Forbidden triples generating a finite set of 3-connected graphs

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Abstract

For a graph G and a set $\mathcal{F}$ of connected graphs, G is said be $\mathcal{F}$-free if G does not contain any member of $\mathcal{F}$ as an induced subgraph. We let $\mathcal{G}_3(\mathcal{F})$ denote the set of all 3-connected $\mathcal{F}$-free graphs. This paper is concerned with sets $\mathcal{F}$ of connected graphs such that $|\mathcal{F}| = 3$ and $\mathcal{G}_3(\mathcal{F})$ is finite. Among other results, we show that for an integer $m \geqslant 3$ and a connected graph T of order greater than or equal to 4, $\mathcal{G}_3(\{K_4, K_{2,m}, T\})$ is finite if and only if T is a path of order 4 or 5.

Keywords: forbidden subgraph; forbidden triple; 3-connected graph

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1 Introduction

In this paper, we consider only finite undirected simple graphs.

Let $\mathcal{G}$ denote the set of connected graphs with order greater than or equal to three. For a graph G and for $F \in \mathcal{G}$, G is said to be F*-free* if G does not contain F as an induced subgraph and, for $\mathcal{F} \subseteq \mathcal{G}$, G is said to be $\mathcal{F}$*-free* if G is F-free for every $F \in \mathcal{F}$. For an integer $k \geqslant 1$ and $\mathcal{F} \subseteq \mathcal{G}$, we let $\mathcal{G}_k$ denote the set of all k-connected graphs, and let $\mathcal{G}_k(\mathcal{F})$ denote the set of all $\mathcal{F}$-free graphs belonging to $\mathcal{G}_k$. Thus

$$\mathcal{G}_k(\mathcal{F}) := \{G \mid G \text{ is a } k\text{-connected } \mathcal{F}\text{-free graph}\}.$$

This paper is concerned with subsets $\mathcal{F}$ of $\mathcal{G}$ such that $\mathcal{G}_k(\mathcal{F})$ is a finite set. In this context, members of $\mathcal{F}$ are often referred to as forbidden subgraphs. For detailed historical background and related results, we refer the reader to [3].

The following result can be found in [1]. (Here K_n denotes the complete graph of order n, P_l denotes the path of order l and, in general, K_{m_1,m_2} denotes the complete bipartite graph with partite sets having cardinalities m_1 and m_2.)

Theorem A (Diestel [1]; Chapter 9)**.** *For $\mathcal{F} \subseteq \mathcal{G}$, $\mathcal{G}_1(\mathcal{F})$ is finite if and only if K_n, $K_{1,m}$, $P_l \in \mathcal{F}$ for some integers $n \geqslant 3$, $m \geqslant 2$ and $l \geqslant 3$.*

For $k \geqslant 2$, it is unlikely that a general result like Theorem A holds. Thus we confine ourselves to the case where $|\mathcal{F}|$ is "small". It is known that for any $k \geqslant 2$, there is no $\mathcal{F} \subseteq \mathcal{G}$ with $|\mathcal{F}| = 1$ such that $\mathcal{G}_k(\mathcal{F})$ is finite. (See [3]; Theorem 2.) Further, those subsets $\mathcal{F}$ of $\mathcal{G}$ with $|\mathcal{F}| = 2$ for which $\mathcal{G}_k(\mathcal{F})$ is finite are determined for $k \leqslant 6$ in [3]. Here we are interested in the case where $|\mathcal{F}| = 3$. Note that a connected $K_{1,2}$-free graph is a complete graph. Hence if $K_{1,2} \in \mathcal{F}$, then $\mathcal{G}_k(\mathcal{F})$ is finite if and only if $K_n \in \mathcal{F}$ for some $n \geqslant 3$, and there is no point in forbidding two more graphs. Thus when we discuss $\mathcal{G}_k(\mathcal{F})$ with $|\mathcal{F}| = 3$, we usually assume $K_{1,2} \notin \mathcal{F}$. For $k = 2$, the following theorem is proved in [3].

Theorem B (Fujisawa, Plummer and Saito [3])**.** *Let $\mathcal{F}$ be a subset of $\mathcal{G}$ with $|\mathcal{F}| = 3$ and $K_{1,2} \notin \mathcal{F}$. Then $\mathcal{G}_2(\mathcal{F})$ is finite if and only if one of the following holds:*

(i) $\mathcal{F} = \{K_3, K_{2,m}, P_l\}$ *for some integers m and l with $m \geqslant 3$ and $4 \leqslant l \leqslant 5$;*

(ii) $\mathcal{F} = \{K_3, K_{2,2}, P_6\}$*; or*

(iii) $\mathcal{F} = \{K_n, K_{1,m}, P_l\}$ *for some integers n, m and l with $n \geqslant 3$, $m \geqslant 3$ and $l \geqslant 4$.*

In the present paper, we investigate the case where $|\mathcal{F}| = 3$ and $k = 3$. It is easy to see $\mathcal{G}_3(\{K_3, K_{1,3}\}) = \varnothing$. (See [3].) Thus when we consider $\mathcal{G}_3(\mathcal{F})$, we assume that $\{K_3, K_{1,3}\} \not\subseteq \mathcal{F}$, in addition to the condition that $K_{1,2} \notin \mathcal{F}$. Before stating our results, we make some more definitions.

Let n be an integer with $n \geqslant 2$. Let $P = x_1x_2 \cdots x_n$ be the path of order n, and let y_1, y_2, z_1 and z_2 be four distinct vertices different from $x_1, \ldots, x_n$. We let Y_n, Y_n^*, Q_n and Q_n^* denote the graphs defined by

$$V(Y_n) = V(Q_n) = V(P) \cup \{y_1, y_2\}, \quad V(Y_n^*) = V(Q_n^*) = V(P) \cup \{y_1, y_2, z_1, z_2\},$$

$$E(Y_n) = E(P) \cup \{x_1y_1, x_1y_2\}, \quad E(Y_n^*) = E(P) \cup \{x_1y_1, x_1y_2, x_nz_1, x_nz_2\},$$
$$E(Q_n) = E(Y_n) \cup \{y_1y_2\} \quad \text{and} \quad E(Q_n^*) = E(Y_n^*) \cup \{y_1y_2, z_1z_2\}$$

(see Figure 1).

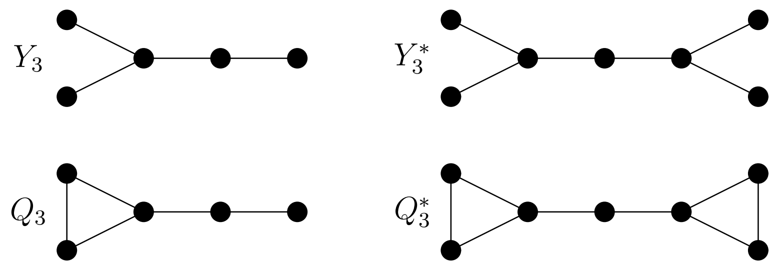

Figure 1: Graphs Y_3, Y_3^*, Q_3 and Q_3^*

A *caterpillar* is a tree for which the removal of all endvertices leaves a path. A complete bipartite graph of the form $K_{1,m}$ with $m \geqslant 1$ is called a *star*. Let $\mathcal{T}_0$ be the set of trees in $\mathcal{G} - \{K_{1,2}, K_{1,3}\}$ having maximum degree at most 3. Note that $\mathcal{T}_0$ does not contain a star. Let $\mathcal{T}_1$ be the set of those caterpillars belonging to $\mathcal{T}_0$ in which no two vertices of degree 3 are adjacent. Let $\mathcal{T}_2 = \{P_l, Y_m, Y_n^* \mid l \geqslant 4, m \geqslant 3, n \geqslant 3\}$. We have $\mathcal{T}_0 \supseteq \mathcal{T}_1 \supseteq \mathcal{T}_2$.

Let G be a connected graph. A vertex v of G is called a *cutvertex* if $G - v$ is disconnected. If G has a cutvertex, G is said to be *separable*; otherwise, it is said to be *nonseparable*. Note that K_1 is a nonseparable graph. A maximal nonseparable subgraph of G is called a *block* of G. When G is separable, the *block-cutvertex graph* of G is defined to be the bipartite graph Z such that Z has as its partite sets the set of all cutvertices of G and the set of all blocks of G and, for a cutvertex v and a block B, v and B are adjacent in Z if and only if v is a vertex of B in G. It is a well-known fact that the block-cutvertex graph of a connected graph is a tree. A *cactus* is a connected graph every block of which is a complete graph of order two or a cycle. Let $\mathcal{T}_0^*$ be the set of those cacti T in $\mathcal{G} - \{K_{1,2}, K_3\}$ such that all cycles of T are triangles and in the block-cutvertex graph of T, the distance between any two vertices corresponding to triangles of T is a multiple of 4. Let $\mathcal{T}_1^*$ be the set of those members of $\mathcal{T}_0^*$ whose block-cutvertex graph is a path. Let $\mathcal{T}_2^* = \{P_l, Q_m, Q_{2n}^* \mid l \geqslant 4, m \geqslant 2, n \geqslant 1\}$. We have $\mathcal{T}_0^* \supseteq \mathcal{T}_1^* \supseteq \mathcal{T}_2^*$.

Our main result is as follows.

Theorem 1.1. *Let $\mathcal{F}$ be a subset of $\mathcal{G}$ with $|\mathcal{F}| = 3$, $K_{1,2} \notin \mathcal{F}$ and $\{K_{1,3}, K_3\} \not\subseteq \mathcal{F}$, and suppose that $\mathcal{G}_3(\mathcal{F})$ is finite. Then one of the following holds:*

(i) $\mathcal{F} = \{K_3, K_{3,m}, T\}$ *with* $m \geqslant 3$*, where* $T \in \mathcal{T}_2$*;*

(ii) $\mathcal{F} = \{K_4, K_{2,m}, T\}$ *with* $m \geqslant 2$*, where* T *is a path;*

(iii) $\mathcal{F} = \{K_3, K_{2,m}, T\}$ *with* $m \geqslant 2$*, where* $T \in \mathcal{T}_0$ *in the case where* $m = 2$*,* $T \in \mathcal{T}_1$ *in the case where* $3 \leqslant m \leqslant 4$*, and* $T \in \mathcal{T}_2$ *in the case where* $m \geqslant 5$*;*

(iv) $\mathcal{F} = \{K_n, K_{1,m}, T\}$ *with* $n \geqslant 4$ *and* $m \geqslant 4$*, where* T *is a path;*

(v) $\mathcal{F} = \{K_3, K_{1,m}, T\}$ *with* $m \geqslant 4$*, where* $T \in \mathcal{T}_1$ *in the case where* $m = 4$*, and* $T \in \mathcal{T}_2$ *in the case where* $m \geqslant 5$*; or*

(vi) $\mathcal{F} = \{K_n, K_{1,3}, T\}$ *with* $n \geqslant 4$*, where* $T \in \mathcal{T}_1^*$ *in the case where* $n = 4$*, and* $T \in \mathcal{T}_2^*$ *in the case where* $n \geqslant 5$*.*

The converse of Theorem 1.1 does not hold. However, if (iv) of Theorem 1.1 holds, then $\mathcal{G}_3(\mathcal{F})$ is finite by Theorem A. Also, as we shall state below in Theorem 1.5, if (v) holds with $m \geqslant 5$, then $\mathcal{G}_3(\mathcal{F})$ is finite. Further when m is "large" in cases (i) through (iii) of Theorem 1.1, we can determine T as follows.

Theorem 1.2. *Let* m *be an integer with* $m \geqslant 4$*, and let* $T \in \mathcal{G} - \{K_{1,2}, K_{1,3}\}$*. Then* $\mathcal{G}_3(\{K_3, K_{3,m}, T\})$ *is finite if and only if* T *is a path of order* 4 *or* 5 *or* $T = Y_3$*.*

Theorem 1.3. *Let* m *be an integer with* $m \geqslant 3$*, and let* $T \in \mathcal{G} - \{K_{1,2}\}$*. Then* $\mathcal{G}_3(\{K_4, K_{2,m}, T\})$ *is finite if and only if* T *is a path of order* 4 *or* 5*.*

Theorem 1.4. *Let* m *be an integer with* $m \geqslant 5$*, and let* $T \in \mathcal{G} - \{K_{1,2}, K_{1,3}\}$*. Then* $\mathcal{G}_3(\{K_3, K_{2,m}, T\})$ *is finite if and only if* T *is either a path of order at most* 7 *or an induced subgraph of* Y_3^**.*

Theorem 1.5. *Let* m *be an integer with* $m \geqslant 5$*, and let* $T \in \mathcal{G} - \{K_{1,2}, K_{1,3}\}$*. Then* $\mathcal{G}_3(\{K_3, K_{1,m}, T\})$ *is finite if and only if* $T \in \mathcal{T}_2$*.*

We prove Theorem 1.1 in Section 2. We prove Theorem 1.2 in Section 3, Theorem 1.3 in Section 4, Theorem 1.5 in Section 5, and Theorem 1.4 in Section 6. Our notation and terminology are standard, and mostly taken from [1]. Exceptions are as follows. Let G be a graph. For $u, v \in V(G)$, $d(u, v)$ denotes the distance between u and v. When G is connected, we define the *diameter* $\mathrm{diam}(G)$ of G by $\mathrm{diam}(G) = \max\{d(u, v) \mid u, v \in V(G)\}$. Let $u \in V(G)$. For an integer $i \geqslant 1$, we let $N_i(u) = \{x \in V(G) \mid d(u, x) = i\}$. We write $N(u)$ for $N_1(u)$. We let $d(u)$ denote the degree of u; thus $d(u) = |N(u)|$. When we need to specify that the underlying graph is G, we write $N_G(u)$ and $d_G(u)$ for $N(u)$ and $d(u)$, respectively. We let $\Delta(G) = \max\{d(u) \mid u \in V(G)\}$. For $Y \subseteq V(G)$, we let $N(Y)$ denote the union of $N(u)$ as u ranges over Y. For $X, Y \subseteq V(G)$ with $X \cap Y = \varnothing$, $E(X, Y)$ denotes the set of edges joining a vertex in X and a vertex in Y. When G is connected, a block of G containing at most one cutvertex of G is called an *endblock* of G. When G is not necessarily connected, by a cutvertex of G, we mean a cutvertex of a component of G. Similarly, by a block (resp. an endblock) of G, we mean a block (resp. an endblock) of a component of G. Note that isolated vertices of G are endblocks of G. For an endblock B of G, a vertex of B which is not a cutvertex of G is called an *internal vertex* of B. For a graph H and an integer $s \geqslant 2$, we let sH denote the disjoint union of s copies of H. For two graphs H_1 and H_2, we let $H_1 + H_2$ denote the join of H_1 and H_2. Finally for $s \geqslant 4$, C_s denotes the cycle of order s and, for $t \geqslant 5$, we let $W_t = C_{t-1} + K_1$ denote the *wheel graph* of order t.

In subsequent arguments, when we prove the finiteness of $\mathcal{G}_3(\mathcal{F})$ for a given family $\mathcal{F}$, we bound the diameter and the maximum degree of a graph G in $\mathcal{G}_3(\mathcal{F})$ from above, and then bound the order in terms of the diameter and the maximum degree. For this purpose, we make one easy observation.

Lemma 1.6. *Let $m \geqslant 2$ and $k \geqslant 3$, and let G be a graph with $\Delta(G) \leqslant m$ and $diam(G) \leqslant k$. Then $|V(G)| \leqslant m^k$.*

Proof. Let $w \in V(G)$. Then $|N_i(w)| \leqslant m(m-1)^{i-1}$ for each $1 \leqslant i \leqslant k$. Hence $|V(G)| \leqslant 1 + m + (\sum_{2 \leqslant i \leqslant k-1} m(m-1)^{i-1}) + m(m-1)^{k-1} \leqslant m^{k-1} + (\sum_{2 \leqslant i \leqslant k-1} m^{k-i}(m-1)^{i-1}) + m(m-1)^{k-1} = m^k$. □

The bound m^k in the above lemma is far from sharp, but we use it for the sake of brevity.

2 A necessary condition

In this section, we prove Theorem 1.1. We start with several lemmas. The first two lemmas are proved in [4] and [3], respectively.

Lemma 2.1 (Kochol [4])**.** *For every integer $g \geqslant 3$, there exists a 3-connected 3-regular graph with girth g. In particular, for every integer $g \geqslant 3$, there exist infinitely many 3-connected 3-regular graphs with girth at least g.*

Lemma 2.2 (Fujisawa, Plummer and Saito [3])**.** *For $\mathcal{F} \subseteq \mathcal{G}$, if $\mathcal{G}_3(\mathcal{F})$ is finite, then $\{K_n, K_{m_1,m_2}\} \subseteq \mathcal{F}$ for some integers n, m_1 and m_2 with $n \geqslant 3$, $m_2 \geqslant m_1 \geqslant 1$ and $m_1 \leqslant 3$.*

Lemma 2.3. *Let $\mathcal{F}$ be a finite subset of $\mathcal{G} - \{K_{1,2}, K_{1,3}\}$, and suppose that $\mathcal{G}_3(\mathcal{F})$ is finite. Then $\mathcal{F}$ contains a member of $\mathcal{T}_0$.*

Proof. Let $t = \max\{|V(F)| \mid F \in \mathcal{F}\}$, and let $\mathcal{H} = \{G \in \mathcal{G}_3 \mid G$ is a 3-regular graph with girth at least $t+1\}$. By Lemma 2.1, $\mathcal{H}$ is an infinite set. Since $\mathcal{G}_3(\mathcal{F})$ is finite and $\mathcal{H}$ is infinite, there exists $G \in \mathcal{H}$ such that G contains a graph F in $\mathcal{F}$ as an induced subgraph. Since the girth of G is strictly greater than $|V(F)|$, F is a tree. Since G is 3-regular and since $F \neq K_{1,2}, K_{1,3}$ by the assumption that $\mathcal{F} \subseteq \mathcal{G} - \{K_{1,2}, K_{1,3}\}$, it follows that $F \in \mathcal{T}_0$. □

Lemma 2.4. *Let m_1 and m_2 be integers such that $m_2 \geqslant m_1 \geqslant 1$, $m_1 \leqslant 3$ and $(m_1, m_2) \neq (1,2), (1,3), (2,2)$, and let $T \in \mathcal{G} - \{K_{1,2}, K_{1,3}\}$. Set $\mathcal{F} = \{K_3, K_{m_1,m_2}, T\}$, and suppose that $\mathcal{G}_3(\mathcal{F})$ is finite. Then the following hold.*

(i) *We have $T \in \mathcal{T}_1$.*

(ii) *If in addition, $(m_1, m_2) \notin \{(1,4), (2,3), (2,4)\}$, then $T \in \mathcal{T}_2$.*

Proof. By Lemma 2.3, $T \in \mathcal{T}_0$.

(i) For each $s \geqslant 5$, the Cartesian product $C_s \times K_2$ is 3-connected and $\{K_3, K_{m_1,m_2}\}$-free. Since $\mathcal{G}_3(\mathcal{F})$ is finite, this implies that there exists $s \geqslant 5$ such that $C_s \times K_2$ contains T as an induced subgraph. Since every member of $\mathcal{T}_0$ contained in $C_s \times K_2$ as an induced subgraph belongs to $\mathcal{T}_1$, $T \in \mathcal{T}_1$.

(ii) By (i), $T \in \mathcal{T}_1$. For each $s \geqslant 5$, let C'_s denote the so-called lexicographic product of C_s and the null graph of order two; that is to say, $V(C'_s) = \{x_{i,j} \mid 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant 2\}$ and $E(C'_s) = \{x_{i,j}x_{i+1,h} \mid 1 \leqslant i \leqslant s, 1 \leqslant j, h \leqslant 2\}$, where first indices of the letter x are to be read modulo s. Then C'_s is 3-connected and $\{K_3, K_{m_1,m_2}\}$-free. Hence there exists $s \geqslant 5$ such that C'_s contains T as an induced subgraph. Since every member of $\mathcal{T}_1$ contained in C'_s as an induced subgraph belongs to $\mathcal{T}_2$, $T \in \mathcal{T}_2$. □

Proof of Theorem 1.1. Let $\mathcal{F}$ be as in Theorem 1.1. By Lemma 2.2, $\{K_n, K_{m_1,m_2}\} \subseteq \mathcal{F}$ for some integers n, m_1 and m_2 with $n \geqslant 3$, $m_2 \geqslant m_1 \geqslant 1$ and $m_1 \leqslant 3$. Write $\mathcal{F} = \{K_n, K_{m_1,m_2}, T\}$.

Case 1: $\mathcal{F}$ contains no star

In this case, $2 \leqslant m_1 \leqslant 3$, and we have $T \in \mathcal{T}_0$ by Lemma 2.3.

Subcase 1.1: $m_1 = 3$

For each $s \geqslant 3$, $P_3 + sK_1$ is 3-connected and K_{3,m_2}-free. Since T is not a star and every tree contained in $P_3 + sK_1$ as an induced subgraph is a star, $P_3 + sK_1$ is also T-free. Since $\mathcal{G}_3(\mathcal{F})$ is finite, this implies that there exists $s \geqslant 3$ such that $P_3 + sK_1$ contains K_n as an induced subgraph. Since $P_3 + sK_1$ is K_4-free, this forces $n = 3$. Now by Lemma 2.4(ii), $T \in \mathcal{T}_2$, and hence (i) of Theorem 1.1 holds.

Subcase 1.2: $m_1 = 2$ and $n \geqslant 4$

For each $s \geqslant 3$, $K_3 + sK_1$ is 3-connected and K_{2,m_2}-free. Since every tree contained in $K_3 + sK_1$ as an induced subgraph is a star, $K_3 + sK_1$ is also T-free. Hence there exists $s \geqslant 3$ such that $K_3 + sK_1$ contains K_n as an induced subgraph, which implies $n = 4$. For each $t \geqslant 6$, W_t is 3-connected and $\{K_4, K_{2,m_2}\}$-free. Hence there exists $t \geqslant 6$ such that W_t contains T as an induced subgraph. Since $T \in \mathcal{T}_0$ and every member of $\mathcal{T}_0$ contained in W_t as an induced subgraph is a path, T is a path. Consequently (ii) of Theorem 1.1 holds.

Subcase 1.3: $m_1 = 2$ and $n = 3$

Recall that $T \in \mathcal{T}_0$. Also if $3 \leqslant m_2 \leqslant 4$, then $T \in \mathcal{T}_1$ by Lemma 2.4(i); if $m_2 \geqslant 5$, then $T \in \mathcal{T}_2$ by Lemma 2.4(ii). Thus (iii) of Theorem 1.1 holds.

Case 2: $\mathcal{F}$ contains a star

Interchanging the roles of K_{m_1,m_2} and T with each other if necessary, we may assume that K_{m_1,m_2} is the star of the smallest order contained in $\mathcal{F}$. Then $m_1 = 1$, and $m_2 \geqslant 3$ by the assumption that $K_{1,2} \notin \mathcal{F}$, and $T \neq K_{1,2}, K_{1,3}$ by the minimality of m_2.

Subcase 2.1: $m_2 \geqslant 4$

By Lemma 2.3, $T \in \mathcal{T}_0$.

Subcase 2.1.1: $n \geqslant 4$

For each $s \geqslant 7$, let C_s^2 denote the square of C_s; that is to say $V(C_s^2) = \{x_i \mid 1 \leqslant i \leqslant s\}$ and $E(C_s^2) = \{x_i x_{i+1}, x_i x_{i+2} \mid 1 \leqslant i \leqslant s\}$, where indices of the letter x are to be read modulo s. Then C_s^2 is 3-connected and $\{K_n, K_{1,m_2}\}$-free. Hence there exists $s \geqslant 7$ such that C_s^2 contains T as an induced subgraph. Since every tree contained in C_s^2 as an induced subgraph is a path, T is a path. Hence (iv) of Theorem 1.1 holds.

Subcase 2.1.2: $n = 3$

If $m_2 = 4$, then $T \in \mathcal{T}_1$ by Lemma 2.4(i); if $m_2 \geqslant 5$, then $T \in \mathcal{T}_2$ by Lemma 2.4(ii). Thus (v) of Theorem 1.1 holds.

Subcase 2.2: $m_2 = 3$

By the assumption that $\{K_{1,3}, K_3\} \not\subseteq \mathcal{F}$, we have $n \geqslant 4$ and $T \neq K_3$. Thus $T \in \mathcal{G} - \{K_{1,2}, K_{1,3}, K_3\}$.

For a 3-connected 3-regular graph G, let H_G be the graph obtained by expanding each vertex of G to a triangle (see Figure 2); more precisely, we define H_G by $V(H_G) = \{x_{u,v} \mid (u,v) \in V(G) \times V(G), uv \in E(G)\}$ and $E(H_G) = \{x_{u,v}x_{u,w} \mid u \in V(G), v, w \in N(u), v \neq w\} \cup \{x_{u,v}x_{v,u} \mid uv \in E(G)\}$. Then H_G is 3-connected 3-regular and $\{K_n, K_{1,3}\}$-free and, for $u \in V(G)$, $B_u = x_{u,v_1}x_{u,v_2}x_{u,v_3}x_{u,v_1}$ ($\{v_1, v_2, v_3\} = N(u)$) is a triangle of H_G. Also each cycle of H_G which is not of the form B_u with $u \in V(G)$ has length at least twice as large as the girth of G and, for any $u, u' \in V(G)$ with $u \neq u'$, every induced path in H_G joining B_u and $B_{u'}$ has even order. Hence every induced connected subgraph of H_G having order greater than or equal to four and strictly less than twice the girth of G belongs to $\mathcal{T}_0^*$. On the other hand, by Lemma 2.1, the set $\{H_G \mid G$ is a 3-connected 3-regular graph with girth at least $(|V(T)|+1)/2\}$ is an infinite set. Hence there exists a 3-connected 3-regular graph G with girth at least $(|V(T)|+1)/2$ such that H_G contains T as an induced subgraph. Note that $|V(T)|$ is less than twice the girth of G. Consequently $T \in \mathcal{T}_0^*$.

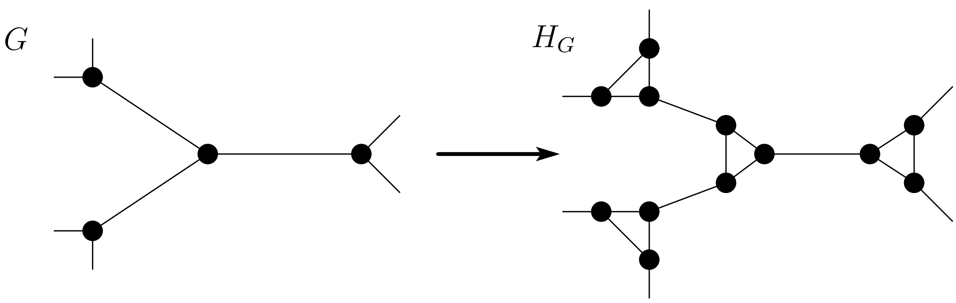

Figure 2: Graph H_G

Further for each $s \geqslant 7$, C_s^2 is 3-connected and $\{K_n, K_{1,3}\}$-free. Hence there exists $s \geqslant 7$ such that C_s^2 contains T as an induced subgraph. Since every member of $\mathcal{T}_0^*$ contained in C_s^2 as an induced subgraph belongs to $\mathcal{T}_1^*$, $T \in \mathcal{T}_1^*$.

Now assume $n \geqslant 5$. For each $s \geqslant 4$, let C_s^* denote the lexicographic product of C_s and K_2; that is to say, $V(C_s^*) = \{x_{i,j} \mid 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant 2\}$ and $E(C_s^*) = \{x_{i,j}x_{i+1,h}, x_{i,1}x_{i,2} \mid 1 \leqslant i \leqslant s, 1 \leqslant j, h \leqslant 2\}$, where first indices of the letter x are to be read modulo s. Then

C_s^* is 3-connected and $\{K_n, K_{1,3}\}$-free. Hence there exists $s \geqslant 4$ such that C_s^* contains T as an induced subgraph. Since every member of $\mathcal{T}_1^*$ contained in C_s^* as an induced subgraph belongs to $\mathcal{T}_2^*$, $T \in \mathcal{T}_2^*$.

Thus we have $T \in \mathcal{T}_1^*$ if $n = 4$, and $T \in \mathcal{T}_2^*$ if $n \geqslant 5$. Hence (vi) of Theorem 1.1 holds.

This completes the proof of Theorem 1.1. □

3 $K_{3,m}$-free graphs

In this section, we prove Theorem 1.2. We first show that both $\mathcal{G}_3(\{K_3, K_{3,m}, Y_3\})$ and $\mathcal{G}_3(\{K_3, K_{3,m}, P_5\})$ are finite.

Proposition 3.1. *Let $m \geqslant 4$. Then $\mathcal{G}_3(\{K_3, K_{3,m}, Y_3\})$ is finite.*

Proof. Let $G \in \mathcal{G}_3(\{K_3, K_{3,m}, Y_3\})$. We show that $|V(G)| \leqslant (m+1)^3$.

Claim 3.1. *$diam(G) \leqslant 3$.*

Proof. Suppose that $\mathrm{diam}(G) \geqslant 4$. Let $x, y \in V(G)$ be vertices with $d(x, y) = 4$, and let $x_0x_1 \cdots x_4$ be a shortest x-y path in G. Since G is 3-connected, $N(x_2) - \{x_1, x_3\} \neq \varnothing$. Let $z \in N(x_2) - \{x_1, x_3\}$. Since G is K_3-free, $zx_1, zx_3 \notin E(G)$. Since $d(x_0, x_4) = 4$, z is adjacent to at most one of x_0 and x_4. Hence $\{x_0, x_1, x_2, x_3, z\}$ or $\{x_4, x_3, x_2, x_1, z\}$ induces Y_3, which is a contradiction. □

In view of Claim 3.1 and Lemma 1.6, it suffices to show that $\Delta(G) \leqslant m+1$. Suppose that $\Delta(G) \geqslant m+2$. Let $w \in V(G)$ be a vertex such that $d(w) = \Delta(G)$, and let $x \in N(w)$. Since G is K_3-free, both $N(w)$ and $N(x)$ are independent. Since G is 3-connected, $d(x) \geqslant 3$. Take $y_1, y_2 \in N(x) - \{w\}$. If $|N(w) - N(y_i)| \geqslant 2$ for $i = 1$ or 2, say $a, b \in N(w) - N(y_i)$, then $\{y_i, x, w, a, b\}$ induces Y_3, a contradiction. Thus $|N(y_i) \cap N(w)| \geqslant d(w) - 1$ for each $i = 1, 2$. Consequently $|N(w) \cap N(y_1) \cap N(y_2)| \geqslant d(w) - 2 \geqslant m$, which implies that $G[\{w, y_1, y_2\} \cup (N(w) \cap N(y_1) \cap N(y_2))]$ contains $K_{3,m}$ as an induced subgraph, a contradiction. □

Proposition 3.2. *Let $m \geqslant 4$. Then $\mathcal{G}_3(\{K_3, K_{3,m}, P_5\})$ is finite.*

Proof. Let $G \in \mathcal{G}_3(\{K_3, K_{3,m}, P_5\})$. We show that $|V(G)| \leqslant (4m-1)^3$. Since G is P_5-free, $\mathrm{diam}(G) \leqslant 3$. Thus in view of Lemma 1.6, it suffices to show that $\Delta(G) \leqslant 4m - 1$. Suppose that $\Delta(G) \geqslant 4m$, and let $w \in V(G)$ be a vertex with $d(w) = \Delta(G)$. Since G is K_3-free, $N(w)$ is an independent set.

Claim 3.2. *Let $X \subseteq N(w)$, and let Y be a minimal subset of $N_2(w)$ such that $N(Y) \supseteq X$. Then $|Y| \leqslant 2$.*

Proof. Suppose that $|Y| \geqslant 3$. Since G is K_3-free, there exist vertices $y_1, y_2 \in Y$ such that $y_1y_2 \notin E(G)$. By the minimality of Y, $(N(y_i) \cap X) - N(y_{3-i}) \neq \varnothing$ for each $i = 1, 2$. For $i = 1, 2$, let $x_i \in (N(y_i) \cap X) - N(y_{3-i})$. Then $\{y_1, x_1, w, x_2, y_2\}$ induces P_5, a contradiction. □

Since G is 3-connected and $N(w)$ is independent, $N(N_2(w)) \supseteq N(w)$. Let Y be a minimal subset of $N_2(w)$ such that $N(Y) \supseteq N(w)$. Then $|Y| \leqslant 2$ by Claim 3.2.

Hence there exists $y \in Y$ such that $|N(w) \cap N(y)| \geqslant d(w)/2 \geqslant 2m$. Since G is 3-connected and $N(w)$ is independent, $N(N_2(w) - \{y\}) \supseteq N(w) \cap N(y)$. Let Y^* be a minimal subset of $N_2(w) - \{y\}$ such that $N(Y^*) \supseteq N(w) \cap N(y)$ (see Figure 3). Then $|Y^*| \leqslant 2$ by Claim 3.2. Hence there exists $y' \in Y^*$ such that $|N(w) \cap N(y) \cap N(y')| \geqslant m$. Since $N(y) \cap N(y') \neq \varnothing$, we have $yy' \notin E(G)$ by the assumption that G is K_3-free. Consequently $G[\{w, y, y'\} \cup (N(w) \cap N(y) \cap N(y'))]$ contains $K_{3,m}$ as an induced subgraph, a contradiction. ☐

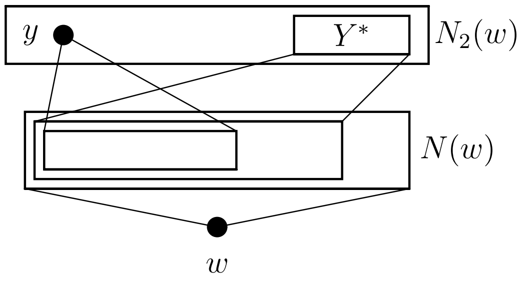

Figure 3: Vertex y and set Y^*

Proof of Theorem 1.2. The 'if' part follows from Propositions 3.1 and 3.2. Thus it suffices to prove the 'only if' part. Suppose that $\mathcal{G}_3(\{K_3, K_{3,m}, T\})$ is finite. From Theorem 1.1, it follows that $T \in \mathcal{T}_2$. For each $s \geqslant 2$, let H_s be the graph defined by $V(H_s) = \{x_{i,j} \mid 1 \leqslant i, j \leqslant 2\} \cup \{y_{i,j} \mid 1 \leqslant i \leqslant 2, 1 \leqslant j \leqslant s\}$ and $E(H_s) = \{x_{1,j}x_{2,h} \mid 1 \leqslant j, h \leqslant 2\} \cup \{x_{i,j}y_{i,h} \mid 1 \leqslant i, j \leqslant 2, 1 \leqslant h \leqslant s\} \cup \{y_{1,j}y_{2,j} \mid 1 \leqslant j \leqslant s\}$. Then H_s is 3-connected and $\{K_3, K_{3,m}\}$-free since $m \geqslant 4$. Since $\mathcal{G}_3(\{K_3, K_{3,m}, T\})$ is finite, there exists $s \geqslant 2$ such that H_s contains T as an induced subgraph. By inspection, we see that if F is a member of $\mathcal{T}_2$ contained in H_s as an induced subgraph, then F is a path of order 4 or 5 or $F = Y_3$. Hence T is a path of order 4 or 5 or $T = Y_3$. ☐

4 $\{K_4, K_{2,m}\}$-free graphs

In this section, we prove Theorem 1.3.

Proposition 4.1. *Let* $m \geqslant 3$. *Then* $\mathcal{G}_3(\{K_4, K_{2,m}, P_5\})$ *is finite.*

Proof. By part (i) of Theorem B, there exists a positive integer $t = t(m)$ such that every 2-connected $\{K_3, K_{2,m}, P_5\}$-free graph has order at most t. Let $G \in \mathcal{G}_3(\{K_4, K_{2,m}, P_5\})$. We show that $|V(G)| \leqslant (3(3m-1)t/2)^3$. Note that $\mathrm{diam}(G) \leqslant 3$. Thus in view of Lemma 1.6, it suffices to show that $\Delta(G) \leqslant 3(3m-1)t/2$. Suppose that $\Delta(G) > 3(3m-1)t/2$, and let $w \in V(G)$ be a vertex with $d(w) = \Delta(G)$. Since G is $\{K_4, K_{2,m}, P_5\}$-free, $G[N(w)]$ is $\{K_3, K_{2,m}, P_5\}$-free.

Let F be a component of $G[N(w)]$. If F has two or more blocks which are not endblocks, then F contains P_5 as an induced subgraph, a contradiction. Thus F has at most one block which is not an endblock, which implies that at least two thirds of the blocks of F are endblocks. Since F is arbitrary, it follows that at least two thirds of the blocks of $G[N(w)]$ are endblocks. Now suppose that $G[N(w)]$ has at most $3m-1$

endblocks. Then $G[N(w)]$ has at most $3(3m-1)/2$ blocks. Since $d(w) > 3(3m-1)t/2$, it follows that there exists a block B of $G[N(w)]$ with $|V(B)| > t$. Since $G[N(w)]$ is $\{K_3, K_{2,m}, P_5\}$-free, this contradicts the definition of t. Thus $G[N(w)]$ has at least $3m$ endblocks. Let $B_1, \dots, B_{3m}$ be endblocks of $G[N(w)]$. Since G is 3-connected, we see that for each $1 \leqslant i \leqslant 3m$, there exists an internal vertex x_i of B_i such that $N(x_i) \cap (V(G) - (\{w\} \cup N(w))) \neq \varnothing$ (see Figure 4). Set $X = \{x_1, \dots, x_{3m}\}$. Then X is an independent set of G, and we have $N(N_2(w)) \supseteq X$. Let Y be a minimal subset of $N_2(w)$ such that $N(Y) \supseteq X$. If $|Y| \leqslant 3$, then there exists $y \in Y$ such that $|N(y) \cap X| \geqslant m$, and hence $G[\{w, y\} \cup (N(y) \cap X)]$ contains $K_{2,m}$ as an induced subgraph, a contradiction. Thus $|Y| \geqslant 4$. Since G is K_4-free, there exist vertices $y_1, y_2 \in Y$ such that $y_1y_2 \notin E(G)$. By the minimality of Y, $(N(y_i) \cap X) - N(y_{3-i}) \neq \varnothing$ for each $i = 1, 2$. For $i = 1, 2$, let $x'_i \in (N(y_i) \cap X) - N(y_{3-i})$. Then $\{y_1, x'_1, w, x'_2, y_2\}$ induces P_5, a contradiction. ☐

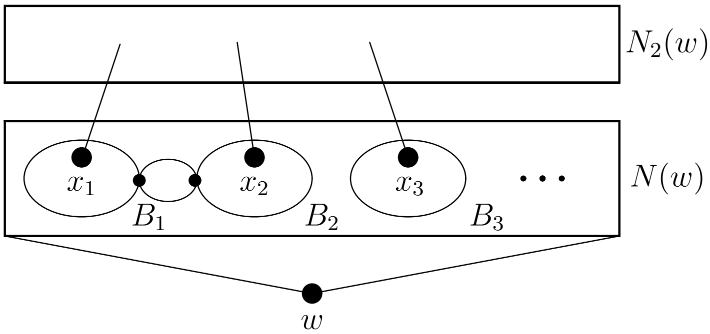

Figure 4: Endblocks B_i and vertices x_i

Proof of Theorem 1.3. The 'if' part follows from Proposition 4.1. Thus it suffices to prove the 'only if' part. Suppose that $\mathcal{G}_3(\{K_4, K_{2,m}, T\})$ is finite. From Theorem 1.1, it follows that T is a path. For each $s \geqslant 2$, let H_s be the graph defined by $V(H_s) = \{x_i, y_{i,j} \mid 1 \leqslant i \leqslant 3, 1 \leqslant j \leqslant s\}$ and $E(H_s) = \{x_ix_j, y_{i,h}y_{j,h} \mid i \neq j, 1 \leqslant h \leqslant s\} \cup \{x_iy_{j,h} \mid i \neq j, 1 \leqslant h \leqslant s\}$ (see Figure 5). Then H_s is 3-connected and $\{K_4, K_{2,m}\}$-free. Hence there exists $s \geqslant 2$ such that H_s contains T as an induced subgraph. Since every induced path of H_s has order at most 5, T has order at most 5. Since $T \neq K_{1,2}$ by assumption, it follows that T is a path of order 4 or 5. ☐

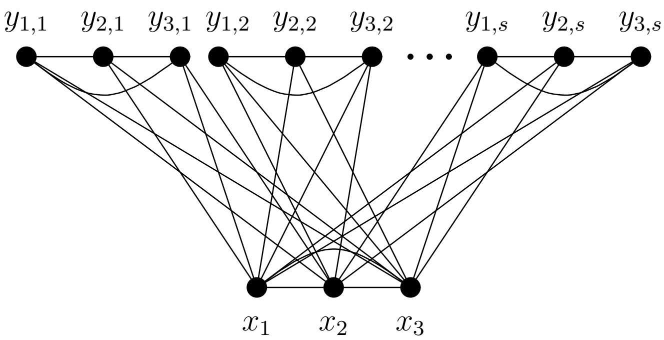

Figure 5: Graph H_s

5 $K_{1,m}$-free graphs

In this section, we prove Theorem 1.5. We first prove a lemma, which we also use in Section 6.

Lemma 5.1. *Let $n \geqslant 3$, and let G be a 3-connected $\{K_3, Y_n^*\}$-free graph. If $n = 3$, let $t(n) = 7$; if $n \geqslant 4$, let $t(n) = 3n + 10$. Then $diam(G) \leqslant t(n)$. In particular, $diam(G) \leqslant 3n + 10$.*

Proof. Let $t = t(n)$, and suppose that $\mathrm{diam}(G) \geqslant t + 1$. Let $x, y \in V(G)$ be vertices such that $d(x, y) = t + 1$, and let $P = x_0x_1 \cdots x_{t+1}$ be a shortest x-y path in G. Since G is 3-connected, $N(x_i) - \{x_{i-1}, x_{i+1}\} \neq \varnothing$ for each $1 \leqslant i \leqslant t$. Let $x'_i \in N(x_i) - \{x_{i-1}, x_{i+1}\}$.

Case 1: $n = 3$

We first show that $N(x_3) \cap N(x_5) = \{x_4\}$. By way of contradiction, suppose that $|N(x_3) \cap N(x_5)| \geqslant 2$. Then we may assume we have chosen x'_3 and x'_5 so that $x'_3 = x'_5 \in (N(x_3) \cap N(x_5)) - \{x_4\}$. Since $d(x_0, x_8) = 8$ and G is K_3-free, we get $N(x'_3) \cap V(P) = \{x_3, x_5\}$, $N(x'_1) \cap V(P) \subseteq \{x_1, x_3\}$, $N(x'_7) \cap V(P) \subseteq \{x_5, x_7\}$, and $x'_1x'_3, x'_3x'_7 \notin E(G)$. Since neither $\{x_0, x'_1, x_1, x_2, x_3, x_4, x'_3\}$ nor $\{x_4, x'_3, x_5, x_6, x_7, x_8, x'_7\}$ induces Y_3^*, this forces $N(x'_1) \cap V(P) = \{x_1, x_3\}$ and $N(x'_7) \cap V(P) = \{x_5, x_7\}$. But then $\{x_2, x'_1, x_3, x_4, x_5, x_6, x'_7\}$ induces Y_3^*, a contradiction. Thus $N(x_3) \cap N(x_5) = \{x_4\}$.

Choose $z \in N(x_3) - \{x_2, x_4\}$. Since $\{x_2, z, x_3, x_4, x_5, x_6, x'_5\}$ does not induce Y_3^* and $N(x_3) \cap N(x_5) = \{x_4\}$, we get $zx'_5 \in E(G)$. Since $d(x_1, x_5) = 4$, it follows that $x_1z \notin E(G)$. Since $z \in N(x_3) - \{x_2, x_4\}$ is arbitrary, this implies $N(x_1) \cap N(x_3) = \{x_2\}$; in particular, $x'_1x_3 \notin E(G)$. Letting $z = x'_3$, we also get $x'_3x'_5 \in E(G)$ and $x_1x'_3 \notin E(G)$. Since $\{x_0, x'_1, x_1, x_2, x_3, x_4, x'_3\}$ does not induce Y_3^*, we now obtain $x'_1x'_3 \in E(G)$. Similarly, $x'_7x'_5 \in E(G)$. But then $x_0x_1x'_1x'_3x'_5x'_7x_7x_8$ is an x_0-x_8 path of length 7, which contradicts the fact that $d(x_0, x_8) = 8$.

Case 2: $n \geqslant 4$

Claim 5.1. *For $0 \leqslant i \leqslant t - n - 1$, if $|N(x_i) \cap N(x_{i+2})| \geqslant 2$, then $x'_{i+n+1}x_{i+n-1} \in E(G)$. For $n \leqslant i \leqslant t - 1$, if $|N(x_i) \cap N(x_{i+2})| \geqslant 2$, then $x'_{i-n+1}x_{i-n+3} \in E(G)$.*

Proof. Let $0 \leqslant i \leqslant t - n - 1$, and assume that $|N(x_i) \cap N(x_{i+2})| \geqslant 2$. Let $u \in (N(x_i) \cap N(x_{i+2})) - \{x_{i+1}\}$. Since $d(x_0, x_{t+1}) = t + 1$ and G is K_3-free, we have $N(u) \cap V(P) = \{x_i, x_{i+2}\}$, $N(x'_{i+n+1}) \cap \{x_0, x_1, \ldots, x_{i+n+2}\} \subseteq \{x_{i+n-1}, x_{i+n+1}\}$ and $ux'_{i+n+1} \notin E(G)$. Since $\{x_{i+1}, u, x_{i+2}, \ldots, x_{i+n+1}, x_{i+n+2}, x'_{i+n+1}\}$ does not induce Y_n^*, this implies $x'_{i+n+1}x_{i+n-1} \in E(G)$. Thus the first assertion is proved. We can similarly verify the second assertion. ☐

Claim 5.2. *There is no integer i with $n \leqslant i \leqslant t - n - 2$ such that $|N(x_i) \cap N(x_{i+2})| \geqslant 2$ and $|N(x_{i+1}) \cap N(x_{i+3})| \geqslant 2$.*

Proof. Suppose that there exists an integer i with $n \leqslant i \leqslant t - n - 2$ such that $|N(x_i) \cap N(x_{i+2})| \geqslant 2$ and $|N(x_{i+1}) \cap N(x_{i+3})| \geqslant 2$. Then by Claim 5.1, $x'_{i-n+1}x_{i-n+3}, x'_{i+n+2}x_{i+n} \in E(G)$. This implies $|N(x_{i-n+1}) \cap N(x_{i-n+3})| \geqslant 2$ and $|N(x_{i+n}) \cap N(x_{i+n+2})| \geqslant 2$. Let $\mathcal{Q}$ be the set of x_i-x_{i+3} paths of order 4, and let $X = (\bigcup_{Q \in \mathcal{Q}} V(Q)) - \{x_i, x_{i+3}\}$. Since

$G - \{x_i, x_{i+3}\}$ is connected and $V(G) - (X \cup \{x_i, x_{i+3}\}) \neq \varnothing$, there exists $z \in X$ such that $N(z) - (X \cup \{x_i, x_{i+3}\}) \neq \varnothing$. By the definition of X, z is adjacent to x_i or x_{i+3}. By symmetry, we may assume $zx_i \in E(G)$. Then there exists $z' \in X$ such that $x_i zz' x_{i+3}$ is an x_i-x_{i+3} path. Now take $w \in N(z) - (X \cup \{x_i, x_{i+3}\})$. Since $|N(x_{i+n}) \cap N(x_{i+n+2})| \geqslant 2$, we obtain $wx_{i+3} \in E(G)$ by applying the second assertion of Claim 5.1 to the path $x_0 x_1 \cdots x_i zz' x_{i+3} x_{i+4} \cdots x_{t+1}$. But then $x_i zwx_{i+3} \in \mathcal{Q}$, which contradicts the fact that $w \notin X$. □

Claim 5.3. *For $n + 2 \leqslant i \leqslant t - 2n$, we have $x'_{i-1}x'_{i+n-2} \in E(G)$ or $x'_i x'_{i+n-1} \in E(G)$ or $x'_{i+1}x'_{i+n} \in E(G)$.*

Proof. Suppose that $x'_i x'_{i+n-1} \notin E(G)$. Since $\{x_{i-1}, x'_i, x_i, \ldots, x_{i+n-1}, x_{i+n}, x'_{i+n-1}\}$ does not induce Y^*_n, $x'_i x_{i+2} \in E(G)$ or $x'_{i+n-1}x_{i+n-3} \in E(G)$. First assume that $x'_i x_{i+2} \in E(G)$. Since $0 \leqslant i \leqslant t-n-1$, we have $x'_{i+n+1}x_{i+n-1} \in E(G)$ by Claim 5.1. Since $n \leqslant i \leqslant t-n-2$ and $n \leqslant i + n - 2 \leqslant t - n - 2$, $x'_{i+1}x_{i+3}, x'_{i+n}x_{i+n-2} \notin E(G)$ by Claim 5.2. Since $\{x_i, x'_{i+1}, x_{i+1}, \ldots, x_{i+n}, x_{i+n+1}, x'_{i+n}\}$ does not induce Y^*_n and G is K_3-free, it follows that $x'_{i+1}x'_{i+n} \in E(G)$, as desired. Next assume that $x'_{i+n-1}x_{i+n-3} \in E(G)$. Arguing as above, we get $x'_{i-2}x_i \in E(G)$ by Claim 5.1, and $x'_{i-1}x_{i+1}, x'_{i+n-2}x_{i+n-4} \notin E(G)$ by Claim 5.2. Since $\{x_{i-2}, x'_{i-1}, x_{i-1}, \ldots, x_{i+n-2}, x_{i+n-1}, x'_{i+n-2}\}$ does not induce Y^*_n and G is K_3-free, it follows that $x'_{i-1}x'_{i+n-2} \in E(G)$, as desired. □

Let $j = n + 2$. Since $t \geqslant 3n + 10$, we have $n + 2 \leqslant j \leqslant t - 2n$. Hence it follows from Claim 5.3 that $x'_{j-1}x'_{j+n-2} \in E(G)$ or $x'_j x'_{j+n-1} \in E(G)$ or $x'_{j+1}x'_{j+n} \in E(G)$. Since $d(x_{j-1}, x_{j+n-2}) = d(x_j, x_{j+n-1}) = d(x_{j+1}, x_{j+n}) = n - 1$, this forces $n = 4$, and hence $t = 22$ and $j = 6$. In particular, $x'_5 x'_8 \in E(G)$ or $x'_6 x'_9 \in E(G)$ or $x'_7 x'_{10} \in E(G)$. Let $s \in \{5, 6, 7\}$ be an integer such that $x'_s x'_{s+3} \in E(G)$.

Claim 5.4. *For $4 \leqslant i \leqslant 11$, if $x'_i x'_{i+3} \in E(G)$, then $x'_{i+2}x'_{i+5} \in E(G)$.*

Proof. If $x'_{i+3}x'_{i+6} \in E(G)$, then $x_i x'_i x'_{i+3} x'_{i+6} x_{i+6}$ is a path, which contradicts the fact that $d(x_i, x_{i+6}) = 6$. Thus $x'_{i+3}x'_{i+6} \notin E(G)$. Suppose that $x'_{i+2}x'_{i+5} \notin E(G)$. Then $x'_{i+2}x'_{i+5}, x'_{i+3}x'_{i+6} \notin E(G)$. Since $n + 2 = 6 \leqslant i + 2 < i + 3 \leqslant 14 = t - 2n$, this together with Claim 5.3 implies $x'_{i+1}x'_{i+4} \in E(G)$ and $x'_{i+4}x'_{i+7} \in E(G)$. It now follows that $x_{i+1}x'_{i+1}x'_{i+4}x'_{i+7}x_{i+7}$ is a path, which contradicts the fact that $d(x_{i+1}, x_{i+7}) = 6$. □

Recall that $x'_s x'_{s+3} \in E(G)$. Since $4 < s < s + 2 < s + 4 \leqslant 11$, by repetitively applying Claim 5.4, we obtain $x'_{s+2}x'_{s+5}, x'_{s+4}x'_{s+7}, x'_{s+6}x'_{s+9} \in E(G)$. Since $x'_s x'_{s+3}, x'_{s+6}x'_{s+9} \in E(G)$ and $d(x_s, x_{s+9}) = 9$, we get $N(x'_{s+3}) \cap \{x_{s+5}, x_{s+6}, x_{s+7}, x'_{s+6}\} = \varnothing$ and $N(x'_{s+6}) \cap \{x_{s+2}, x'_{s+3}, x_{s+3}, x_{s+4}\} = \varnothing$. Since G is K_3-free, it follows that $\{x_{s+2}, x'_{s+3}, x_{s+3}, x_{s+4}, x_{s+5}, x_{s+6}, x_{s+7}, x'_{s+6}\}$ induces Y^*_4, which is a contradiction.

This completes the proof of Lemma 5.1. □

Proof of Theorem 1.5. Let $n \geqslant 3$, and let $G \in \mathcal{G}_3(\{K_3, K_{1,m}, Y^*_n\})$. Then by Lemma 5.1, $\mathrm{diam}(G) \leqslant 3n + 10$. Since G is $\{K_3, K_{1,m}\}$-free, we also have $\Delta(G) \leqslant m - 1$. Hence $|V(G)| \leqslant (m - 1)^{3n+10}$ by Lemma 1.6. Thus $\mathcal{G}_3(\{K_3, K_{1,m}, Y^*_n\})$ is finite. Since $n \geqslant 3$ is arbitrary, this proves the 'if' part. The 'only if' part follows from Theorem 1.1. □

6 $\{K_3, K_{2,m}\}$-free graphs

In this section, we prove Theorem 1.4. We first prove several lemmas.

Lemma 6.1. *Let $m \geqslant 2$, and let G be a $\{K_3, K_{2,m}\}$-free graph. Let H be a connected induced subgraph of G with order $n \geqslant 2$ and let $x \in V(H)$, and suppose that $d_G(x) \geqslant (m-1)(n-2)+t+1$. Then G contains as an induced subgraph the graph obtained from H by adding t pendant edges to x.*

Proof. Since G is K_3-free, $N_G(x) - V(H)$ is independent and no vertex in $N_G(x) - V(H)$ is adjacent to a vertex in $N_G(x) \cap V(H)$. Since G is $K_{2,m}$-free, we see that for each $x' \in V(H) - (\{x\} \cup N_G(x))$, x' is adjacent to at most $m-1$ vertices in $N_G(x) - V(H)$. Set $Y = N_G(x) \cap ((\bigcup_{x' \in V(H)-\{x\}} N_G(x')) \cup V(H))$. It follows that $|Y| \leqslant (m-1)(n-1-d_H(x)) + d_H(x) \leqslant (m-1)(n-2)+1$. Hence $|N_G(x) - Y| \geqslant t$. Now if we let Z be a subset of $N_G(x) - Y$ with $|Z| = t$, then $V(H) \cup Z$ induces the desired graph. □

Lemma 6.2. *Let $m \geqslant 2$, and let G be a $\{K_3, K_{2,m}\}$-free graph. Let P be an induced path of G with order $n \geqslant 2$, and suppose that both endvertices of P have degree at least $(m-1)(n-1)+2$. Then G contains P_{n+2} as an induced subgraph.*

Proof. Applying Lemma 6.1 to one of the endvertices of P with $H = P$, we get an induced path P' of order $n+1$ and, applying Lemma 6.1 to the other endvertices of P with $H = P'$, we obtain a path of the type desired. □

Similarly, we obtain the following lemma.

Lemma 6.3. *Let $m \geqslant 2$, and let G be a $\{K_3, K_{2,m}\}$-free graph, and let P be an induced path of G with order $n \geqslant 2$, and suppose that both endvertices of P have degree at least $(m-1)n+3$. Then G contains Y_n^* as an induced subgraph.*

Proposition 6.4. *Let $m \geqslant 5$. Then $\mathcal{G}_3(\{K_3, K_{2,m}, Y_3^*\})$ is finite.*

Proof. Let $G \in \mathcal{G}_3(\{K_3, K_{2,m}, Y_3^*\})$. We show that $|V(G)| \leqslant ((6m-3)(3m-2)(m-2)+1)^7$. By Lemma 5.1, $\mathrm{diam}(G) \leqslant 7$. Thus it suffices to show that $\Delta(G) \leqslant (6m-3)(3m-2)(m-2)+1$. Suppose that $\Delta(G) \geqslant (6m-3)(3m-2)(m-2)+2$, and let $w \in V(G)$ be a vertex with $d(w) = \Delta(G)$. Since G is K_3-free, $N(w)$ is independent. In particular, for any two vertices x, x' in $N(w)$, $\{x, w, x'\}$ induces P_3. Choose $a \in N(w)$ so that $d(a) \geqslant d(x)$ for all $x \in N(w)$. If two vertices in $N(w)$ have degree at least $3m$, then by Lemma 6.3, G contains Y_3^* as an induced subgraph, a contradiction. Thus all vertex in $N(w) - \{a\}$ have degree at most $3m-1$. Let $B \subseteq N(w) - \{a\}$ be a maximal set such that $N(b) \cap N(b') = \{w\}$ for any $b, b' \in B$ with $b \neq b'$.

Suppose that $|B| \leqslant 6m-3$. Since $B \subseteq N(w) - \{a\}$, $|(\bigcup_{b \in B} N(b)) - \{w\}| \leqslant (6m-3)(3m-2)$. Since G is $K_{2,m}$-free and $(\bigcup_{b \in B} N(b)) - \{w\} \subseteq N_2(w)$, $|N(y) \cap N(w)| \leqslant m-1$ for all $y \in (\bigcup_{b \in B} N(b)) - \{w\}$. Hence $|\{x \in N(w) \mid N(x) \cap ((\bigcup_{b \in B} N(b)) - \{w\}) \neq \varnothing\}| \leqslant (6m-3)(3m-2)(m-1)$. Since $|N(w) - \{a\}| \geqslant (6m-3)(3m-2)(m-1)+1$, there exists $b^* \in N(w) - \{a\}$ such that $(N(b^*) \cap (\bigcup_{b \in B} N(b))) - \{w\} = \varnothing$. Then $B \cup \{b^*\}$ satisfies the condition that $N(b) \cap N(b') = \{w\}$ for any $b, b' \in B \cup \{b^*\}$ with $b \neq b'$, which contradicts the maximality of B. Thus $|B| \geqslant 6m-2$.

Since G is 3-connected, $|N(b) - \{w\}| \geqslant 2$ for every $b \in B$. For each $b \in B$, let $v_b, u_b \in N(b)-\{w\}$. Fix a vertex $b_0 \in B$. For each $b \in B-\{b_0\}$, we have $E(\{v_{b_0}, u_{b_0}\}, \{v_b, u_b\}) \neq \varnothing$ because $\{v_{b_0}, u_{b_0}, b_0, w, b, v_b, u_b\}$ does not induce Y_3^*. Since $|B-\{b_0\}| \geqslant 6m-3$, this implies that v_{b_0} or u_{b_0} is adjacent to at least $3m-1$ vertices in $\{v_b, u_b \mid b \in B - \{b_0\}\}$. We may assume $|N(v_{b_0}) \cap \{v_b, u_b \mid b \in B - \{b_0\}\}| \geqslant 3m-1$. Then $d(v_{b_0}) \geqslant 3m$. Since $d(w) \geqslant 3m$ and $wb_0v_{b_0}$ is an induced path of G of order 3, G contains Y_3^* as an induced subgraph by Lemma 6.3, a contradiction.

This completes the proof of Proposition 6.4. □

Before proving the finiteness of $\mathcal{G}_3(\{K_3, K_{2,m}, P_7\})$, we state an important part of the proof as a separate lemma.

Lemma 6.5. *Let $m \geqslant 5$, and let $G \in \mathcal{G}_3(\{K_3, K_{2,m}, P_7\})$. Then there exist no vertices $a, b \in V(G)$ such that $d(a,b) = 2$, $d(a) \geqslant 2(m-1)^7$ and $d(b) \geqslant 2(m-1)^7$.*

Proof. Suppose that there exist two vertices $a, b \in V(G)$ such that $d(a,b) = 2$, $d(a) \geqslant 2(m-1)^7$ and $d(b) \geqslant 2(m-1)^7$. Since $d(a,b) = 2$, $ab \notin E(G)$. Let $w \in N(a) \cap N(b)$, and let $A = N(a) - N(b)$ and $B = N(b) - N(a)$. Since G is K_3-free, $N(w)$, $N(a)$ and $N(b)$ are independent sets, and hence $N(a) - \{w\}, N(b) - \{w\} \subseteq N_2(w)$. Since G is $K_{2,m}$-free, $|N(a) \cap N(b)| \leqslant m-1$, and hence $|A| \geqslant 2(m-1)^7 - m + 1$ and $|B| \geqslant 2(m-1)^7 - m + 1$. Since G is 3-connected, $G - w$ is 2-connected. Let k be the maximum order of an induced a-b path in $G - w$.

We consider three cases according to the value of k.

Case 1: $k = 3$

Note that it follows from the assumption of this case that $N(a) \cup N(b)$ is independent. Since $G - w$ is 2-connected, it also follows that $|(N(a) \cap N(b)) - \{w\}| \geqslant 2$. Let $c_1, c_2 \in (N(a) \cap N(b)) - \{w\}$ with $c_1 \neq c_2$.

Note that we have $N(x) \cap (N(a) \cup N(b)) = \varnothing$ for each $x \in A \cup B$ because $N(a) \cup N(b)$ is independent. Let $A_0 = \{x \in A \mid N(x) \subseteq N(w)\}$ and $A_1 = A - A_0$. For each $x \in A$, we define $u_x \in N(x)$ as follows: if $x \in A_0$, let $u_x \in N(x) - \{a\}$; if $x \in A_1$, let $u_x \in N(x) - N(w)$. For $A' \subseteq A$, let $U_{A'} = \{u_x \mid x \in A'\}$. Let $B_0 = \{y \in B \mid N(y) \subseteq N(w)\}$ and $B_1 = B - B_0$. For each $y \in B$, we define $v_y \in N(y)$ as follows: if $y \in B_0$, let $v_y \in N(y) - \{b\}$; if $y \in B_1$, let $v_y \in N(y) - N(w)$. For $B' \subseteq B$, let $V_{B'} = \{v_y \mid y \in B'\}$.

Claim 6.1. *For $A' \subseteq A$, $|U_{A'}| \geqslant |A'|/(m-1)$. For $B' \subseteq B$, $|V_{B'}| \geqslant |B'|/(m-1)$.*

Proof. Let $A' \subseteq A$. Suppose that $|U_{A'}| < |A'|/(m-1)$. Then there exists $u \in U_{A'}$ such that $u = u_x$ for some m vertices x in A'. Hence $G[\{a, u\} \cup (N(a) \cap N(u))]$ contains $K_{2,m}$ as an induced subgraph, a contradiction. Thus $|U_{A'}| \geqslant |A'|/(m-1)$. Similarly, we get $|V_{B'}| \geqslant |B'|/(m-1)$ for $B' \subseteq B$. □

We consider two subcases.

Subcase 1.1: $A_1 = \varnothing$ or $B_1 = \varnothing$

We may assume that $A_1 = \varnothing$. By Claim 6.1 and the fact that $|A_0| = |A| \geqslant 2(m-1)^7 - m + 1 > (m-1)^2$, $|U_{A_0}| \geqslant m$. Since $|B| \geqslant 2(m-1)^7 - m + 1 > (m-1)^2$, $|V_B| \geqslant m$ by Claim 6.1. Since G is K_3-free, it follows that if both c_1 and c_2 are adjacent to all vertices in V_B, then $G[\{c_1, c_2\} \cup V_B]$ contains $K_{2,m}$ as an induced subgraph, a contradiction. Thus $c_h v_y \notin E(G)$ for some $h \in \{1, 2\}$ and some $y \in B$. Then since $v_y \notin N(a) \cup N(b)$, $ac_h byv_y$ is an induced path. Note that $U_{A_0} \subseteq N(w)$. Since G is $K_{2,m}$-free, $|N(c_h) \cap U_{A_0}| \leqslant |N(c_h) \cap N(w)| \leqslant m-1$. Since $|U_{A_0}| \geqslant m$, there exists $x \in A_0$ such that $u_x c_h \notin E(G)$. Then $u_x xac_h b$ is an induced path. Since $N(a) \cup N(b)$ is independent, $xy \notin E(G)$. Since $k = 3$, we also have $u_x y, xv_y, u_x v_y \notin E(G)$. Consequently the path $u_x xac_h byv_y$ is an induced path of order 7, a contradiction.

Subcase 1.2: $A_1 \neq \varnothing$ and $B_1 \neq \varnothing$

Let $x \in A_1$ and $y \in B_1$. Then since $u_x, v_y \notin N(a) \cup N(b) \cup N(w)$ and $k = 3$, $u_x xawbyv_y$ is an induced path of order 7, a contradiction.

Case 2: $k = 4$

Let $H_1, \ldots, H_p$ be the components of $G[A \cup B]$. Let $1 \leqslant i \leqslant p$. Suppose that there exists $x \in V(H_i)$ such that $d_{H_i}(x) \geqslant m$. We may assume that $x \in A$. Then $N_{H_i}(x) \subseteq B$, which implies that $G[\{b\} \cup V(H_i)]$ contains $K_{2,m}$ as an induced subgraph, a contradiction. Thus $\Delta(H_i) \leqslant m-1$. Since H_i is P_7-free, we also have $\mathrm{diam}(H_i) \leqslant 5$. Consequently $|V(H_i)| \leqslant (m-1)^5$ for each $1 \leqslant i \leqslant p$. Since $|A \cup B| \geqslant 2(2(m-1)^7 - m + 1)$, it follows that $p \geqslant 2(2(m-1)^7 - m + 1)/(m-1)^5 > 3(m-1)^2 + 2$. Since $k = 4$, $|V(H_i)| \geqslant 2$ for some i. We may assume that $|V(H_1)| \geqslant 2$. Let $a_1 b_1 \in E(H_1)$ with $a_1 \in A$ and $b_1 \in B$.

Note that each vertex in $N(a) \cap N(b)$ is an isolated vertex in $G[N(a) \cup N(b)]$. Since $G - \{a, b\}$ is connected, this implies that for each $2 \leqslant i \leqslant p$, there exists $x_i \in V(H_i)$ such that $N(x_i) - (\{a, b\} \cup N(a) \cup N(b)) \neq \varnothing$. Let $X = \{x_i \mid 2 \leqslant i \leqslant p\}$. Let $A_0 = \{x \in A \cap X \mid N(x) - (\{a, b\} \cup N(a) \cup N(b)) \subseteq N(w)\}$ and $A_1 = (A \cap X) - A_0$. For each $x \in A \cap X$, we define $u_x \in N(x)$ as follows: if $x \in A_0$, let $u_x \in N(x) - (\{a, b\} \cup N(a) \cup N(b))$; if $x \in A_1$, let $u_x \in N(x) - (\{a, b\} \cup N(a) \cup N(b)) - N(w)$. For $A' \subseteq A \cap X$, let $U_{A'} = \{u_x \mid x \in A'\}$. Let $B_0 = \{y \in B \cap X \mid N(y) - (\{a, b\} \cup N(a) \cup N(b)) \subseteq N(w)\}$ and $B_1 = (B \cap X) - B_0$. For each $y \in B \cap X$, we define $v_y \in N(y)$ as follows: if $y \in B_0$, let $v_y \in N(y) - (\{a, b\} \cup N(a) \cup N(b))$; if $y \in B_1$, let $v_y \in N(y) - (\{a, b\} \cup N(a) \cup N(b)) - N(w)$. For $B' \subseteq B \cap X$, let $V_{B'} = \{v_y \mid y \in B'\}$. Note that $|A_0| + |A_1| + |B_0| + |B_1| = |X| = p - 1$. Arguing as in the proof of Claim 6.1, we obtain the following claim.

Claim 6.2. *For $A' \subseteq A \cap X$, $|U_{A'}| \geqslant |A'|/(m-1)$. For $B' \subseteq B \cap X$, $|V_{B'}| \geqslant |B'|/(m-1)$.*

Here we consider the following three subcases:
(1) $|A_0| > (m-1)^2$ or $|B_0| > (m-1)^2$;
(2) $|A_0| \leqslant (m-1)^2$ and $|B_0| \leqslant (m-1)^2$ and, moreover, we have $A_1 \neq \varnothing$ and $B_1 \neq \varnothing$;
(3) $|A_0| \leqslant (m-1)^2$ and $|B_0| \leqslant (m-1)^2$, but we have $A_1 = \varnothing$ or $B_1 = \varnothing$.

Subcase 2.1: $|A_0| > (m-1)^2$ or $|B_0| > (m-1)^2$

We may assume that $|A_0| > (m-1)^2$. By Claim 6.2, $|U_{A_0}| \geqslant m$. Note that $U_{A_0} \subseteq N(w)$. Since G is $K_{2,m}$-free, $|N(a_1) \cap U_{A_0}| \leqslant m-1$. Since $|U_{A_0}| \geqslant m$, there exists $x \in A_0$ such that $u_x a_1 \notin E(G)$. We may assume that $x = x_2$. Since x_2 and b_1 belong to distinct components of $G[A \cup B]$, $x_2 b_1 \notin E(G)$. Since $|B| \geqslant 2(m-1)^7 - m + 1 > 2(m-1)^5 \geqslant |V(H_1)| + |V(H_2)|$, $B - (V(H_1) \cup V(H_2)) \neq \varnothing$. Let $y \in B - (V(H_1) \cup V(H_2))$. Then

$ya_1, yx_2 \notin E(G)$. Since $k = 4$, we also have $u_{x_2}b_1, u_{x_2}y \notin E(G)$. Since $u_{x_2} \notin N(a) \cup N(b)$, we now see that the path $u_{x_2}x_2aa_1b_1by$ is an induced path of order 7, a contradiction.

Subcase 2.2: $|A_0| \leqslant (m-1)^2$ and $|B_0| \leqslant (m-1)^2$ and, moreover, we have $A_1 \neq \varnothing$ and $B_1 \neq \varnothing$

Let $x \in A_1$ and $y \in B_1$. By the definition of X, x and y belong to distinct components of $G[A \cup B]$, and hence $xy \notin E(G)$. Since $u_x, v_y \notin N(a) \cup N(b) \cup N(w)$ and $k = 4$, it follows that the path $u_x xawbyv_y$ is an induced path of order 7, a contradiction.

Subcase 2.3: $|A_0| \leqslant (m-1)^2$ and $|B_0| \leqslant (m-1)^2$, but we have $A_1 = \varnothing$ or $B_1 = \varnothing$

We may assume that $A_1 = \varnothing$. Then $3(m-1)^2 + 1 < p - 1 = |A_0| + |B_0| + |B_1| \leqslant |B_1| + 2(m-1)^2$, and hence $|B_1| > (m-1)^2 + 1$. On the other hand, since $|A| \geqslant 2(m-1)^7 - m + 1 > (m-1)^2(m-1)^5 \geqslant |A_0|(m-1)^5 = |A_0 \cup A_1|(m-1)^5 = |A \cap X|(m-1)^5$ and since $|V(H_i)| \leqslant (m-1)^5$ for each $2 \leqslant i \leqslant p$, there exists i with $2 \leqslant i \leqslant p$ such that $V(H_i) \cap A \neq \varnothing$ and $x_i \notin A$. We may assume that $V(H_2) \cap A \neq \varnothing$ and $x_2 \notin A$. Then $x_2 \in B$, which implies $|V(H_2)| \geqslant 2$. Let $a_2b_2 \in E(H_2)$ with $a_2 \in A$ and $b_2 \in B$. Now since $|B_1 - \{x_2\}| > (m-1)^2$, $|V_{B_1 - \{x_2\}}| \geqslant m$ by Claim 6.2. Since G is K_3-free, it follows that if both b_1 and b_2 are adjacent to all vertices in $V_{B_1 - \{x_2\}}$, then $G[\{b_1, b_2\} \cup V_{B_1 - \{x_2\}}]$ contains $K_{2,m}$ as an induced subgraph, a contradiction. Thus $b_hv_y \notin E(G)$ for some $h \in \{1, 2\}$ and some $y \in B_1 - \{x_2\}$. We may assume that $y = x_3$. Since $|A| \geqslant 2(m-1)^7 - m + 1 > 3(m-1)^5 \geqslant |V(H_1)| + |V(H_2)| + |V(H_3)|$, $A - (V(H_1) \cup V(H_2) \cup V(H_3)) \neq \varnothing$. Let $x \in A - (V(H_1) \cup V(H_2) \cup V(H_3))$. Since $v_{x_3} \notin N(a) \cup N(b)$ and $k = 4$, the path $xaa_hb_hbx_3v_{x_3}$ is an induced path of order 7, a contradiction.

Case 3: $k \geqslant 5$

If $k = 5$, then by Lemma 6.2 and the assumption that $d(a) \geqslant 2(m-1)^7 \geqslant 4(m-1) + 2$ and $d(b) \geqslant 2(m-1)^7 \geqslant 4(m-1) + 2$, G contains P_7 as an induced subgraph, a contradiction; if $k \geqslant 6$, then by Lemma 6.1 and the assumption that $d(a) \geqslant 2(m-1)^7 \geqslant 4(m-1) + 2$, G contains P_7 as an induced subgraph, a contradiction.

This completes the proof of Lemma 6.5. □

Proposition 6.6. *Let $m \geqslant 5$. Then $\mathcal{G}_3(\{K_3, K_{2,m}, P_7\})$ is finite.*

Proof. Let $G \in \mathcal{G}_3(\{K_3, K_{2,m}, P_7\})$. We show that $|V(G)| \leqslant (4(m-1)^7(2(m-1)^7 + m - 2)^5)^5$. Since $\operatorname{diam}(G) \leqslant 5$, it suffices to show that $\Delta(G) \leqslant 4(m-1)^7(2(m-1)^7 + m - 2)^5$. Suppose that $\Delta(G) \geqslant 4(m-1)^7(2(m-1)^7 + m - 2)^5 + 1$, and let $w \in V(G)$ be a vertex with $d(w) = \Delta(G)$. Since G is K_3-free, $N(w)$ is independent. Let $L = \{x \in V(G) \mid |N(x) \cap N_2(w)| \geqslant 2(m-1)^7\}$. By Lemma 6.5, $|L \cap N(w)| \leqslant 1$ and $L \cap N_2(w) = \varnothing$.

Case 1: $|L| \geqslant 2$ and $|L \cap N(w)| = 1$

Let $a \in L \cap N(w)$ and $b \in L - N(w)$. Since $b \notin N(w) \cup N_2(w)$, we have $b \in N_3(w)$, and hence $ab \notin E(G)$. By Lemma 6.5, $N(a) \cap N(b) = \varnothing$. Let $y \in N(b) \cap N_2(w)$ and $v \in N(y) \cap N(w)$. Since $ab \notin E(G)$, $N(a) \cap N(b) = \varnothing$, and $N(w)$ is independent, it follows that the path $awvyb$ is an induced path of order 5. Hence by Lemma 6.2 and the assumption that $d(a) \geqslant 2(m-1)^7 \geqslant 4(m-1) + 2$ and $d(b) \geqslant 2(m-1)^7 \geqslant 4(m-1) + 2$, G contains P_7 as an induced subgraph, a contradiction.

Case 2: $|L| \geqslant 2$ and $L \cap N(w) = \varnothing$

Let $a, b \in L$ with $a \neq b$. Then $a, b \in N_3(w)$. If $ab \in E(G)$, then $N(a) \cap N(b) = \varnothing$ by the assumption that G is K_3-free and, if $ab \notin E(G)$, then $N(a) \cap N(b) = \varnothing$ by Lemma 6.5. Hence in either case, we have $N(a) \cap N(b) = \varnothing$. Let $A = N(a) \cap N_2(w)$ and $B = N(b) \cap N_2(w)$. Let $y \in B$ and $v \in N(y) \cap N(w)$. Since G is $K_{2,m}$-free, $|N(y) \cap N(w)| \leqslant m - 1$. Since G is $K_{2,m}$-free, it follows that $|((\bigcup_{u \in N(y) \cap N(w)} N(u)) \cup N(y)) \cap A| \leqslant (|N(y) \cap N(w)| + 1)(m-1) \leqslant m(m-1)$. Since $|A| \geqslant 2(m-1)^7 \geqslant m(m-1)+1$, $A - ((\bigcup_{u \in N(y) \cap N(w)} N(u)) \cup N(y)) \neq \varnothing$. Take $x \in A - ((\bigcup_{u \in N(y) \cap N(w)} N(u)) \cup N(y))$. Since $v \in N(y) \cap N(w)$, we have $xv, xy \notin E(G)$. Take $u \in N(x) \cap N(w)$. By the choice of x, $u \notin N(y) \cap N(w)$, and hence $uy \notin E(G)$. Consequently the path $xuwvyb$ is an induced path of order 6. Now by Lemma 6.1 and the assumption that $d(b) \geqslant 2(m-1)^7 \geqslant 4(m-1)+2$, G contains P_7 as an induced subgraph, a contradiction.

Case 3: $|L| \leqslant 1$

Let $H_1, \ldots, H_p$ be the components of $G[(N(w) \cup N_2(w)) - L]$. Let $1 \leqslant i \leqslant p$. Take $x \in V(H_i)$. Since G is $\{K_3, K_{2,m}\}$-free, $|N(x) \cap N(w)| \leqslant m - 1$. Since $x \notin L$, we also have $|N(x) \cap N_2(w)| \leqslant 2(m-1)^7 - 1$. Hence $d_{H_i}(x) \leqslant |N(x) \cap N(w)| + |N(x) \cap N_2(w)| \leqslant 2(m-1)^7 + m - 2$. Since $x \in V(H_i)$ is arbitrary, $\Delta(H_i) \leqslant 2(m-1)^7 + m - 2$. Since H_i is P_7-free, we also have $\mathrm{diam}(H_i) \leqslant 5$. Thus $|V(H_i)| \leqslant (2(m-1)^7 + m - 2)^5$ for each $1 \leqslant i \leqslant p$. We have $V(H_i) \cap N_2(w) \neq \varnothing$ for each $1 \leqslant i \leqslant p$ because $N(w)$ is independent. Let $q = |\{i \mid V(H_i) \cap N(w) \neq \varnothing\}|$. (Note that in the case where $\varnothing \neq L \subseteq N(w)$, it is possible that $V(H_i) \cap N(w) = \varnothing$ for some i.) Without loss of generality, we may assume that $V(H_i) \cap N(w) \neq \varnothing$ for each $1 \leqslant i \leqslant q$ and $V(H_i) \cap N(w) = \varnothing$ for each $q + 1 \leqslant i \leqslant p$. Since $|N(w) - L| \geqslant d(w) - 1 \geqslant 4(m-1)^7(2(m-1)^7 + m - 2)^5$, we have $q \geqslant |N(w) - L|/(2(m-1)^7 + m - 2)^5 \geqslant 4(m-1)^7$. Since G is 3-connected, $G - (\{w\} \cup L)$ is connected. Let P be a shortest $V(H_1)$-$((N(w) \cup N_2(w)) - L - V(H_1))$ path in $G - (\{w\} \cup L)$. Then $|V(P)| \geqslant 3$. Let y_1, z_1 and z_2 be the first three vertices of P. Then $y_1 \in V(H_1) \cap N_2(w)$ and $z_1 \in N_3(w)$. Let wx_1y_1 be a shortest w-y_1 path in $G[V(H_1) \cup \{w\}]$. Then $wx_1y_1z_1z_2$ is an induced w-z_2 path of order 5. Since $z_1 \notin L$, $|N(z_1) \cap N_2(w)| \leqslant 2(m-1)^7 - 1$. In particular, z_1 is adjacent to at most $2(m-1)^7 - 1$ of the H_i, $1 \leqslant i \leqslant q$. If $z_2 \notin N_2(w)$, then similarly, z_2 is adjacent to at most $2(m-1)^7 - 1$ of the H_i, $1 \leqslant i \leqslant q$; if $z_2 \in N_2(w)$, then clearly z_2 is adjacent to at most one of the H_i, $1 \leqslant i \leqslant q$ (note that it is possible that z_2 belongs to H_i for some i with $q + 1 \leqslant i \leqslant p$). Thus z_2 is adjacent to at most $2(m-1)^7 - 1$ of the H_i. Since $q \geqslant 4(m-1)^7$, there exists j with $2 \leqslant j \leqslant q$ such that neither z_1 nor z_2 is adjacent to H_j. Since $V(H_j) \cap N(w) \neq \varnothing$ and $V(H_j) \cap N_2(w) \neq \varnothing$, there exist $x_2 \in V(H_j) \cap N(w)$ and $y_2 \in V(H_j) \cap N_2(w)$ such that $x_2y_2 \in E(G)$. Then $y_2x_2wx_1y_1z_1z_2$ is an induced path of order 7, a contradiction.

This completes the proof of Proposition 6.6. □

Proof of Theorem 1.4. The 'if' part follows from Propositions 6.4 and 6.6. Thus it suffices to prove the 'only if' part. Suppose that $\mathcal{G}_3(\{K_3, K_{2,m}, T\})$ is finite. From Theorem 1.1, it follows that $T \in \mathcal{T}_2$. For each $s \geqslant 2$, let $H_s^{(1)}$ be the graph defined by $V(H_s^{(1)}) = \{x_1, x_2, x_3\} \cup \{y_{i,h}, z_{i,h} \mid 1 \leqslant i \leqslant 2, 1 \leqslant h \leqslant s\}$ and $E(H_s^{(1)}) = \{x_1x_2\} \cup \{x_iy_{i,h}, x_3z_{i,h} \mid 1 \leqslant i \leqslant 2, 1 \leqslant h \leqslant s\} \cup \{y_{i,h}z_{j,h} \mid 1 \leqslant i, j \leqslant 2, 1 \leqslant h \leqslant s\}$ (see Figure 6). Then $H_s^{(1)}$ is

3-connected and $\{K_3, K_{2,m}\}$-free. Since $\mathcal{G}_3(\{K_3, K_{2,m}, T\})$ is finite, there exists $s \geqslant 2$ such that $H_s^{(1)}$ contains T as an induced subgraph. Since every tree contained in $H_s^{(1)}$ as an induced subgraph has diameter at most 6, $\mathrm{diam}(T) \leqslant 6$. Now for each $t \geqslant 3$, let $P = x_1x_2$ be a path of order 2 and $C = y_1y_2 \cdots y_{2t}y_1$ be a cycle of order $2t$, and let $H_t^{(2)}$ be the graph defined by $V(H_t^{(2)}) = V(C) \cup V(P)$ and $E(H_t^{(2)}) = E(C) \cup E(P) \cup \{x_1y_{2i-1}, x_2y_{2i} \mid 1 \leqslant i \leqslant t\}$ (see Figure 6). Then $H_t^{(2)}$ is 3-connected and $\{K_3, K_{2,m}\}$-free. Hence there exists $t \geqslant 3$ such that $H_t^{(2)}$ contains T as an induced subgraph. Observe that if F is a member of $\mathcal{T}_2$ contained in $H_t^{(2)}$ as an induced subgraph, then F is a path or an induced subgraph of Y_3^*. Therefore T is either a path of order at most 7 or an induced subgraph of Y_3^*. □

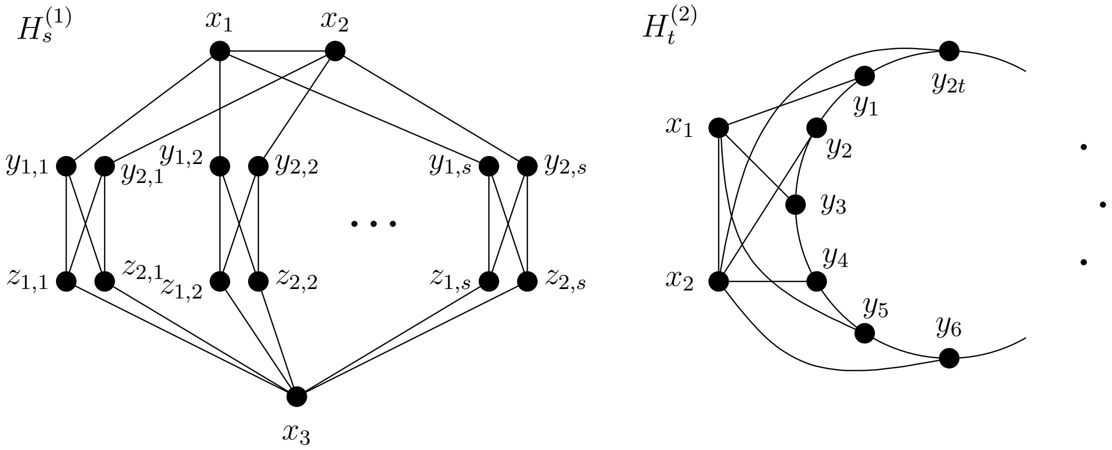

Figure 6: Graphs $H_s^{(1)}$ and $H_t^{(2)}$

7 Concluding remarks

In this paper, we have considered three forbidden subgraphs which generate a finite set in the class of 3-connected graphs. As we have seen in Theorem 1.1, there are six types. For many of them, we have given a characterization. The cases which remain uncharacterized are the following:

(a) $\mathcal{F} = \{K_3, K_{3,3}, T\}$, where $T \in \mathcal{T}_2$;
(b) $\mathcal{F} = \{K_4, K_{2,2}, T\}$, where T is a path of order at least 4;
(c) $\mathcal{F} = \{K_3, K_{2,4}, T\}$ and $\{K_3, K_{2,3}, T\}$, where $T \in \mathcal{T}_1$;
(d) $\mathcal{F} = \{K_3, K_{2,2}, T\}$, where $T \in \mathcal{T}_0$;
(e) $\mathcal{F} = \{K_3, K_{1,4}, T\}$, where $T \in \mathcal{T}_1$;
(f) $\mathcal{F} = \{K_n, K_{1,3}, T\}$, where $n \geqslant 5$ and $T \in \mathcal{T}_2^*$;
(g) $\mathcal{F} = \{K_4, K_{1,3}, T\}$, where $T \in \mathcal{T}_1^*$.

In cases (a)–(d), $\mathcal{F}$ does not contain a star. In these cases, we can give a bound on the diameter of T.

Proposition 7.1. *Let $\mathcal{F}$ be a subset of $\mathcal{G}$ with $|\mathcal{F}| = 3$. Suppose $\mathcal{F}$ does not contain a star and $\mathcal{G}_3(\mathcal{F})$ is finite. Then $\mathcal{F}$ contains a tree of diameter at most* 8.

Proof. Since $\mathcal{F}$ does not contain a star, it follows from Theorem 1.1 that $\mathcal{F}$ can be written in the form $\mathcal{F} = \{K_n, K_{m_1,m_2}, T\}$, where $n \in \{3,4\}$, $m_1 \in \{2,3\}$, $m_2 \geqslant m_1$ and $T \in \mathcal{T}_0$.

Let $s \geqslant 2$. Let $C^{(i)} = a_1^{(i)} b_1^{(i)} c_1^{(i)} a_2^{(i)} b_2^{(i)} c_2^{(i)} a_1^{(i)}$ be a cycle of order 6 for each $1 \leqslant i \leqslant s$, and define a graph H_s by $V(H_s) = \{a_0, b_0, c_0\} \cup (\bigcup_{1\leqslant i\leqslant s} V(C^{(i)}))$ and $E(H_s) = \{a_0 a_j^{(i)}, b_0 b_j^{(i)}, c_0 c_j^{(i)} \mid 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant 2\} \cup (\bigcup_{1\leqslant i\leqslant s} E(C^{(i)}))$. Then H_s is 3-connected and $\{K_n, K_{m_1,m_2}\}$-free. Hence there exists $s \geqslant 2$ such that H_s contains T as an induced subgraph. However, H_s does not contain P_{10} as an induced subgraph. Therefore the diameter of T is at most 8. ☐

In (a)–(d), we have $T \in \mathcal{T}_0$ and hence $\Delta(T) \leqslant 3$. By combining this fact and Proposition 7.1, we see that the order of T is bounded. Thus the number of triples in these cases is finite. On the other hand, in cases (e)–(g), where the triple contains a star, Proposition 7.1 gives no further information about $\mathcal{F}$. In fact, Theorem A shows that in these cases, there exist infinitely many $\mathcal{F}$ such that $\mathcal{G}_3(\mathcal{F})$ is finite.

We add that for (a) and (b), and for the case where $\mathcal{F} = \{K_3, K_{2,4}, T\}$ in (c), the determination of T has recently been completed in [2].

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