-1}, \ldots, \alpha_h) = (a, a-1, 2, 1)$ *with* $a \geqslant 4$ *and* $\alpha_1 = \alpha_2 + \alpha_{k-1} + \cdots + \alpha_h - 3$,
  - $(\alpha_{k-1}, \ldots, \alpha_h) = (a, a-1, 3, 1)$ *with* $a \geqslant 5$ *and* $\alpha_1 = \alpha_2 + \alpha_{k-1} + \cdots + \alpha_h - 4$.

Then $\beta := (|\alpha| - \alpha_1, 1^{\alpha_1})$ *is a partition with* $h^{\beta}_{2,1} = \alpha_1$ *and* $\chi^{\beta}_{\alpha} = (-1)^{\alpha_1 - 1}2$.

Proof. From the definition we clearly have that β is a partition with $h^{\beta}_{2,1} = \alpha_1$.

Notice that from the assumptions $\alpha_1 = \alpha_2 + 2a - 1$. Also

$$|\alpha| - \alpha_1 = \alpha_2 + \cdots + \alpha_h > \alpha_2 + 2a - 1 = \alpha_1$$

and so, as $\alpha_2 > \alpha_3 + \cdots + \alpha_h$, so that any partition of $\alpha_2 + \cdots + \alpha_h$ has at most one hook of length α_2,

$$\begin{aligned} \chi^{\beta}_{\alpha} &= (-1)^{\alpha_1 - 1}\chi^{(|\alpha|-\alpha_1)}_{(\alpha_2,\ldots,\alpha_h)} + \chi^{(|\alpha|-2\alpha_1, 1^{\alpha_1})}_{(\alpha_2,\ldots,\alpha_h)} \\ &= (-1)^{\alpha_1 - 1} + (-1)^{\alpha_2 - 1}\chi^{(|\alpha|-2\alpha_1, 1^{\alpha_1-\alpha_2})}_{(\alpha_3,\ldots,\alpha_h)} \\ &= (-1)^{\alpha_1 - 1} + (-1)^{\alpha_1}\chi^{(|\alpha|-2\alpha_1, 1^{2a-1})}_{(\alpha_3,\ldots,\alpha_h)}. \end{aligned}$$

Assume first that either $k = 4$ or $k > 4$ and $\alpha_{k-2} \geqslant 2a$. Then, as $\alpha_{k-1} + \cdots + \alpha_h \geqslant 2a$ it follows that

$$\chi^{(|\alpha|-2\alpha_1, 1^{2a-1})}_{(\alpha_3,\ldots,\alpha_h)} = \chi^{(\alpha_{k-1}+\cdots+\alpha_h-2a+1, 1^{2a-1})}_{(\alpha_{k-1},\ldots,\alpha_h)} = (-1)^{(a-1)+(a-2)} = -1.$$

The second last equality follows from

$$(\alpha_{k-1} + \cdots + \alpha_h - 2a + 1, 1^{2a-1}) = \begin{cases} (1^{2a}) & (\alpha_{k-1}, \ldots, \alpha_h) = (a, a-1, 1), \\ (3, 1^{2a-1}) & (\alpha_{k-1}, \ldots, \alpha_h) = (a, a-1, 2, 1), \\ (4, 1^{2a-1}) & (\alpha_{k-1}, \ldots, \alpha_h) = (a, a-1, 3, 1), \end{cases}$$

so that, by assumption on a, $a - 1 > h^{(\alpha_{k-1}+\cdots+\alpha_h-2a+1, 1^{2a-1})}_{1,2}$ in the last two cases.

Assume now that $k > 4$ and $\alpha_{k-2} < 2a \leqslant \alpha_{k-1} + \cdots + \alpha_h$. Notice that in this case $(\alpha_{k-1}, \ldots, \alpha_h) = (a, a-1, 1)$, as $(\alpha_2, \ldots, \alpha_h) \in \mathrm{Sign}$ and then also $(\alpha_{k-2}, \ldots, \alpha_h) \in \mathrm{Sign}$. From this assumption and the assumption that $(\alpha_{k-2}, \ldots, \alpha_h) \notin \{(3,2,1,1), (5,3,2,1)\}$ it follows that $(\alpha_{k-2}, \ldots, \alpha_h) \in \{(4,3,2,1), (5,4,3,1)\}$. Also, always by assumption of $(\alpha_2, \ldots, \alpha_h) \in \mathrm{Sign}$, if $k \geqslant 6$ then $\alpha_{k-3} > 2a - 1$. In either of the two cases

$$\chi^{(|\alpha|-2\alpha_1, 1^{2a-1})}_{(\alpha_3,\ldots,\alpha_h)} = \chi^{(\alpha_{k-2}+1, 1^{2a-1})}_{(\alpha_{k-2},\ldots,\alpha_h)} = -1.$$

In either case $\chi^{\beta}_{\alpha} = (-1)^{\alpha_1 - 1}2$ and so the theorem is proved. □

Theorem 3.5. *Assume that the following hold:*

- $\alpha \notin \text{Sign}$, $(\alpha_2, \ldots, \alpha_h) \in \text{Sign}$ *and* $\alpha_1 > \alpha_2 > \alpha_3 + \cdots + \alpha_h$,
- $k \leqslant h$,
- $\alpha_1 - \alpha_2$ *is not a part of* α,
- $\alpha_1 = \alpha_2 + \alpha_{k-1} + \cdots + \alpha_h - 1$,
- $(\alpha_{k-2}, \ldots, \alpha_h) \in \{(3, 2, 1, 1), (5, 3, 2, 1)\}$.

Then $\beta := (|\alpha| - \alpha_1, \alpha_1)$ *is a partition with* $h^{\beta}_{2,1} = \alpha_1$ *and* $\chi^{\beta}_{\alpha} = 2$.

Proof. Since, by assumption, $\alpha_1 < \alpha_2 + \cdots + \alpha_h = |\alpha| - \alpha_1$ we have that β is a partition. Also clearly $h^{\beta}_{2,1} = \alpha_1$.

Notice that in this case $k - 2 > 2$, as $\alpha_{k-2} < \alpha_{k-1} + \cdots + \alpha_h$ and by assumption $\alpha_2 > \alpha_3 + \cdots + \alpha_h$. As

$$1 < \alpha_3 + \cdots + \alpha_{k-2} + 3 < \alpha_3 + \cdots + \alpha_h < \alpha_2 < \alpha_1$$

it follows that

$$\begin{aligned} h^{\beta}_{1,\alpha_3+\cdots+\alpha_{k-2}+3} &= |\alpha| - \alpha_1 + 2 - (\alpha_3 + \cdots + \alpha_{k-2} + 3) \\ &= |\alpha| - (\alpha_2 + \alpha_{k-1} + \cdots + \alpha_h - 1) - (\alpha_3 + \cdots + \alpha_{k-2}) - 1 \\ &= |\alpha| - \alpha_2 - \cdots - \alpha_h \\ &= \alpha_1. \end{aligned}$$

So

$$\chi^{\beta}_{\alpha} = \chi^{(|\alpha|-\alpha_1)}_{(\alpha_2,\ldots,\alpha_h)} - \chi^{\delta}_{(\alpha_2,\ldots,\alpha_h)} = 1 - \chi^{\delta}_{(\alpha_2,\ldots,\alpha_h)},$$

with

$$\delta := (\alpha_1 - 1, \alpha_3 + \cdots + \alpha_{k-2} + 2) = (\alpha_2 + \alpha_{k-1} + \cdots + \alpha_h - 2, \alpha_3 + \cdots + \alpha_{k-2} + 2).$$

Also by assumption

$$1 < \alpha_{k-1} + \cdots + \alpha_h < \alpha_{k-2} + 2 \leqslant \alpha_3 + \cdots + \alpha_{k-2} + 2$$

and then

$$h^{\delta}_{1,\alpha_{k-1}+\cdots+\alpha_h} = \alpha_2 + \alpha_{k-1} + \cdots + \alpha_h - 2 + 2 - \alpha_{k-1} + \cdots + \alpha_h = \alpha_2.$$

From the previous $\alpha_3 + \cdots + \alpha_{k-2} + 2 < \alpha_2$ and so

$$\chi^{\delta}_{(\alpha_2,\ldots,\alpha_h)} = -\chi^{\epsilon}_{(\alpha_3,\ldots,\alpha_h)}$$

with

$$\epsilon := (\alpha_3 + \cdots + \alpha_{k-2} + 1, \alpha_{k-1} + \cdots + \alpha_h - 1).$$

As $(\alpha_2, \ldots, \alpha_h) \in \text{Sign}$ by assumption, so that $\alpha_j > \alpha_{k-1} + \cdots + \alpha_h > \epsilon_2$ for $j \leqslant k-3$ and as $\alpha_{k-2} + 1 > \alpha_{k-1} + \cdots + \alpha_h - 1$ by assumption, it follows that

$$\chi^{\epsilon}_{(\alpha_3,\ldots,\alpha_h)} = \chi^{(\alpha_{k-2}+1,\alpha_{k-1}+\cdots+\alpha_h-1)}_{(\alpha_{k-2},\ldots,\alpha_h)} = 1$$

(the last equation follows from the assumption that $(\alpha_{k-2}, \ldots, \alpha_h)$ is either $(3, 2, 1, 1)$ or $(5, 3, 2, 1)$).

In particular $\chi^{\beta}_{\alpha} = 2$ and so the theorem holds. □

Theorem 3.6. *Assume that the following hold:*

- $\alpha \notin \text{Sign}$, $(\alpha_2, \ldots, \alpha_h) \in \text{Sign}$ *and* $\alpha_1 > \alpha_2 > \alpha_3 + \cdots + \alpha_h$*,*
- $k \leqslant h$*,*
- *there exists* i *with* $\alpha_i = \alpha_1 - \alpha_2$*,*
- $\alpha_i \geqslant \alpha_{i+1} + \cdots + \alpha_h$*.*

Then $\beta = (|\alpha| - \alpha_1, \alpha_2 + 1, 1^{\alpha_1-\alpha_2-1})$ *is a partition with* $h^{\beta}_{2,1} = \alpha_1$ *and* $\chi^{\beta}_{\alpha} = (-1)^{\alpha_1-\alpha_2-1}2$*.*

Proof. Since by assumption $\alpha_1 > \alpha_2 + \alpha_h \geqslant \alpha_2 + 1$ and (also using Lemma 1.5)

$$|\alpha| - \alpha_1 \geqslant \alpha_1 > \alpha_2 + \alpha_h \geqslant \alpha_2 + 1$$

it follows that β is partition. Also clearly $h^{\beta}_{2,1} = \alpha_1$.

From the definition of k and from

$$2\alpha_2 > \alpha_2 + \cdots + \alpha_h \geqslant \alpha_1$$

we have that $3 \leqslant i < k \leqslant h$. Then

$$\begin{aligned} h^{\beta}_{1,2} &= |\alpha| - \alpha_1 = \alpha_2 + \cdots + \alpha_h \geqslant \alpha_2 + \alpha_i + \alpha_h > \alpha_1, \\ h^{\beta}_{1,\alpha_2+1} &= |\alpha| - \alpha_1 + 2 - \alpha_2 - 1 = \alpha_3 + \cdots + \alpha_h + 1 \leqslant \alpha_2 < \alpha_1. \end{aligned}$$

In particular there exists $3 \leqslant j \leqslant \alpha_2$ such that $h^{\beta}_{1,j} = \alpha_1$. From the Murnaghan-Nakayama formula it follows that

$$\begin{aligned} \chi^{\beta}_{\alpha} &= (-1)^{\alpha_1-\alpha_2-1}\chi^{(|\alpha|-\alpha_1)}_{(\alpha_2,\ldots,\alpha_h)} - \chi^{(\alpha_2,j-1,1^{\alpha_1-\alpha_2-1})}_{(\alpha_2,\ldots,\alpha_3)} \\ &= (-1)^{\alpha_1-\alpha_2-1} + \chi^{(j-2,1^{\alpha_1-\alpha_2})}_{(\alpha_3,\ldots,\alpha_h)} \\ &= (-1)^{\alpha_1-\alpha_2-1} + \chi^{(\alpha_{i+1}+\cdots+\alpha_h,1^{\alpha_i})}_{(\alpha_i,\ldots,\alpha_h)} \\ &= (-1)^{\alpha_1-\alpha_2-1} + (-1)^{\alpha_i-1}\chi^{(\alpha_{i+1}+\cdots+\alpha_h)}_{(\alpha_{i+1},\ldots,\alpha_h)} \\ &= (-1)^{\alpha_1-\alpha_2-1}2. \end{aligned}$$

The second line follows from $h^{(\alpha_2,j-1,1^{\alpha_1-\alpha_2-1})}_{1,2} = \alpha_2$, as $j \geqslant 3$, and from

$$|(\alpha_2, j-1, 1^{\alpha_1-\alpha_2-1})| = |\alpha| - \alpha_1 < 2\alpha_2,$$

so that $(\alpha_2, j-1, 1^{\alpha_1-\alpha_2-1})$ has at most one hook of length α_2. The third line from $\alpha_j > \alpha_i$ for $j < i$ and from $i < h$, so that

$$\begin{aligned} h_{1,2}^{(j-2,1^{\alpha_1-\alpha_2})} &= |(\alpha_3, \dots, \alpha_h)| - (\alpha_1 - \alpha_2) - 1 \\ &= \alpha_3 + \cdots + \alpha_h - \alpha_i - 1 \\ &\geqslant \alpha_1 + \cdots + \alpha_{i+1}. \end{aligned}$$

The fourth line follows from $\alpha_i \geqslant \alpha_{i+1} + \cdots + \alpha_h$. □

Theorem 3.7. *Assume that the following hold:*

- $\alpha \notin \mathrm{Sign}$, $(\alpha_2, \dots, \alpha_h) \in \mathrm{Sign}$ *and* $\alpha_1 > \alpha_2 > \alpha_3 + \cdots + \alpha_h$,
- $k \leqslant h$,
- *there exists* i *with* $\alpha_i = \alpha_1 - \alpha_2$,
- $\alpha_i < \alpha_{i+1} + \cdots + \alpha_h$.

Then $\beta = (|\alpha| - \alpha_1, \alpha_2 + 2, 1^{\alpha_1-\alpha_2-2})$ *is a partition with* $h_{2,1}^\beta = \alpha_1$ *and* $\chi_\alpha^\beta = (-1)^{\alpha_1-\alpha_2}2$.

Proof. Since by assumption $\alpha_1 > \alpha_2 + \alpha_h \geqslant \alpha_2 + 1$ and

$$|\alpha| - \alpha_1 \geqslant \alpha_1 > \alpha_2 + \alpha_h \geqslant \alpha_2 + 1$$

it follows that β is partition with $h_{2,1}^\beta = \alpha_1$.

From $\alpha_i < \alpha_{i+1} + \cdots + \alpha_h$ and $(\alpha_2, \dots, \alpha_h) \in \mathrm{Sign}$ it follows that

$$\begin{aligned} (\alpha_i, \dots, \alpha_h) \in\ & \{(3,2,1,1), (5,3,2,1)\} \cup \{(a, a-1, 2, 1) : a \geqslant 4\} \\ & \cup \{(a, a-1, 3, 1) : a \geqslant 5\}. \end{aligned}$$

Similar to the previous theorem we have that $3 \leqslant i < k \leqslant h$, from which follows that

$$\begin{aligned} h_{1,2}^\beta &= |\alpha| - \alpha_1 = \alpha_2 + \cdots + \alpha_h \geqslant \alpha_2 + \alpha_i + \cdots + \alpha_h \geqslant \alpha_1 + 2, \\ h_{1,\alpha_2+2}^\beta &= |\alpha| - \alpha_1 + 2 - \alpha_2 - 2 = \alpha_3 + \cdots + \alpha_h < \alpha_2 < \alpha_1. \end{aligned}$$

In particular there exists $4 \leqslant j \leqslant \alpha_2$ such that $h_{1,j}^\beta = \alpha_1$. So

$$\begin{aligned} \chi_\alpha^\beta &= (-1)^{\alpha_1-\alpha_2-2}\chi_{(\alpha_2,\dots,\alpha_h)}^{(|\alpha|-\alpha_1)} - \chi_{(\alpha_2,\dots,\alpha_3)}^{(\alpha_2+1,j-1,1^{\alpha_1-\alpha_2-2})} \\ &= (-1)^{\alpha_1-\alpha_2} + \chi_{(\alpha_3,\dots,\alpha_h)}^{(j-2,2,1^{\alpha_1-\alpha_2-2})} \\ &= (-1)^{\alpha_1-\alpha_2} + \chi_{(\alpha_i,\dots,\alpha_h)}^{(\alpha_{i+1}+\cdots+\alpha_h,2,1^{\alpha_i-2})}. \end{aligned}$$

The second line follows from $\alpha_2 > \alpha_3 + \cdots + \alpha_h$ and, as $j \geqslant 4$,

$$h_{1,3}^{(\alpha_2+1,j-1,1^{\alpha_1-\alpha_2-2})} = \alpha_2 + 1 + 2 - 3 = \alpha_2.$$

The third line follows from $\alpha_j > \alpha_i$ for $j < i$ and from

$$\begin{aligned} h_{1,3}^{(j-2,2,1^{\alpha_1-\alpha_2-2})} &= |(\alpha_3,\ldots,\alpha_h)| - (\alpha_1-\alpha_2) - 2 \\ &= \alpha_3 + \cdots + \alpha_h - \alpha_i - 2 \\ &\geqslant \alpha_1 + \cdots + \alpha_{i+1}. \end{aligned}$$

If $(\alpha_i,\ldots,\alpha_h) \in \{(3,2,1,1),(5,3,2,1)\}$ it is easy to check that

$$\chi_{(\alpha_i,\ldots,\alpha_h)}^{(\alpha_{i+1}+\cdots+\alpha_h,2,1^{\alpha_i-2})} = -1 = (-1)^{\alpha_i} = (-1)^{\alpha_1-\alpha_2}.$$

In particular the theorem holds in this case.

If $(\alpha_i,\ldots,\alpha_h) = (a,a-1,c,1)$ with $c \in \{2,3\}$ then, as $a-1 > c$,

$$\begin{aligned} \chi_{(\alpha_i,\ldots,\alpha_h)}^{(\alpha_{i+1}+\cdots+\alpha_h,2,1^{\alpha_i-2})} &= \chi_{(a,a-1,c,1)}^{(a+c,2,1^{a-2})} \\ &= (-1)^{a-2}\chi_{(a-1,c,1)}^{(a+c)} + \chi^{(c,2,1^{a-2})} \\ &= (-1)^a \\ &= (-1)^{\alpha_1-\alpha_2}, \end{aligned}$$

so that the theorem holds also in this case. □

In the next theorems we will consider the case $k = h+1$, that is $\alpha_1 - \alpha_2 \leqslant \alpha_h$.

Theorem 3.8. *Assume that the following hold:*

- $\alpha \notin \mathrm{Sign}$, $(\alpha_2,\ldots,\alpha_h) \in \mathrm{Sign}$ *and* $\alpha_1 > \alpha_2 > \alpha_3 + \cdots + \alpha_h$,
- $\alpha_1 - \alpha_2 < \alpha_h$.

Then $\beta := (|\alpha| - \alpha_1, 1^{\alpha_1})$ *is a partition with* $h_{2,1}^\beta = \alpha_1$ *and* $\chi_\alpha^\beta = (-1)^{\alpha_1-1}2$.

Proof. Clearly β is a partition and $h_{2,1}^\beta = \alpha_1$. By assumption $|\alpha| - \alpha_1 \geqslant \alpha_2 + \alpha_h > \alpha_1$, from which also follows that $\alpha_1 - \alpha_2 < \alpha_h \leqslant \alpha_j$ for $j \leqslant h$. Also as by assumption $\alpha_2 > \alpha_3 + \cdots + \alpha_h$, so that any partition of $\alpha_2 + \cdots + \alpha_h$ has at most one α_2-hook, it follows from the Murnaghan-Nakayama formula that

$$\begin{aligned} \chi_\alpha^\beta &= (-1)^{\alpha_1-1}\chi_{(\alpha_2,\ldots,\alpha_h)}^{(|\alpha|-\alpha_1)} + \chi_{(\alpha_2,\ldots,\alpha_h)}^{(|\alpha|-2\alpha_1,1^{\alpha_1})} \\ &= (-1)^{\alpha_1-1} + (-1)^{\alpha_2-1}\chi_{(\alpha_3,\ldots,\alpha_h)}^{(|\alpha|-2\alpha_1,1^{\alpha_1-\alpha_2})} \\ &= (-1)^{\alpha_1-1} + (-1)^{\alpha_2-1}\chi_{(\alpha_h)}^{(\alpha_h-\alpha_1+\alpha_2,1^{\alpha_1-\alpha_2})} \\ &= (-1)^{\alpha_1-1}2. \end{aligned}$$

□

Theorem 3.9. *Assume that the following hold:*

- $\alpha \notin \mathrm{Sign}$, $(\alpha_2,\ldots,\alpha_h) \in \mathrm{Sign}$ *and* $\alpha_1 > \alpha_2 > \alpha_3 + \cdots + \alpha_h$,

- $\alpha_1 - \alpha_2 = \alpha_h$,

- $h = 3$.

Then $\beta = (\alpha_1, \alpha_1)$ *is a partitions with* $h^\beta_{2,1} = \alpha_1$ *and* $\chi^\beta_\alpha = 2$.

Proof. Notice that $\alpha_3 \geqslant 2$, since $1 \leqslant \alpha_1 - \alpha_2 = \alpha_3$ and $(\alpha_1, \alpha_2, \alpha_3) \notin \mathrm{Sign}$. Clearly β is a partition with $h^\beta_{2,1} = \alpha_1$.

As $\beta = (\alpha_1, \alpha_1)$ and $\alpha_3 \geqslant 2$ we have that

$$\chi^\beta_\alpha = \chi^{(\alpha_1)}_{(\alpha_2,\alpha_3)} - \chi^{(\alpha_1-1,1)}_{(\alpha_2,\alpha_3)} = 2. \qquad \square$$

Theorem 3.10. *Assume that the following hold:*

- $\alpha \notin \mathrm{Sign}$, $(\alpha_2, \ldots, \alpha_h) \in \mathrm{Sign}$ *and* $\alpha_1 > \alpha_2 > \alpha_3 + \cdots + \alpha_h$,

- $\alpha_1 - \alpha_2 = \alpha_h \geqslant 2$,

- $h \geqslant 4$.

Then $\beta = (|\alpha| - \alpha_1, \alpha_2 + 2, 1^{\alpha_1-\alpha_2-2})$ *is a partition with* $h^\beta_{2,1} = \alpha_1$ *and* $\chi^\beta_\alpha = (-1)^{\alpha_1-\alpha_2}2$.

Proof. As $\alpha_2 + 2 \leqslant \alpha_2 + \alpha_h = \alpha_1$ and $|\alpha| - \alpha_1 \geqslant \alpha_2 + \alpha_h$ we have that β is a partition and that $h^\beta_{2,1} = \alpha_1$. Notice that β'_1, which is the number of parts of β, is given by

$$\beta'_1 = \alpha_1 - \alpha_2 = \alpha_h.$$

As $h \geqslant 4$ and $\alpha_{h-1} > \alpha_h \geqslant 2$ we have that

$$\begin{aligned} h^\beta_{1,2} &= |\alpha| - \alpha_1 \geqslant \alpha_2 + \alpha_h + \alpha_{h-1} \geqslant \alpha_1 + 3, \\ h^\beta_{1,\alpha_2+2} &= |\alpha| - \alpha_1 - \alpha_2 = \alpha_3 + \cdots + \alpha_h \leqslant \alpha_2 - 1 \leqslant \alpha_1 - 2. \end{aligned}$$

In particular there exists $5 \leqslant j \leqslant \alpha_2$ with $h^\beta_{1,j} = \alpha_1$. Such j satisfies $\beta \setminus R^\beta_{1,j} = (\alpha_2 + 1, j - 1, 1^{\alpha_1-\alpha_2-2})$ and then also $h^{\beta\setminus R^\beta_{1,j}}_{1,3} = \alpha_2$ as $j - 1 > 3$ (where $R^\beta_{1,j}$ is the rim hook of β corresponding to node $(1, j)$). As $\alpha_2 > \alpha_3 + \cdots + \alpha_h$, as $\beta'_1 = \alpha_h$ and as $\alpha_i > \alpha_h$ for $i < h$ (since $\alpha_h \geqslant 2$) we then obtain from the Murnaghan-Nakayama formula that

$$\begin{aligned} \chi^\beta_\alpha &= (-1)^{\alpha_1-\alpha_2-2}\chi^{(|\alpha|-\alpha_1)}_{(\alpha_2,\ldots,\alpha_h)} - \chi^{(\alpha_2+1,j-1,1^{\alpha_1-\alpha_2-2})}_{(\alpha_2,\ldots,\alpha_h)} \\ &= (-1)^{\alpha_1-\alpha_2} + \chi^{(j-2,2,1^{\alpha_h-2})}_{(\alpha_3,\ldots,\alpha_h)} \\ &= (-1)^{\alpha_1-\alpha_2} + \chi^{(\alpha_{h-1},2,1^{\alpha_h-2})}_{(\alpha_{h-1},\alpha_h)} \\ &= (-1)^{\alpha_1-\alpha_2} - \chi^{(1^{\alpha_h})}_{(\alpha_{h-1},\alpha_h)} \\ &= (-1)^{\alpha_1-\alpha_2} + (-1)^{\alpha_h} \\ &= (-1)^{\alpha_1-\alpha_2}2. \end{aligned}$$

$\square$

Theorem 3.11. *Assume that the following hold:*

- $\alpha \notin \mathrm{Sign}$, $(\alpha_2, \ldots, \alpha_h) \in \mathrm{Sign}$ *and* $\alpha_1 > \alpha_2 > \alpha_3 + \cdots + \alpha_h$,
- $\alpha_1 - \alpha_2 = \alpha_h = 1 = \alpha_{h-1}$,
- $h \geqslant 4$.

Then $\beta = (|\alpha| - \alpha_1, \alpha_1)$ *is a partition with* $h^{\beta}_{2,1} = \alpha_1$ *and* $\chi^{\beta}_{\alpha} = 2$.

Proof. From Lemma 1.5 it follows from the assumptions that $|\alpha| - \alpha_1 \geqslant \alpha_1$ and so β is a partition. Also $h^{\beta}_{2,1} = \alpha_1$. As

$$3 = \alpha_{h-1} + 2 \leqslant \alpha_3 + \cdots + \alpha_{h-1} + 2 = \alpha_3 + \cdots + \alpha_h + 1 \leqslant \alpha_2 < \alpha_1$$

and

$$|\alpha| - 2\alpha_1 + 2 = \alpha_2 + \cdots + \alpha_h - \alpha_1 + 2 = \alpha_3 + \cdots + \alpha_{h-1} + 2,$$

we have that, for $j = |\alpha| - 2\alpha_1 + 2$,

$$h^{\beta}_{1,j} = |\alpha| - \alpha_1 + 2 - j = \alpha_1.$$

Also $2 \leqslant j - 1 < \alpha_2$ and then, as $\alpha_2 = \alpha_1 - 1$ and $\alpha_{h-2} > \alpha_{h-1} = \alpha_h = 1$,

$$\chi^{\beta}_{\alpha} = \chi^{(|\alpha|-\alpha_1)}_{(\alpha_2 \ldots, \alpha_h)} - \chi^{(\alpha_1-1, j-1)}_{(\alpha_2, \ldots, \alpha_h)} = 1 + \chi^{(j-2,1)}_{(\alpha_3, \ldots, \alpha_h)} = 2. \qquad \square$$

Theorem 3.12. *Assume that the following hold:*

- $\alpha \notin \mathrm{Sign}$, $(\alpha_2, \ldots, \alpha_h) \in \mathrm{Sign}$ *and* $\alpha_1 > \alpha_2 > \alpha_3 + \cdots + \alpha_h$,
- $\alpha_1 - \alpha_2 = \alpha_h = 1 < \alpha_{h-1}$,
- $h = 4$.

Then $\beta = (\alpha_1 - 2, \alpha_3, \alpha_3, 4, 1^{\alpha_1 - \alpha_3 - 2})$ *is a partition with* $h^{\beta}_{2,1} = \alpha_1$ *and* $\chi^{\beta}_{\alpha} = (-1)^{\alpha_1 - \alpha_3} 2$.

Proof. Notice that from the assumptions it follows that $\alpha_3 \geqslant 4$. Also $\alpha_1 > \alpha_2 > \alpha_3$ and so β is a partition with $h^{\beta}_{2,1} = \alpha_1$. As $\alpha_2 = \alpha_1 - 1$ and $\alpha_4 = 1$ we have that

$$\begin{aligned} \chi^{\beta}_{\alpha} &= (-1)^{\alpha_1 - \alpha_3} \chi^{(\alpha_1 - 2, \alpha_3 - 1, 3)}_{(\alpha_1 - 1, \alpha_3, 1)} - \chi^{(\alpha_3 - 1, \alpha_3 - 1, 3, 1^{\alpha_1 - \alpha_3 - 1})}_{(\alpha_1 - 1, \alpha_3, 1)} \\ &= (-1)^{\alpha_1 - \alpha_3} \chi^{(\alpha_3 - 2, 2, 1)}_{(\alpha_3, 1)} + (-1)^{\alpha_1 - \alpha_3 + 1} \chi^{(\alpha_3 - 1, 2)}_{(\alpha_3, 1)} \\ &= (-1)^{\alpha_1 - \alpha_3} 2. \end{aligned}$$

$\square$

Theorem 3.13. *Assume that the following hold:*

- $\alpha \notin \mathrm{Sign}$, $(\alpha_2, \ldots, \alpha_h) \in \mathrm{Sign}$ *and* $\alpha_1 > \alpha_2 > \alpha_3 + \cdots + \alpha_h$,

- $\alpha_1 - \alpha_2 = \alpha_h = 1$,
- $h \geqslant 5$,
- $\alpha_{h-1} = 2$.

Then $\beta = (|\alpha| - \alpha_1 - 2, \alpha_1 - 2, 2, 2)$ *is a partition with* $h^\beta_{2,1} = \alpha_1$ *and* $\chi^\beta_\alpha = -2$.

Proof. As $\alpha_1 > \alpha_2 > \ldots > \alpha_h = 1$ it follows that $\alpha_1 \geqslant h \geqslant 5$. Also, by assumption on α,

$$|\alpha| - \alpha_1 \geqslant \alpha_2 + \alpha_{h-2} + \alpha_h \geqslant \alpha_1 + 3$$

and so it follows that β is a partition. Clearly $h^\beta_{2,1} = \alpha_1$. Since by assumption

$$|\alpha| - 2\alpha_1 + 2 = \alpha_2 + \cdots + \alpha_h - \alpha_1 + 2 = \alpha_3 + \cdots + \alpha_h + 1 \leqslant \alpha_2 < \alpha_1$$

we also have that

$$\begin{aligned} h^\beta_{1,3} &= |\alpha| - \alpha_1 - 2 + 2 - 3 = |\alpha| - \alpha_1 - 3 \geqslant \alpha_1, \\ h^\beta_{1,\alpha_1-2} &= |\alpha| - \alpha_1 - 2 + 2 - \alpha_1 + 2 = |\alpha| - 2\alpha_1 + 2 < \alpha_1. \end{aligned}$$

In particular there exists $3 \leqslant j \leqslant \alpha_1 - 3$ with $h^\beta_{1,j} = \alpha_1$.

From $\alpha_{h-1} = 2$ and $\alpha_h = 1$ it follows that $\alpha_j + \cdots + \alpha_h - 3 \geqslant \alpha_j$ for $j \leqslant h - 2$. Since $\alpha_j \geqslant 3$ for $j \leqslant h - 2$ we then have that

$$\begin{aligned} \chi^\beta_\alpha &= \chi^{(|\alpha|-\alpha_1-2,1,1)}_{(\alpha_2,\ldots,\alpha_h)} - \chi^{(\alpha_1-3,j-1,2,2)}_{(\alpha_2,\ldots,\alpha_h)} \\ &= \chi^{(|\alpha|-\alpha_1-\alpha_2-2,1,1)}_{(\alpha_3,\ldots,\alpha_h)} + \chi^{(j-2,1,1)}_{(\alpha_3,\ldots,\alpha_h)} \\ &= 2\chi^{(1,1,1)}_{(2,1)} \\ &= -2. \end{aligned}$$

□

Theorem 3.14. *Assume that the following hold:*

- $\alpha \notin$ Sign, $(\alpha_2, \ldots, \alpha_h) \in$ Sign *and* $\alpha_1 > \alpha_2 > \alpha_3 + \cdots + \alpha_h$,
- $\alpha_1 - \alpha_2 = \alpha_h = 1$,
- $h \geqslant 5$,
- $\alpha_{h-1} \geqslant 3$.

Then $\beta = (|\alpha| - \alpha_1 - \alpha_{h-1} + 1, 3, 3, 2^{\alpha_{h-1}-3}, 1^{\alpha_1-\alpha_{h-1}-1})$ *is a partition with* $h^\beta_{2,1} = \alpha_1$ *and* $\chi^\beta_\alpha = (-1)^{\alpha_1+\alpha_{h-1}-1}2$.

Proof. As $h \geqslant 5$, so that

$$\beta_1 = |\alpha| - \alpha_1 - \alpha_{h-1} + 1 \geqslant \alpha_2 + \alpha_3 + 1 > \alpha_1 + 3,$$

and as $\alpha_1 > \alpha_{h-1} \geqslant 3$ it follows that β is a partition with $h^\beta_{2,1} = \alpha_1$. Also $\beta_1 \geqslant 4$ and $h^\beta_{1,4} \geqslant \alpha_1$. From the assumptions we also have

$$|\alpha| - 2\alpha_1 - \alpha_{h-1} = \alpha_2 + \cdots + \alpha_h - \alpha_1 - \alpha_{h-1} = \alpha_3 + \cdots + \alpha_{h-2} > \alpha_3 + \cdots + \alpha_{h-3} + 2.$$

Since $\alpha_j > \alpha_{h-1}$ for $j < h-1$ and again any partition of $\alpha_2 + \cdots + \alpha_h$ has at most one α_2-hook, we have that

$$\begin{aligned}
\chi^\beta_\alpha &= (-1)^{\alpha_1-3}\chi^{(|\alpha|-\alpha_1-\alpha_{h-1}+1,2,1^{\alpha_{h-1}-3})}_{(\alpha_2,\ldots,\alpha_h)} + \chi^{(|\alpha|-2\alpha_1-\alpha_{h-1}+1,3,3,2^{\alpha_{h-1}-3},1^{\alpha_1-\alpha_{h-1}-1})}_{(\alpha_2,\ldots,\alpha_h)} \\
&= (-1)^{\alpha_1-1}\chi^{(|\alpha|-2\alpha_1-\alpha_{h-1}+2,2,1^{\alpha_{h-1}-3})}_{(\alpha_3,\ldots,\alpha_h)} + (-1)^{\alpha_1-4}\chi^{(|\alpha|-2\alpha_1-\alpha_{h-1}+1,3,1^{\alpha_{h-1}-3})}_{(\alpha_3,\ldots,\alpha_h)} \\
&= (-1)^{\alpha_1-1}\chi^{(\alpha_{h-2}+2,2,1^{\alpha_{h-1}-3})}_{(\alpha_{h-2},\alpha_{h-1},\alpha_h)} + (-1)^{\alpha_1}\chi^{(\alpha_{h-2}+1,3,1^{\alpha_{h-1}-3})}_{(\alpha_{h-2},\alpha_{h-1},\alpha_h)} \\
&= (-1)^{\alpha_1-1}2\chi^{(2,2,1^{\alpha_{h-1}-3})}_{(\alpha_{h-1},\alpha_h)} \\
&= (-1)^{\alpha_1+\alpha_{h-1}-1}2.
\end{aligned}$$

□

4 The partitions $(\gamma_{s+1}, \ldots, \gamma_r)$ are sign partitions

In this section we will prove that

- $()$, $(1,1)$, $(3,2,1,1)$, $(5,3,2,1)$,
- $(a, a-1, 1)$ with $a \geqslant 2$,
- $(a, a-1, 2, 1)$ with $a \geqslant 4$,
- $(a, a-1, 3, 1)$ with $a \geqslant 5$

are all sign partitions. For $()$, $(1,1)$, $(3,2,1,1)$ and $(5,3,2,1)$ this can be done by just looking at the corresponding character table. For the other partitions we will use the next lemma.

Lemma 4.1. *Let $a \geqslant 2$ and $\gamma = (a, a-1, \gamma_3, \ldots, \gamma_r)$ be a partition. Assume that the following hold.*

- *$(a-1, \gamma_3, \ldots, \gamma_r)$ is a sign partition,*
- *$\gamma_3 + \cdots + \gamma_r \leqslant a$.*

If β is a partition of $|\gamma|$ for which $\chi^\beta_\gamma \notin \{0, \pm 1\}$ then β has two a-hooks. Also if δ is obtained from β by removing an a-hook then $\chi^\delta_{(a-1,\gamma_3,\ldots,\gamma_r)} \neq 0$. In particular each such δ has an $(a-1)$-hook.

Proof. By assumption

$$|\gamma| = 2a - 1 + \gamma_3 + \cdots + \gamma_r < 3a.$$

In particular any partition of $|\gamma|$ has at most two a-hooks. As

$$\chi^\beta_\gamma = \sum_{(i,j):h^\beta_{i,j}=a} \pm\chi^{\beta\backslash R^\beta_{i,j}}_{(a-1,\gamma_3,\ldots,\gamma_r)}$$

and, since $(a-1,\gamma_3,\ldots,\gamma_r)$ is a sign partition, so that $\chi^{\beta\backslash R^\beta_{i,j}}_{(a-1,\gamma_3,\ldots,\gamma_r)} \in \{0,\pm 1\}$ for each $(i,j) \in [\beta]$, the Young diagram of β, with $h^\beta_{i,j} = a$, the lemma follows. □

Theorem 4.2. *If $a \geqslant 2$ then $(a,a-1,1)$ is a sign partition.*

Proof. As $(a-1,1)$ is a sign partition for $a \geqslant 2$, from Lemma 4.1 we only need to check that $\chi^\beta_{(a,a-1,1)} \in \{0,\pm1\}$ for partitions β of $2a$ with two a-hooks and such that if μ and ν are the partitions obtained from β by removing an a-hook then μ and ν both have an an $(a-1)$-hook. From β having two a-hooks it follows that μ and ν also have an a-hook. The only partitions of a having both an a-hook and an $(a-1)$-hook are (a) and (1^a). As $\mu \neq \nu$ it then follows that $\{\mu,\nu\} = \{(a),(1^a)\}$. Looking at the a-quotients and a-cores of β, μ and ν we have that there exists a unique such β, which is given by $\beta = (a,2,1^{a-2})$. We have

$$\chi^{(a,2,1^{a-2})}_{(a,a-1,1)} = (-1)^{a-2}\chi^{(a)}_{(a-1,1)} - \chi^{(1^a)}_{(a-1,1)} = (-1)^a + (-1)^{a-1} = 0$$

and so $(a,a-1,1)$ is a sign partition. □

Theorem 4.3. *If $a \geqslant 4$ then $(a,a-1,2,1)$ is a sign partition.*

Proof. For $a = 4$ we can check that $(a,a-1,2,1) = (4,3,2,1)$ is a sign partition by looking at the character table of S_{10}. So assume that $a \geqslant 5$. As $(a-1,2,1)$ is a sign partition for $a \geqslant 5$ from Lemma 1.5, from Lemma 4.1 we only need to check that $\chi^\beta_{(a,a-1,2,1)} \in \{0,\pm1\}$ for partitions β of $2a+2$ with two a-hooks and such that if μ and ν are the partitions obtained from β by removing an a-hook then μ and ν have both an a-hook and an $(a-1)$-hook.

So let β have two a-hook. Then, as $|\beta| = 2a+2 < 3a$, we have that $\beta_{(a)}$, the a-core of β, is either (2) or (1^2). We will assume that $\beta_{(a)} = (2)$, since for any partitions λ,ρ with $|\lambda| = |\rho|$ and any positive integer q, we have that $\chi^\lambda_\rho = \pm\chi^{\lambda'}_\rho$ and $\lambda'_{(q)} = (\lambda_{(q)})'$, where λ' is the adjoint partition of λ and similarly for $\lambda_{(q)}$. Then μ and ν can be obtained by adding an a-hook to (2) and so

$$\mu,\nu \in \{(a+2),(2,2,1^{a-2}),(2,1^a)\} \cup \{(a-i,3,1^{i-1}) : 1 \leqslant i \leqslant a-3\},$$

as all these partitions can be obtained by adding an a-hook to (2) and, since $2 < a$, there are exactly a such partitions. As μ and ν have an $(a-1)$-hook we then have that

$$\mu,\nu \in \{(a+2),(2,1^a),(a-1,3),(3,3,1^{a-4})\}.$$

Notice that since $a \geqslant 5$ the four above partitions are distinct. As $a \geqslant 5$

$$\begin{aligned} \chi^{(2,1^a)}_{(a-1,2,1)} &= (-1)^{a-2}\chi^{(2,1)}_{(2,1)} = 0, \\ \chi^{(a-1,3)}_{(a-1,2,1)} &= -\chi^{(2,1)}_{(2,1)} = 0, \end{aligned}$$

we only need to consider, from Lemma 4.1, the partition β corresponding to $\{\mu,\nu\} = \{(a+2),(3,3,1^{a-4})\}$, that is for $\beta = (a+2,4,1^{a-4})$. As

$$\chi^{(a+2,4,1^{a-4})}_{(a,a-1,2,1)} = -\chi^{(3,3,1^{a-4})}_{(a-1,2,1)} + (-1)^{a-4}\chi^{(a+2)}_{(a-1,2,1)} = (-1)^{a-3}\chi^{(3)}_{(2,1)} + (-1)^a = 0$$

it follows that $(a,a-1,2,1)$ is a sign partition. □

Theorem 4.4. *If $a \geqslant 5$ then $(a,a-1,3,1)$ is a sign partition.*

Proof. If $a = 5$ then $(a,a-1,3,1) = (5,4,3,1)$ and by looking at the character table of S_{13} we can easily check that this is a sign partition. So assume now that $a \geqslant 6$. As $(a-1,3,1)$ is a sign partition for $a \geqslant 6$ from Lemma 1.5, from Lemma 4.1 we only need to check that $\chi^{\beta}_{(a,a-1,3,1)} \in \{0,\pm 1\}$ for partitions β of $2a+3$ with two a-hooks and such that if μ and ν are the partitions obtained from β by removing an a-hook then μ and ν have both an a-hook and an $(a-1)$-hook.

So let β have two a-hook. Then $\beta_{(a)}$ is (3), $(2,1)$ or (1^3). Similarly to the previous theorem we will assume that $\beta_{(a)}$ is either (3) or $(2,1)$.

Assume first that $\beta_{(a)} = (3)$. Then, as μ and ν can be obtained by adding an a-hook to (3) and as there exists exactly a such partitions since $a > 3$,

$$\mu,\nu \in \{(a+3),(3,3,1^{a-3}),(3,2,1^{a-2}),(3,1^a)\} \cup \{(a-i,4,1^{i-1}) : 1 \leqslant i \leqslant a-4\}.$$

As μ and ν also have an $(a-1)$-hook it then follows that

$$\mu,\nu \in \{(a+3),(3,1^a),(a-1,4),(4,4,1^{a-5})\}.$$

As $a \geqslant 6$

$$\begin{aligned} \chi^{(3,1^a)}_{(a-1,3,1)} &= (-1)^{a-2}\chi^{(3,1)}_{(3,1)} = 0, \\ \chi^{(a-1,4)}_{(a-1,3,1)} &= -\chi^{(3,1)} = 0 \end{aligned}$$

and so, from Lemma 4.1, we can assume that $\{\gamma,\delta\} = \{(a+3),(4,4,1^{a-5})\}$, that is that $\beta = (a+3,5,1^{a-5})$ and then

$$\chi^{\beta}_{(a,a-1,3,1)} = -\chi^{(4,4,1^{a-5})}_{(a-1,3,1)} + (-1)^{a-5}\chi^{(a+3)}_{(a-1,3,1)} = (-1)^{a-4}\chi^{(4)}_{(3,1)} + (-1)^{a-5} = 0.$$

Assume now that $\beta_{(a)} = (2,1)$. Also in this case, as $a > 3$, there exist exactly a partitions which can be obtained by adding an a-hook to $(2,1)$ and μ and ν are two of them. So

$$\mu,\nu \in \{(a+2,1),(a,3),(2,2,2,1^{a-3}),(2,1^{a+1})\} \cup \{(a-i,3,2,1^{i-2}) : 2 \leqslant i \leqslant a-3\}.$$

As μ and ν have an $(a-1)$-hook it follows that

$$\mu, \nu \in \{(a+2,1), (a,3), (2,2,2,1^{a-3}), (2,1^{a+1}), (a-2,3,2), (3,3,2,1^{a-5})\}.$$

Since $a \geqslant 6$

$$\begin{aligned} \chi^{(a+2,1)}_{(a-1,3,1)} &= \chi^{(3,1)}_{(3,1)} = 0, \\ \chi^{(2,1^{a+1})}_{(a-1,3,1)} &= (-1)^{a-2}\chi^{(2,1,1)}_{(3,1)} = 0, \\ \chi^{(a-2,3,2)}_{(a-1,3,1)} &= \chi^{(2,1,1)}_{(3,1)} = 0, \\ \chi^{(3,3,2,1^{a-5})}_{(a-1,3,1)} &= (-1)^{a-4}\chi^{(3,1)}_{(3,1)} = 0 \end{aligned}$$

we again only need to consider one partition β. In this case $\{\mu, \nu\} = \{(a,3), (2,2,2,1^{a-3})\}$ and then $\beta = (a,3,3,1^{a-3})$. As

$$\chi^{(a,3,3,1^{a-3})}_{(a,a-1,3,1)} = \chi^{(2,2,2,1^{a-3})}_{(a-1,3,1)} + (-1)^{a-3}\chi^{(a,3)}_{(a-1,3,1)} = (-1)^{a-3}\chi^{(2,2)}_{(3,1)} + (-1)^{a-2}\chi^{(2,2)}_{(3,1)} = 0,$$

it follows that $(a, a-1, 3, 1)$ is a sign partition also for $a \geqslant 6$. □

5 Proof of Theorem 1.3

For $r \leqslant 2$ Theorem 1.3 follows from Lemmas 1.4 and 1.5. So assume now that $r \geqslant 3$.

From Lemma 1.5 and Section 4 it easily follows that if $\gamma \in \mathrm{Sign}$ then γ is a sign partition.

Assume now that $\gamma = (\gamma_1, \ldots, \gamma_r)$ is a sign partition. From Lemma 1.4 it follows that $(\gamma_{r-1}, \gamma_r) \in \mathrm{Sign}$. Also from Lemma 1.5, $\gamma_{i-1} > \gamma_i$ for $2 \leqslant i \leqslant r-1$. Fix $2 \leqslant i \leqslant r-1$ and assume that $(\gamma_i, \ldots, \gamma_r) \in \mathrm{Sign}$.

Assume that $(\gamma_{i-1}, \ldots, \gamma_r) \neq (5,4,3,2,1)$ and that $(\gamma_{i-1}, \ldots, \gamma_r) \notin \mathrm{Sign}$. From Theorem 1.6 we can find β such that $\chi^{\beta}_{(\gamma_{i-1},\ldots,\gamma_r)} \notin \{0, \pm 1\}$ and $h^{\beta}_{2,1} = \gamma_{i-1}$. Let

$$\delta := (\beta_1 + \gamma_1 + \cdots + \gamma_{i-2}, \beta_2, \beta_3, \ldots).$$

Then δ is a partition of $|\gamma|$. If $i-1 = 1$ then

$$\chi^{\delta}_{\gamma} = \chi^{\beta}_{(\gamma_{i-1},\ldots,\gamma_r)} \notin \{0, \pm 1\},$$

in contradiction to γ being a sign partition. If $i-1 \geqslant 2$ then $(1, \beta_1+1) \in [\delta]$ and

$$h^{\delta}_{1,\beta_1+1} = \gamma_1 + \cdots + \gamma_{i-2}.$$

Since $\beta_2 < \beta_1 + 1$ and $h^{\delta}_{2,1} = h^{\beta}_{2,1} = \gamma_{i-1} < \gamma_j$ for $j \leqslant i-2$, we have that also in this case

$$\chi^{\delta}_{\gamma} = \chi^{\beta}_{(\gamma_{i-1},\ldots,\gamma_r)} \notin \{0, \pm 1\},$$

which again gives a contradiction.

Assume now that $(\gamma_{i-1}, \dots, \gamma_r) = (5,4,3,2,1)$. If $i-1 = 1$ or $i-1 \geqslant 2$ and $\gamma_{i-2} \geqslant 7$, then similarly to the previous case

$$\chi_\gamma^{(4+\gamma_1+\cdots+\gamma_{i-2},4,4,3)} = \chi_{(5,4,3,2,1)}^{(4,4,4,3)} = -2.$$

If $i-1 \geqslant 2$ and $\gamma_{i-1} = 6$ we have similarly that

$$\chi_\gamma^{(15+\gamma_1+\cdots+\gamma_{i-3},2,1,1,1,1)} = \chi_{(6,5,4,3,2,1)}^{(15,2,1,1,1,1)} = 2.$$

In either case we have a contradiction with γ being a sign partition.

So $(\gamma_{i-1}, \dots, \gamma_r) \in$ Sign. By induction $\gamma \in$ Sign and so Theorem 1.3 is proved.

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