Absolute differences along Hamiltonian paths

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Abstract

We prove that if the vertices of a complete graph are labeled with the elements of an arithmetic progression, then for any given vertex there is a Hamiltonian path starting at this vertex such that the absolute values of the differences of consecutive vertices along the path are pairwise distinct. In another extreme case where the label set has small additive energy, we show that the graph actually possesses a Hamiltonian cycle with the property just mentioned. These results partially solve a conjecture by Z.-W. Sun.

1 Introduction

In this paper we consider the following conjecture posed by Z.-W. Sun, formulated among other open problems in [5, Conjecture 3.1].

Conjecture 1. Let $A = \{a_1, a_2, \dots, a_n\}$ be a set of n distinct real numbers. Then there is a permutation $b_1, b_2, \dots, b_n$ of $a_1, \dots, a_n$ with $b_1 = a_1$ such that the $n-1$ numbers

$$|b_2 - b_1|, |b_3 - b_2|, \dots, |b_n - b_{n-1}|$$

are pairwise distinct.

Similar problems have been studied in [2] and [1, 3, 4], see Section 4.

If we consider the complete graph on $\{a_1, a_2, \dots, a_n\}$, in order for the conjecture to be true we need to find for every element a_i a Hamiltonian path starting at a_i, such that the absolute differences of consecutive vertices along the path are pairwise distinct.

As Z.-W. Sun pointed out in [5, Theorem 1.1], ordering the elements $a_1 < a_2 < \cdots < a_n$ we can easily find such an Hamiltonian path starting from a_1 or a_n: if $n = 2k$ is even we can consider the permutations

$$(a_1, a_n, a_2, a_{n-1}, \ldots, a_{k-1}, a_{k+2}, a_k, a_{k+1})$$
$$(a_n, a_1, a_{n-1}, a_2, \ldots, a_{k+2}, a_{k-1}, a_{k+1}, a_k),$$

and if $n = 2k-1$ is odd consider the permutations

$$(a_1, a_n, a_2, a_{n-1}, \ldots, a_{k-1}, a_{k+1}, a_k)$$
$$(a_n, a_1, a_{n-1}, a_2, \ldots, a_{k+1}, a_{k-1}, a_k).$$

If the cardinality of $A - A = \{a_i - a_j : a_i, a_j \in A\}$ is large, then heuristically it should be easy to find Hamiltonian paths as required, whereas this should be an harder task for structured sets, where $|A - A|$ can be as small as $|A| - 1$.

However, we are able to prove that the conjecture holds in these cases.

Theorem 2. *Conjecture 1 holds if A is an arithmetic progression.*

Moreover, as expected, the conjecture holds if the set A does not have a particular additive structure.

Theorem 3. *If A is a set of real numbers with $E(A, A) = c|A|^2$ for $c < 5/2$, where $E(A, A)$ is the additive energy of A, then there exists a Hamiltonian cycle on the complete graph whose vertices are labeled with the elements of A such that the absolute values of the differences of consecutive vertices along the path are pairwise distinct. In particular, Conjecture 1 holds in this case.*

2 Proof of Theorem 2

Without loss of generality let $A = [n] := [1, n] \cap \mathbb{Z}$ be the set of the first n positive integers.

Fix an element $s \in [n]$. In order to prove Theorem 2, we need a permutation $a = (a_1, \ldots, a_n)$ of $[1, n]$ with $a_1 = s$ such that the $n - 1$ differences

$$|a_2 - a_1|, \ldots, |a_n - a_{n-1}|$$

are pairwise distinct.

Denote the set of absolute differences of the permutation a as $d(a) := \{|a_{i+1} - a_i| : 1 \leqslant i \leqslant n-1\}$. We want to find an a of $[1, n]$ such that $|d(a)| = n - 1$.

Definition 4. Let S_n be the set of permutations on $[n]$. We say that a permutation $a = (a_1, \ldots, a_n) \in S_n$ is a *good permutation of the first kind* if the absolute values of the differences $|a_2 - a_1|, \ldots, |a_n - a_{n-1}|$ are pairwise distinct and

$$a_{2i} < \left\lceil \frac{n+1}{2} \right\rceil \leqslant a_{2i+1} \text{ whenever } 0 \leqslant i \leqslant n/2.$$

Similarly, we say that a permutation $a = (a_1, \ldots, a_n) \in S_n$ is a *good permutation of the second kind* if the absolute values of the differences $|a_2 - a_1|, \ldots, |a_n - a_{n-1}|$ are pairwise distinct and

$$a_{2i+1} \leqslant \left\lfloor \frac{n+1}{2} \right\rfloor < a_{2i} \text{ whenever } 0 \leqslant i \leqslant n/2.$$

The set of good permutations will be denoted by G_n.

The next lemma shows some properties of the set G_n which will be needed for the proof of Theorem 2.

Lemma 5. *Let $a = (a_1, \ldots, a_n) \in G_n$. Then the following hold:*

1. *The permutation $b = \{b_i\}$ given by $b_i = n + 1 - a_i$, for $1 \leqslant i \leqslant n$, is in G_n.*
2. *If a is a good permutation of the first kind, then $b = (b_1, \ldots, b_n)$ defined by*

$$\begin{aligned} b_{2i-1} &= a_{2i-1} - \left\lfloor \frac{n}{2} \right\rfloor, && 1 \leqslant i \leqslant (n+1)/2, \\ b_{2i} &= a_{2i} + \left\lfloor \frac{n+1}{2} \right\rfloor, && 1 \leqslant i \leqslant n/2 \end{aligned}$$

   is a good permutation of the second kind.

3. *If a is a good permutation of the second kind, then $b = (b_1, \ldots, b_n)$ defined by*

$$\begin{aligned} b_{2i-1} &= a_{2i-1} + \left\lfloor \frac{n}{2} \right\rfloor, && 1 \leqslant i \leqslant (n+1)/2, \\ b_{2i} &= a_{2i} - \left\lfloor \frac{n+1}{2} \right\rfloor, && 1 \leqslant i \leqslant n/2 \end{aligned}$$

   is a good permutation of the first kind.

Proof. 1. Suppose a is a good permutation of the first kind. Then, for $b = (b_1, \ldots, b_n)$ given by $b_i = n + 1 - a_i$, for $1 \leqslant i \leqslant n$, we have

$$\begin{aligned} b_{2i+1} &\leqslant n + 1 - \left\lceil \frac{n+1}{2} \right\rceil = \left\lfloor \frac{n+1}{2} \right\rfloor, && 0 \leqslant i \leqslant n/2 \\ b_{2i} &> n + 1 - \left\lceil \frac{n+1}{2} \right\rceil = \left\lfloor \frac{n+1}{2} \right\rfloor, && 0 \leqslant i \leqslant n/2, \end{aligned}$$

and so b is a good permutation of the second kind.

Similarly, if a is a good permutation of the second kind, $b = (b_1, \ldots, b_n)$ given by $b_i = n + 1 - a_i$, for $1 \leqslant i \leqslant n$ is a good permutation of the first kind.

2. Since $0 < a_{2i} < \lceil \frac{n+1}{2} \rceil \leqslant a_{2i+1} \leqslant n$, for $0 \leqslant i \leqslant n/2$, we have

$$1 = \left\lceil \frac{n+1}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor \leqslant b_{2l+1} \leqslant n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor,$$

$$\left\lfloor \frac{n+1}{2} \right\rfloor < b_{2l} < \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lceil \frac{n+1}{2} \right\rceil = n+1,$$

so that, in order to show that that b is a good permutation of the second kind, we are left to check that for $0 \leqslant i \leqslant n-1$, $|b_{i+1} - b_i|$ are pairwise distinct. This is true since

$$b_{i+1} - b_i = \begin{cases} a_{i+1} - a_i - n & \text{if } i \in [1, n-1] \text{ is even,} \\ a_{i+1} - a_i + n & \text{if } i \in [1, n-1] \text{ is odd.} \end{cases}$$

In either case, $|b_{i+1} - b_i| = n - |a_{i+1} - a_i|$, and, since for $0 \leqslant i \leqslant n-1$ $|a_{i+1} - a_i|$ are pairwise distinct, so are $|b_{i+1} - b_i|$.

3. Same proof as in point 2.

□

We can now prove the main result, which clearly implies Theorem 2.

Theorem 6. *If $n \not\equiv 1 \mod 4$ then for every $s \in [n]$ there exists a good permutation $a = (a_1, \ldots, a_n)$ of $[n]$ with $a_1 = s$.*

If $n \equiv 1 \mod 4$ then for every $s \in [n]$ there exists a permutation $a = (a_1, \ldots, a_n)$ of $[n]$ with $a_1 = s$ and $|d(a)| = n-1$. Moreover, if $s \neq \frac{1}{2}\left(\left\lfloor \frac{n+1}{2} \right\rfloor + 1\right)$ one can find a good permutation starting from s.

Proof. The proof goes by induction on n. Because of the first part of Lemma 5 we can prove it just for starting points $s \leqslant \left\lfloor \frac{n+1}{2} \right\rfloor$.

If $s = 1$ the permutation $(1, n, 2, n-1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor + \delta)$, where $\delta = 1$ if n is even and $\delta = 0$ if n is odd, is clearly a good permutation.

Take $2 \leqslant s \leqslant \frac{1}{2} \left\lfloor \frac{n+1}{2} \right\rfloor$. We consider two cases:

Case 1: $n - 2s \not\equiv 1 \mod 4$.

Then we consider the following permutation on the set $[1, s] \cup [n-s+1, n]$:

$$b = (s, n-s+1, s-1, n-s+2, \ldots, 1, n),$$

with $d(b) = [n-2s+1, n-1]$.

We now want to complete b with the remaining elements from A in order to get the required good permutation $a \in G_n$.

Choose the next element as $\alpha = 2s \leqslant \left\lfloor \frac{n+1}{2} \right\rfloor$ in order to get the absolute difference $n - 2s$.

By the induction hypothesis we can find a good permutation c of $[1, n-2s]$ starting from s, so that $d(c) = [1, n-2s-1]$.

Let a be the juxtaposition of b and $s + c = (c_1 + s = 2s, c_2 + s, \ldots, c_{n-2s} + s)$, i.e. a is the permutation

$$(s, n-s+1, s-1, n-s+2, \ldots, 1, n, c_1 + s, c_2 + s, \ldots, c_{n-2s} + s).$$

Since

$$s \leqslant \frac{1}{2}\left\lfloor \frac{n+1}{2} \right\rfloor = \frac{1}{2}\left\lfloor \frac{n-2s+1}{2} \right\rfloor + \frac{s}{2}$$

implies

$$s \leqslant \left\lfloor \frac{n-2s+1}{2} \right\rfloor,$$

we have that c is a good permutation of the second kind and hence a satisfies

$$a_{2i+1} \leqslant \left\lfloor \frac{n+1}{2} \right\rfloor < a_{2i} \quad \text{for } 0 \leqslant i \leqslant n/2,$$

so that a is a good permutation of the second kind starting from s.

Case 2: $n - 2s + 1 \not\equiv 1 \bmod 4$.

Arguing in similar way as Case 1, start from the permutation b on the elements of $[1, s] \cup [n - s + 2, n]$ defined as

$$b = (s, n - s + 2, s - 1, n - s + 3, \dots, 2, n, 1),$$

so that $d(b) = [n - 2s + 2, n - 1]$, and take the next element as $\alpha = n - 2s + 2$ in order to get the absolute difference $n - 2s + 1$.

Using the inductive hypothesis we find a good permutation c of $[1, n-2s+1]$ starting from $n - 3s + 2$, with $d(c) = [1, n - 2s]$.

Let a be the juxtaposition of b and $s + c = (c_1 + s = 2s, c_2 + s, \dots, c_{n-2s+1} + s)$, i.e. a is the permutation

$$(s, n - s + 2, s - 1, n - s + 3, \dots, 2, n, 1, c_1 + s, c_2 + s, \dots, c_{n-2s+1} + s).$$

Since our hypotheses on s imply that $n - 3s + 2 \geqslant \left\lceil \frac{n-2s+2}{2} \right\rceil$ we get that c is a good permutation of the first kind and, since

$$s + c_{2i} < \left\lfloor \frac{n+1}{2} \right\rfloor + 1 = \left\lceil \frac{n+2}{2} \right\rceil \leqslant s + c_{2i+1},$$

then a is a good permutation of the second kind, as required.

Since for every n and $s \leqslant \frac{1}{2}\left\lfloor \frac{n+1}{2} \right\rfloor$ either $n - 2s$ or $n - 2s + 1$ is not congruent to 1 modulo 4, the result is proven in these cases.

Suppose now $\frac{1}{2}\left\lfloor \frac{n+1}{2} \right\rfloor < s \leqslant \left\lfloor \frac{n+1}{2} \right\rfloor$. Then $s' = n + 1 - s - \left\lfloor \frac{n}{2} \right\rfloor < \frac{1}{2}\left\lfloor \frac{n+1}{2} \right\rfloor + 1$, and by Lemma 5 we are done unless $n \equiv 1, 2$ modulo 4 and $s = \frac{1}{2}\left\lfloor \frac{n+1}{2} \right\rfloor + \frac{1}{2}$.

In these cases however, we can exhibit explicit permutations satisfying the requests of the Theorem.

Case 1: $n \equiv 1 \bmod 4$.

Consider the permutation

$$(s, n - s + 2, s - 1, n - s + 3, \dots, 2, n, 1, n - 2s + 2 = \left\lceil \frac{n+1}{2} \right\rceil,$$

$$\left\lceil\frac{n+1}{2}\right\rceil+1,\left\lceil\frac{n+1}{2}\right\rceil-1,\dots,n-s+1).$$

Case 2: $n \equiv 2 \bmod 4$.
Consider the permutation

$$(s,n-s+2,s-1,n-s+3,\dots,2,n,1,n-2s+2=\left\lceil\frac{n+1}{2}\right\rceil,$$

$$\left\lceil\frac{n+1}{2}\right\rceil-1,\left\lceil\frac{n+1}{2}\right\rceil+1,\dots,s+1,n-s+1). \qquad \square$$

3 Proof of Theorem 3

In order to prove 3, we show that for a randomly chosen Hamiltonian cycle on the graph under consideration, with positive probability all absolute differences along the cycle are pairwise distinct.

Let $(A-A)_+ := (A-A)\cap\mathbb{N}$. For a random circular permutation $a=(a_1,\dots,a_n)$ of A, let $d(a)=\{|a_2-a_1|,\dots,|a_n-a_{n-1}|,|a_1-a_n|\}$.

Then

$$E(|d(a)|)=\sum_{d\in(A-A)_+}P(d\in d(a)). \tag{1}$$

Fix $d\in(A-A)_+$. Let X_i be the event $|a_i-a_{i-1}|=d$ for $i=2,\dots,n$, and X_1 be the event $|a_1-a_n|=d$.

Then

$$P(d\in d(a))=P(X_1\cup\dots\cup X_n)\geqslant\sum_{i=1}^{n}P(X_i)-\sum_{1\leqslant i<j\leqslant n}P(X_i\cap X_j) \tag{2}$$

by inclusion-exclusion.

Let $s(d)=|\{a\in A: a-d,a+d\in A\}|$ be the number of 3-terms arithmetic progressions of difference d contained in A, and $r(x):=|\{(a,a')\in A\times A: a-a'=x\}|$. We have the following elementary estimate.

Lemma 7. *Let $|A|=n$. Then $\sum_{d\in(A-A)_+}s(d)\leqslant n^2/4$.*

Proof. If $A=\{a_1<\dots<a_n\}$ then a_i can be the middle term of no more than $\min(i-1,n-i)$ three terms arithmetic progressions. Hence

$$\sum_{d\in(A-A)_+}s(d)\leqslant 2\sum_{i=1}^{\lfloor\frac{n+1}{2}\rfloor}(i-1)\leqslant\frac{n^2}{4}. \qquad \square$$

We can now compute the probabilities of the events X_i and $X_i \cap X_j$ in order to get a lower bound for $E(|d(a)|)$.

$$P(X_i) = \frac{2r(d)}{n(n-1)} \qquad \text{for } 1 \leqslant i \leqslant n,$$

$$P(X_i \cap X_j) = \begin{cases} \frac{2s(d)}{n(n-1)(n-2)} & \text{if } 1 \leqslant i = j-1 \leqslant n-1, \\ \frac{4r(d)(r(d)-1)}{n(n-1)(n-2)(n-3)} & \text{if } 1 \leqslant i < j-1 \leqslant n-1. \end{cases}$$

Putting these equalities in 1 and 2 we get

$$\begin{aligned} E(|d(a)|) \quad &\geqslant \quad \frac{2\sum_{d\in(A-A)_+} r(d)}{n-1} - \frac{2\sum_{d\in(A-A)_+} s(d)}{(n-1)(n-2)} + \\ &\quad - \frac{2\sum_{d\in(A-A)_+} r(d)(r(d)-1)}{n(n-3)} \\ &\geqslant \quad n - \frac{1}{2}\frac{n^2}{(n-1)(n-2)} - \frac{E(A,A) - 2n^2 + n}{n(n-3)}. \end{aligned}$$

Then, if n is sufficiently large and A satisfies $E(A,A) = cn^2$ for some $c < 5/2$, we have $E(|d(a)|) > n-1$, and hence there exists a Hamiltonian cycle a of A with $|d(a)| = n$ as required.

4 A problem of Marco Buratti

In [1, 3, 4], the authors consider the following variant of Conjecture 1.

Fix an odd prime p, and consider the complete graph K_p on $\{0, \ldots, p-1\}$. Define the length of an edge $[x, y]$ of K_p as $l(x, y) = \min(|x-y|, p-|x-y|)$.

Marco Buratti posed the following conjecture:

Conjecture 8. Let $p = 2n+1$ be an odd prime, and L be a list of $2n$ elements, each from the set $\{1, 2, \ldots, n\}$. Then there exists an Hamiltonian path $a = (a_1, \ldots, a_p)$ on K_p such that $l(a) := \{l(a_1, a_2), \ldots, l(a_{p-1}, a_p)\} = L$.

A solution to this conjecture seems, at present day, very far. Only partial results, showing the existence of such Hamiltonian paths for specific families of lists have been obtained so far.

Clearly, Theorem 2 implies the validity of Conjecture 8 for the list $L = \{1, 1, 2, 2, \ldots, n, n\}$, that is the list where each element in $\{1, \ldots, n\}$ is present exactly two times.

Moreover, not only the conjecture holds in this case, but there exist at least p Hamiltonian paths which realize the list L, one for each starting point in $\{0, \ldots, p-1\}$.

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