

Generalized Small Schröder numbers

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Abstract

We study generalized small Schröder paths in the sense of arbitrary sizes of steps. A generalized small Schröder path is a generalized lattice path from $(0, 0)$ to $(2n, 0)$ with the step set of $\{(k, k), (l, -l), (2r, 0) \mid k, l, r \in \mathbf{P}\}$, where \mathbf{P} is the set of positive integers, which never goes below the x -axis, and with no horizontal steps at level 0. We find a bijection between 5-colored Dyck paths and generalized small Schröder paths, proving that the number of generalized small Schröder paths is equal to $\sum_{k=1}^n N(n, k)5^{n-k}$ for $n \geq 1$.

Keywords: generalized large and small Schröder paths; colored Dyck paths

1 Introduction

Lattice paths have been studied extensively by many mathematicians over a long period of time. Recently, several attempts of generalization have been successful. (See [2], [4], [7], and [3].) Schröder paths are also interesting objects, but have been rarely studied for any type of generalization until J.P.S Kung and A. De Mier [5] found the generating functions for the number of generalized Dyck and Schröder paths, which are called rook and queen paths, with a given right boundary and steps satisfying a natural slope condition. In this paper we try to enumerate combinatorially the number of the generalized small Schröder paths.

In the next section we give some basic definitions and facts on lattice paths in general. Section 3 talks about a bijection between colored Dyck paths and generalized small Schröder paths.

2 Preliminaries

We list two sets of definitions.

Definition 1. The following are fairly well known, but we state them anyway.

- (1) An n -Dyck path is a path from $(0, 0)$ to $(2n, 0)$ with up $U = (1, 1)$ and down $D = (1, -1)$ steps that never goes below the x -axis.
- (2) An n -Dyck path is called a prime if it never touches the x -axis except at the starting and terminal vertices.
- (3) A peak in a Dyck path is a vertex between an up step and a down step, and a valley is a vertex between a down step and an up step.
- (4) A double ascent in a Dyck path is an intermediate vertex of a consecutive UU pair, and a double descent is defined similarly.
- (5) The Narayana number $N(n, k)$ counts n -Dyck paths with k peaks.
- (6) An n -Schröder path is a path from $(0, 0)$ to $(2n, 0)$ with up $U = (1, 1)$, down $D = (1, -1)$, and flat $F = (2, 0)$ steps that never goes below the x -axis.
- (7) An n -small Schröder path is a Schröder path having no flat steps at level 0.

Definition 2. We introduce the new ones that we need throughout this paper.

- (1) A 5-colored n -Dyck path is an n -Dyck path only whose double ascents are colored in one of five colors, b (black), w (white), f (flame), g (green), and y (yellow).
- (2) A bicolored n -small Schröder path is an n -small Schröder path whose vertices between distinct steps of U, D , and F are colored in b only and vertices between the same kind of steps of UU, DD , and FF are colored in either b or w . The starting vertex and ending vertex are not colored.
- (3) A generalized (large) n -Schröder path is a lattice path from $(0, 0)$ to $(2n, 0)$ using steps from $\{(k, k), (l, -l), (2r, 0)\}$ for any positive integers k, l , and r , that never goes below the x -axis.
- (4) A generalized n -small Schröder path is a generalized n -Schröder path with no flat steps at level 0.

Then the following lemma is straightforward from the definitions 2.1 (5) and 2.2 (1). (The number is listed as A078009 in OEIS [6].)

Lemma 3. *The number of 5-colored n -Dyck paths is $\sum_{k=1}^n N(n, k)5^{n-k}$.*

3 A bijection between 5-colored Dyck paths and generalized small Schröder paths

In this section we provide a bijective relation between 5-colored n -Dyck paths and bicolored n -small Schröder paths by Bijection I. Then we give a natural bijection between bicolored n -small Schröder paths and generalized n -small Schröder paths by Bijection II. In other words, we prove the following theorem bijectively.

Theorem 4. *The number of generalized n -small Schröder paths is*

$$\sum_{k=1}^n N(n, k)5^{n-k}.$$

Proof. **Bijection I.**

Let P be a 5-colored n -Dyck path. We construct the corresponding bicolored n -small Schröder paths by taking four steps described as follow:

STEP 1. (coloring)

For each double ascent of P find the first double descent right of the ascent at the same level. We call this the *matching* double descent. Note that each double ascent of P is colored in one of five colors, b, w, f, g , and y . Let P' be the path obtained from P by the following process:

- (i) If a double ascent of P is colored in one of three colors, b, w , and f , then color its matching double descent in the same color.
- (ii) If a double ascent of P is colored in one of two colors, g and y , then replace the color with b and w , respectively, and color its matching double descent in w and b , respectively.
- (iii) Color each peak and valley of P in b .

Notice that every f colored double ascent in P' has the matching double descent also colored in f .

STEP 2. (factorizing)

Find the first vertex v colored in f . Assume that v is the ending vertex of an up step U at level i . Then one can find the corresponding first down step D at the same level. Let Q be the subpath of P' above the level i from U to D . Then we can factor P' in general as $LUfQxDR$, where L and R are subpaths of P' , f is the color of the vertex v , and x is a color of the starting vertex of D which is either b or f . Initially, this x is f , but as processes repeated it can be b if Q ends with a flat step. (See Figure 1 (i).)

STEP 3. (creating flats)

In the subpath $UfQxD$, Q can be factored as Q_1bQ_2 where Q_1 is the first prime path at level i and Q_2 is the rest of the path of Q . If Q is a prime then take $Q_1 = Q$. (See Figure 1 (ii).)

Now find the lowest level k of Q_1 at which a black colored vertex lies.

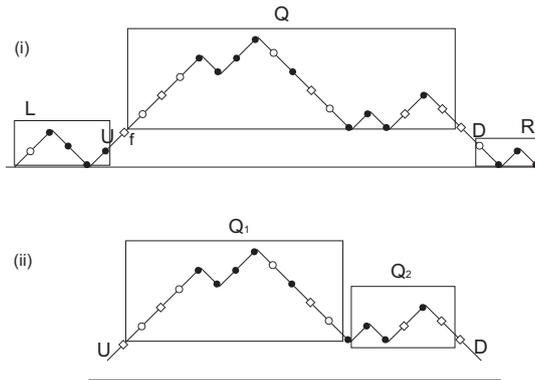


Figure 1: The factorizations of P' , (i) and Q , (ii). A square vertex represents a vertex of color f .

Note that

- (i) Q_1 starts with $k - i$ U steps and ends with $k - i$ D steps.
- (ii) First $k - i - 1$ double ascents are colored in w or f
- (iii) Each of the last $k - i - 1$ double descents is colored in the same color of its matching double ascent.

Thus we can factor Q_1 as $Q_1 = U_1 c_1 U_2 \cdots c_{k-i-1} U_{k-i} c_{k-i} Q_1^* c_{k-i+1} D_{k-i} c_{k-i-1} \cdots D_2 c_1 D_1$, where Q_1^* is a subpath of Q_1 above the level k , U_j is the up step ending at the level $i + j$, and D_j is the down step starting at the level $i + j$ for $j = 1, 2, \dots, k - i$.

We have four cases to construct flat steps depending on the colors of vertices c_{k-i} and c_{k-i+1} , i.e., the starting and ending vertices of the subpath Q_1^* , which are:

- Case 1: The color c_{k-i+1} is b .
- Case 2: The colors of c_{k-i} and c_{k-i+1} are both f .
- Case 3: The color c_{k-i} is b and c_{k-i+1} is w .
- Case 4: The colors of c_{k-i} and c_{k-i+1} are both w .

We explain these four cases one at a time, all of which will be concluded by the last process of changing colors. We start at the subpath $UfQxD$ of the path P' .

Case 1.

- (i) Erase the color f of $UfQxD$ and delete $U_1 c_1 U_2 \cdots c_{k-i-1} U_{k-i}$ of Q_1 , so that the color c_{k-i} takes over the color f .

- (ii) Replace each down step D_j ($1 \leq j \leq k - i$) of Q_1 by F_j at level i with colors unchanged.

- (iii) Replace the x of $UfQxD$ by b if it is an f .

Then the path $UfQxD$ is now transformed to

$$U c_{k-i} Q_1^* c_{k-i+1} F_{k-i} c_{k-i-1} \cdots F_2 c_1 F_1 b Q_2 b D. \quad (\text{See Figure 2.})$$

If the Q_1^* is empty, then $c_{k-i} = c_{k-i+1}$ and then flat steps followed.

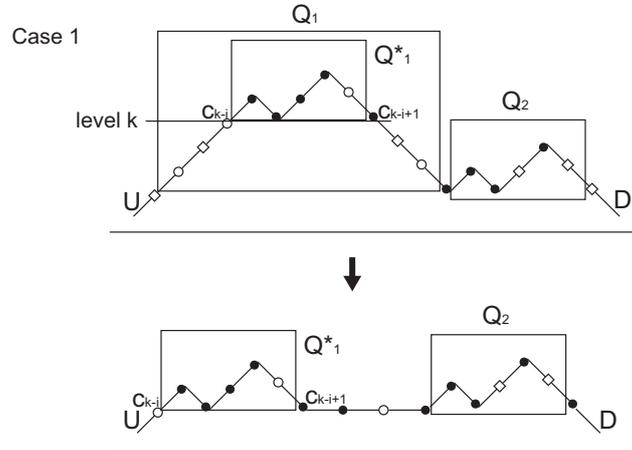


Figure 2: Case 1.

Case 2.

The first three processes are the same as Case 1. We just change the color of the first flat step from $c_{k-i+1} = f$ to b .

Then the path $UfQxD$ is transformed to $U c_{k-i} Q_1^* b F_{k-i} c_{k-i-1} \cdots F_2 c_1 F_1 b Q_2 b D$. (See Figure 3.)

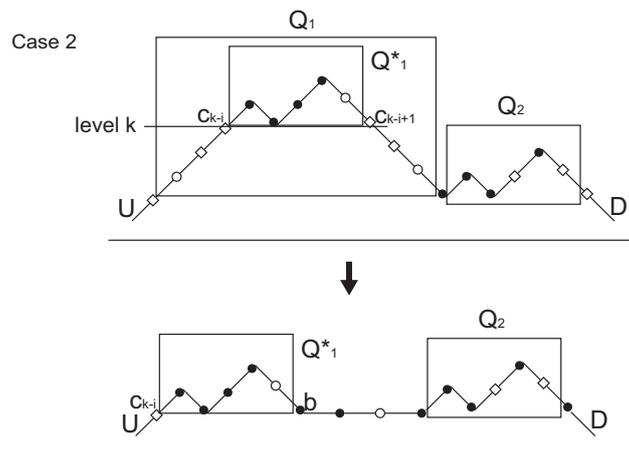


Figure 3: Case 2.

Case 3.

(i) Replace the color f of $UfQxD$ with b and erase the color x .

(ii) Replace each up step U_j ($1 \leq j \leq k - i$) of Q_1 by F_j at level i with colors unchanged.

(iii) Delete $D_{k-i}c_{k-i-1} \cdots D_2c_1D_1$ of Q_1 .

(iv) Cut off bQ_2 of $Q = Q_1bQ_2$ and then paste it after turning 180 degrees (denoted by \bar{Q}_2b) in between the first color b in (i) and the first flat step F_1 in (ii). (See Figure 4).

Then the path $UfQxD$ is transformed to $Ub\bar{Q}_2bF_1c_1 \cdots F_{k-i}c_{k-i}Q_1^*c_{k-i+1}D$.

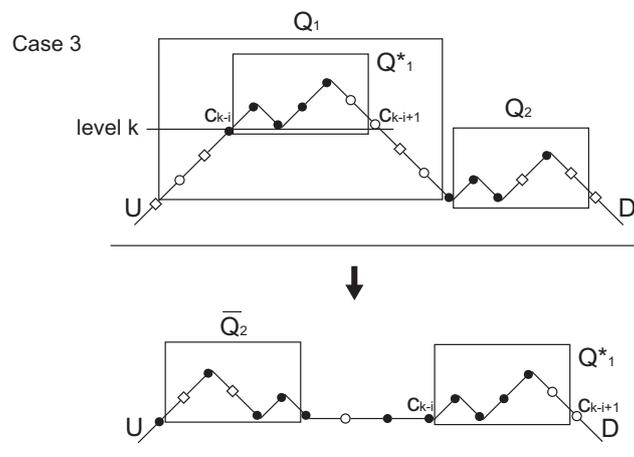


Figure 4: Case 3.

Case 4.

In this case, Q_1^* is clearly neither empty nor prime. So we can factor Q_1^* as $Q_{11}^*bQ_{12}^*$, where Q_{11}^* is the first prime of Q_1^* and Q_{12}^* is the rest of Q_1^* .

(i) Erase the color f of $UfQxD$ and replace the x with b .

(ii) Delete $U_1c_1U_2 \cdots c_{k-i-1}U_{k-i}$.

(iii) Replace each down step D_j ($1 \leq j \leq k - i$) of Q_1 by F_j at level i with colors unchanged.

(iv) Let Q'_2 represent the newly created flat steps by (iii) along with Q_2b from (i), and after turning 180 degrees of Q'_2 let us denote it by \bar{Q}'_2 . Now cut off Q'_2 and then paste \bar{Q}'_2 after Q_{11}^* . (See Figure 5.)

Then the path $UfQxD$ is transformed to $Uc_{k-i}Q_{11}^*\bar{Q}'_2bQ_{12}^*c_{k-i+1}D$.

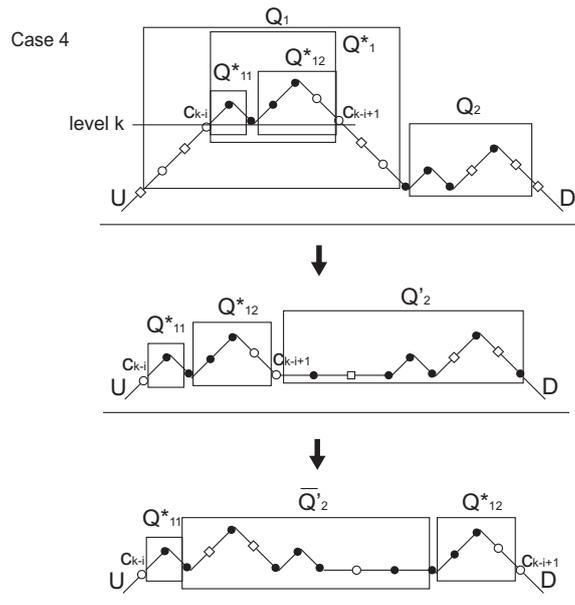


Figure 5: Case 4.

For the last process of Step 3 we replace the color f 's between the flat steps with b 's.

STEP 4. Repeat Step 2 and 3 until there is no vertex colored in f .

Then clearly the outcome is a bicolored n -small Schröder path.

Now we give an example, which can be obtained by applying STEPs 1 through 4 along with Figures 1 through 5.

Let a 5-colored 16-Dyck path P be given:

$$P = U f U f U w U f U f U f U w U D U D D U D D D U f U D^5 U f f U y U D^5,$$

where D^5 represents $DDDDD$. We will construct the corresponding bicolored 16-small Schröder path \hat{P} :

$$\hat{P} = U w U b F b U b F b U b D w D b F w F b F b U b F b D b F b F b U b D b F b U b D w D.$$

Step by step applications of STEPs and Cases are showed in the Figure 6.

For recovery processes we first note that from STEP 3:

Now find the flat steps whose subscript is the largest. These steps can be appeared as blocks of consecutive flat steps. Then look for the starting vertex v_2 of the label down step for these flat steps. Let v_1 be the ending vertex of the corresponding label up step, for convenience. Then we take the following processes to recover the corresponding Dyck path just before the STEP 1.

1. If v_2 has color b , then the leftmost block of flat steps are changed first. If the color of v_1 is b or w , then we apply the reverse process of Case 1 of STEP 3. If the color of v_1 is f , then apply the reverse process of Case 2 of STEP 3.
2. If v_2 has the color w , then the rightmost block of flat steps are changed first. If the color of v_1 is b , then we apply the reverse process of Case 3 of STEP 3. If the color of v_1 is w , then apply the reverse process of Case 4 of STEP.
3. Repeat these processes until we have no flats.

Then we have an n -Dyck path whose double ascents and double descents are colored in one of three colors, b , w , and f , and all peaks and valleys are colored in b . By the following recovery processes of colors, we can fully recover the 5-colored n -Dyck path P :

1. Delete the colors of peaks and valleys.
2. If a double ascent and its matching double descent are in the same color, then delete the color of the double descent.
3. If not, give each double ascent a new color by the following:
 - (a) Color the double ascent in g if a pair of the colors of a double ascent and its matching double descent is (b, w) and then delete the color of the double descent.
 - (b) Color the double ascent in y if a pair of the colors of a double ascent and its matching double descent is (w, b) and then delete the color of the double descent.

Now we have the 5-colored n -Dyck path as we desired. Figure 7 shows full recovery processes.

Let us consider the second bijection. As we mentioned earlier, this bijection gives one-to-one correspondence between bicolored n -small Schröder paths and generalized n -small Schröder paths.

Bijection II.

Let \hat{P} be a given bicolored n -small Schröder path. Recall that in \hat{P} all peaks and valleys are colored in black, and the starting and ending vertices of all blocks of flats are colored in black. And other vertices can be black within blocks of up, down, and flat

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