Zeros of Jones polynomials of graphs

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Abstract

In this paper, we introduce the Jones polynomial of a graph $G = (V, E)$ with k components as the following specialization of the Tutte polynomial:

$$J_G(t) = (-1)^{|V|-k} t^{|E|-|V|+k} T_G(-t, -t^{-1}).$$

We first study its basic properties and determine certain extreme coefficients. Then we prove that $(-\infty, 0]$ is a zero-free interval of Jones polynomials of connected bridgeless graphs while for any small $\epsilon > 0$ or large $M > 0$, there is a zero of the Jones polynomial of a plane graph in $(0, \epsilon)$, $(1 - \epsilon, 1)$, $(1, 1 + \epsilon)$ or $(M, +\infty)$. Let $r(G)$ be the maximum moduli of zeros of $J_G(t)$. By applying Sokal's result on zeros of Potts model partition functions and Lucas's theorem, we prove that

$$\frac{q_s - |V| + 1}{|E|} \leqslant r(G) < 1 + 6.907652\Delta_G$$

for any connected bridgeless and loopless graph $G = (V, E)$ of maximum degree Δ_G with q_s parallel classes. As a consequence of the upper bound, X.-S. Lin's conjecture holds if the positive checkerboard graph of a connected alternating link has a fixed maximum degree and a sufficiently large number of edges.

Keywords: graph; Jones polynomial; real zeros; complex bound

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1 Introduction

Let $T_G(x, y)$ be the Tutte polynomial of the graph G [27]. We shall first give the definition of *the Jones polynomial* of a graph. Let $G = (V, E)$ be a graph with k components. The Jones polynomial of G is defined to be

$$J_G(t) = (-1)^{|V|-k} t^{|E|-|V|+k} T_G(-t, -t^{-1}). \tag{1}$$

This definition is motivated by the connection between the Jones polynomial [15] of an oriented connected alternating link and the Tutte polynomial of the positive plane graph formed from an alternating link diagram of the link via the classical checkerboard coloring. To be precise, let L be a connected oriented link which admits an alternating link diagram D. Let $V_L(t)$ be the Jones polynomial of L. Let $T_{G_+(D)}$ be the Tutte polynomial of $G_+(D)$, the positive checkerboard graph constructed from D. Then [24]

$$V_L(t) = (-1)^w t^{\frac{b-a+3w}{4}} T_{G_+(D)}(-t, -t^{-1}), \tag{2}$$

where w is the writhe of the link diagram D, a (resp. b) is the number of vertices (resp. regions) of $G_+(D)$.

Zeros of Jones polynomials of links (i.e. plane graphs) have been studied. In [28, 5, 9, 10], some families of links are considered. In [4, 13], the unit-circle theorem is obtained. In [14], the authors proved that zeros of Jones polynomials of pretzel links are dense in the whole complex plane. There has been a great deal of work on zeros of the chromatic polynomial (more generally, the Potts model partition function), which is also a partial evaluation of the Tutte polynomial. See, for example, [23] and references therein. In this paper, we try to use the method of studying (di)chromatic zeros to study zeros of the Jones polynomial of graphs, hoping to obtain some similar results.

In the next section we provide some basic properties of the Jones polynomial of graphs, some of which are specializations of properties of the Tutte polynomial or generalizations of properties of the Jones polynomial of alternating links (equivalently, planar graphs). We obtain several results on zeros of the Jones polynomial of graphs in Section 3. In particular, we prove that $(-\infty, 0]$ is a zero-free interval of the Jones polynomial of connected bridgeless graphs while for any small $\epsilon > 0$ or large $M > 0$, there is a zero of the Jones polynomial of a plane graph in $(0, \epsilon)$, $(1 - \epsilon, 1)$, $(1, 1 + \epsilon)$ or $(M, +\infty)$.

Let G be a connected graph. Let $z_1, z_2, \cdots, z_q$ be all the zeros of $J_G(t)$. Define the Jones spectral radius of G to be

$$r(G) = \max_{1 \leqslant i \leqslant q} \{|z_i|\}.$$

By using Sokal's result [22] and imitating Brown's method [3], we succeed in obtaining an upper bound and a lower bound for $r(G)$. Let $G = (V, E)$ be a connected bridgeless and loopless graph of maximum degree Δ_G with q_s parallel classes. In Section 4, we prove that

$$\frac{q_s - |V| + 1}{|E|} \leqslant r(G) < 1 + 6.907652\Delta_G. \tag{3}$$

As an application of the upper bound, we discuss a conjecture stated by X.-S. Lin's in [17], a paper partially written before he passed away in 2007. We conclude the paper in Section 5 with two unsolved problems for further study.

2 Basic properties

It is obvious that $J_{E_n}(t) = 1$ for the edgeless graph E_n with n vertices and when a graph contains at least one edge, we have the following deletion-contraction recurrence which can be used to compute the Jones polynomial for any graph and thus can also be viewed as an alternative definition.

Theorem 1. *Let $G = (V, E)$ be a graph and $e \in E$. Then*

$$J_G(t) = \begin{cases} tJ_{G/e}(t) & \text{if } e \text{ is a bridge of } G, \\ -J_{G-e}(t) & \text{if } e \text{ is a loop of } G, \\ tJ_{G-e}(t) - J_{G/e}(t) & \text{otherwise,} \end{cases}$$

where $G - e$ and G/e are graphs obtained from G by deleting and contracting the edge e, respectively. In particular, $J_G(t)$ is a polynomial with integer coefficients.

Proof. The result follows directly from the following deletion-contraction formula for the Tutte polynomial:

$$T_G(-t, -t^{-1}) = \begin{cases} -t\, T_{G/e}(-t, -t^{-1}) & \text{if } e \text{ is a bridge of } G, \\ -t^{-1}\, T_{G-e}(-t, -t^{-1}) & \text{if } e \text{ is a loop of } G, \\ T_{G-e}(-t, -t^{-1}) + T_{G/e}(-t, -t^{-1}) & \text{otherwise.} \end{cases}$$

See, for example, [2]. □

Theorem 2. *Let G be a plane graph and G^* be the planar dual of G. Then*

$$J_{G^*}(t) = (-t)^{|E|} J_G(t^{-1}). \tag{4}$$

Proof. This follows directly from the following duality formula for the Tutte polynomial:

$$T_{G^*}(x, y) = T_G(y, x).$$

See, for example, [2]. □

Theorem 3. *Let $G = (V, E)$ be a graph. Then the terms in $J_G(t)$ alternate in sign.*

Proof. Let $K_G(t) = (-1)^{|E|} J_G(-t)$. From Theorem 1, we obtain

$$K_G(t) = \begin{cases} tK_{G/e}(t) & \text{if } e \text{ is a bridge of } G, \\ K_{G-e}(t) & \text{if } e \text{ is a loop of } G, \\ tK_{G-e}(t) + K_{G/e}(t) & \text{otherwise.} \end{cases}$$

Now it suffices to show that the coefficients in $K_G(t)$ are all non-negative. We shall prove this by induction on the number of edges of G. It holds trivially when $G = E_n$. We suppose that it holds for all graphs with fewer than m edges and let G be a connected graph with m edges. By the above deletion-contraction formula for $K_G(t)$ and the inductive hypothesis, it is obvious that the coefficients of $K_G(t)$ are all non-negative. □

Remark 4. It is possible for some coefficient of $K_G(t)$ to be zero. For example, $K_{C_n}(t) = t^n + t^{n-1} + \cdots + t^2 + 1$ for the n-cycle C_n ($n \geqslant 2$). Furthermore, it is not difficult for us to derive from Theorems 14 and 15 in [2] that the coefficients of $K_G(t)$ are all positive for all connected non-separable graphs which are neither cycles nor dual graphs of cycles.

Let $G = (V, E)$ be a graph. A *parallel class* of G is a maximal subset of E all of which have the same endvertices. A *series class* of G is a maximal subset of E such that the removal of any two edges of the subset will increase the number of components of the graph. Both parallel classes and series classes partition the edge set E. In the case of a plane graph, a parallel class is exactly a series class of its dual graph. A parallel or series class will be called *trivial* if it contains only one edge. Let the Tutte polynomial evaluation of the graph G

$$T_G(-t, -t^{-1}) = a_n t^n + a_{n+1} t^{n+1} + \cdots + a_{m-1} t^{m-1} + a_m t^m, \tag{5}$$

with $a_n \neq 0, a_m \neq 0$ and $n \leqslant m$. In [6], Dasbach and Lin proved:

Lemma 5. *Let $G = (V, E)$ be a connected loopless graph with q_s parallel classes and $\tilde{q}_s$ nontrivial parallel classes. Then in (5), $m = |V| - 1$ and*

(1) $a_m = (-1)^{|V|-1}$,

(2) $a_{m-1} = (-1)^{|V|-1}(|V| - 1 - q_s)$,

(3) $a_{m-2} = (-1)^{|V|-1}\left[\binom{q_s - |V| + 2}{2} + \tilde{q}_s - tri\right]$, *where tri is the number of triangles of the parallel reduction obtained from G by deleting all edges but one for every parallel class of G.*

Proof. See Proposition 2.1 of [6]. By checking the proof, one finds that the 'planarity' condition in Proposition 2.1 is actually not needed and thus Lemma 5 follows. □

Now we investigate the value of n and the two coefficients a_n and a_{n+1} in Eq. (5), to obtain a 'dual' result of Lemma 5. Let $G = (V, E)$ be a connected bridgeless graph. $S \subset E$ is said to be a *pairwise-disconnecting set* if $|S| \geqslant 2$ and the deletion of any two members of S disconnects the graph.. The notion of pairwise-disconnecting set was introduced in [1]. The following three statements on pairwise-disconnecting sets are all obvious.

ST1: Any k-edge connected graph ($k \geqslant 3$) does not contain any pairwise-disconnecting set.

ST2: When $|S| = 2$, S is a pairwise-disconnecting set if and only if S is a 2-edge cut of G.

ST3: Any subset with cardinality greater than 1 of a pairwise-disconnecting set S is also a pairwise-disconnecting set.

Proposition 6. *Let $G = (V, E)$ be a connected bridgeless graph, $S \subset E$ and $|S| \geqslant 2$. Then the following are equivalent:*

- *S is a pairwise-disconnecting set.*
- *$S = \{e_1, e_2, \cdots, e_k\}$ is a set of edges in a cycle such that $G - S = G_1 \cup G_2 \cup \cdots \cup G_k$ and each G_i $(i = 1, 2, \cdots, k)$ is connected, as shown in Fig. 1.*
- *$k(G - S) = |S|$.*

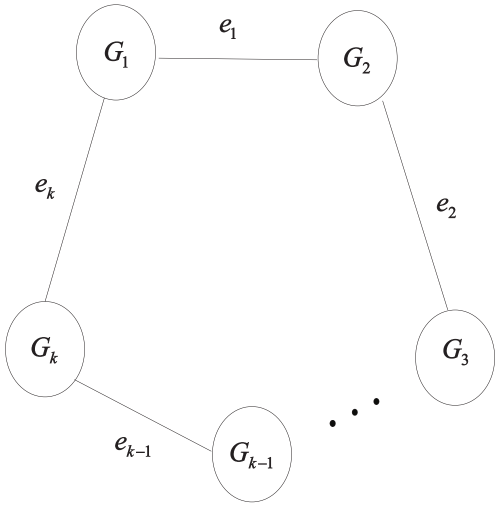

Fig. 1: $S = \{e_1, e_2, \cdots, e_k\}$ and $G - S = G_1 \cup G_2 \cup \cdots \cup G_k$ and each G_i $(i = 1, 2, \cdots, k)$ is connected.

Proof. We first prove that if $k(G - S) = |S|$, then all edges of S occur on a cycle of G as shown in Fig. 1. It holds when $|S| = 2$ and now we suppose $|S| \geqslant 3$ and $f \in S$. Because $k(G - S) = |S|$ we have $k(G - S + f) = |S - f|$ and f is a bridge of $G - S + f$. By the inductive hypothesis the edges $S - f$ occur on a cycle of G. Suppose that G becomes $G_1, G_2, \cdots, G_{|S-f|}$ when $S - f$ is deleted. Then f belongs to some G_i and is also a bridge of G_i. Hence, all edges of S occur on a cycle of G.

It is clear that if all edges of S occur on a cycle of G as shown in Fig. 1, then S is a pairwise-disconnecting set.

Finally we prove that if S is pairwise-disconnecting set, then $k(G - S) = |S|$. It holds when $|S| = 2$ and now we suppose $|S| \geqslant 3$ and $f \in S$. Then $S - f$ is also a pairwise-disconnecting set. By the inductive hypothesis we have $k(G - S + f) = |S - f|$. Let $g \in S - f$. Then $\{f, g\}$ is a 2-edge cut of G. Hence, f is a bridge of $G - g$ and also a bridge of $G - S + f$. Therefore, we have $k(G - S) = k(G - S + f) + 1 = |S - f| + 1 = |S|$. □

A pairwise-disconnecting set S_M is said to be *maximal* if no pairwise-disconnecting set contains it as a proper subset. Note that maximal pairwise-disconnecting sets are exactly nontrivial maximal series classes.

Proposition 7. *Let G be a connected bridgeless graph. For any given pairwise-disconnecting set S of G, there exists a unique maximal pairwise-disconnecting set S_M of G containing S.*

Proof. The existence follows from the definition of pairwise-disconnecting sets directly. To prove the uniqueness, we suppose that there are two distinct maximal pairwise-disconnecting sets S_M^1 and S_M^2 of G such that $S \subset S_M^i$ $(i = 1, 2)$. Let $e, f \in S$ and $g \in S_M^2 - S_M^1$. Then $\{e, f\}$ is a 2-edge cut of G. Suppose that $G - e - f = G_1 \cup G_2$. Without loss of generality we suppose that $g \in E(G_2)$. Since $e, f, g \in S_M^2$, $\{e, g\}$ is a 2-edge cut of G, which implies that g must be a bridge of G_2. We suppose that $G_2 - g = G_2' \cup G_2''$. See Fig. 2 (a). Let $h \in S_M^1 - e - f$. We shall show that $\{h, g\}$ is a 2-edge cut of G. Since $e, f, h \in S_M^1$, $\{e, h\}$ is a 2-edge cut of G, which implies that h must be a bridge of G_1 or G_2. There are two cases.
Case 1. h is a bridge of G_1. Suppose that $G_1 - h = G_1' \cup G_1''$, then $G - g - h$ is disconnected as shown in Fig. 2 (b).
Case 2. h is a bridge of G_2. Without loss of generality, we suppose that $h \in G_2'$, then h is also a bridge of G_2'. Suppose that $G_2' - h = G_{21}' \cup G_{22}'$, then $G - g - h$ is also disconnected as shown in Fig. 2 (c).
Hence, $S_M^1 \cup \{g\}$ is a pairwise-disconnecting set, which contradicts the maximality of S_M^1. $\square$

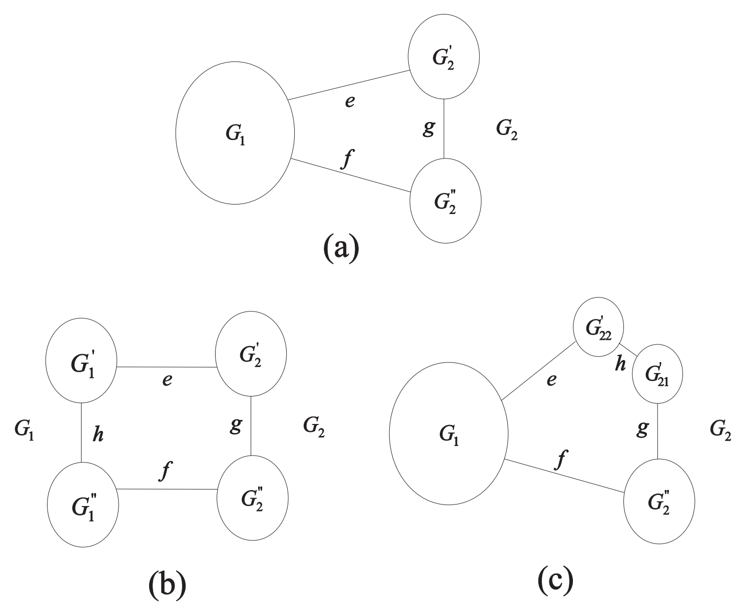

Fig. 2: The proof of Proposition 7.

Lemma 8. *Let $G = (V, E)$ be a connected bridgeless graph with q_r series classes. Then in (5), $n = -|E| + |V| - 1$ and*

(1) $a_n = (-1)^{|E|-|V|+1}$,

(2) $a_{n+1} = (-1)^{|E|-|V|+1}(-|V| + 1 + |E| - q_r)$.

Proof. Recall that

$$\begin{aligned} T_G(-t,-t^{-1}) &= \sum_{F\subseteq E}(-t-1)^{k(F)-1}(-t^{-1}-1)^{|F|-|V|+k(F)} \\ &= \sum_{F\subseteq E}(-1)^{|F|-|V|+1}(1+t)^{k(F)-1}(t^{-1}+1)^{|F|-|V|+k(F)}. \end{aligned}$$

It is clear that $k(F)-1 \geqslant 0$. Thus we obtain

$$(1+t)^{k(F)-1} = 1+(k(F)-1)t+\binom{k(F)-1}{2}t^2+\cdots.$$

Since $|F|-|V|+k(F)$ is the nullity of the subgraph (V,F) of $G=(V,E)$, $0 \leqslant |F|-|V|+k(F) \leqslant |E|-|V|+1$. Now

$$\begin{aligned} (t^{-1}+1)^{|F|-|V|+k(F)} &= t^{-(|F|-|V|+k(F))}+(|F|-|V|+k(F))t^{-(|F|-|V|+k(F)-1)} \\ &+ \binom{|F|-|V|+k(F)}{2}t^{-(|F|-|V|+k(F)-2)}+\cdots. \end{aligned}$$

Note that G is connected and bridgeless, so $|F|-|V|+k(F)=|E|-|V|+1$ if and only if $F=E$. Hence, $n=-|E|+|V|-1$ and $a_n=(-1)^{|E|-|V|+1}$. Furthermore, $|F|-|V|+k(F)=|E|-|V|$ if and only if $F=E-e$ for $e\in E$ or, by Proposition 6 $F=E-S$, where S is a pairwise-disconnecting set of G. Thus,

$$\begin{aligned} a_{n+1} &= (-1)^{|E|-|V|+1}(|E|-|V|+1)+(-1)^{|E|-|V|}|E|+\sum_{E-S}(-1)^{|E-S|-|V|+1} \\ &= (-1)^{|E|-|V|+1}(-|V|+1)+\sum_{E-S_M}\sum_{S\subset S_M}(-1)^{|E-S|-|V|+1} \\ &\quad \text{(By Proposition 7)} \\ &= (-1)^{|E|-|V|+1}(-|V|+1+\sum_{S_M}(|S_M|-1)) \\ &= (-1)^{|E|-|V|+1}(-|V|+1+|E|-q_r). \end{aligned}$$

□

Theorem 9. *Let $G=(V,E)$ be a connected bridgeless and loopless graph. Then, with notation as above,*

$$J_G(t) = b_0+b_1t+b_2t^2+\cdots+b_{|E|-2}t^{|E|-2}+b_{|E|-1}t^{|E|-1}+b_{|E|}t^{|E|}, \tag{6}$$

where $(-1)^{|E|-i}b_i$ is a non-negative integer for $i=0,1,2,\cdots,|E|$ and in particular,

$$\begin{aligned} b_0 &= (-1)^{|E|}, \\ b_1 &= (-1)^{|E|}(-|V|+1+|E|-q_r), \\ b_{|E|-2} &= \binom{q_s-|V|+2}{2}+\tilde{q}_s-tri, \\ b_{|E|-1} &= |V|-1-q_s, \\ b_{|E|} &= 1. \end{aligned}$$

Proof. This follows from Theorems 1 and 3, and Lemmas 5 and 8. □

Lemma 10. *Let G be a connected plane graph. Then*

$$J_G(1) = (-1)^{|E|-|V|+1}(-2)^{\mu(G)-1}, \tag{7}$$

where $\mu(G)$ is the number of left-right paths of G (or equivalently, the number of crossing circuits of the medial graph $M(G)$ of G and the number of components of the link represented by $D(G)$).

Lemma 10 follows from results by Martin [18] and Las Vergnas [16], see also [19]. In fact, $\mu(G) - 1$ is exactly the dimension of the bicycle space of G [20, 21].

3 Real and non-real zeros

Let G be a connected bridgeless and loopless graph with p vertices, q edges, and q_s parallel classes. Let $z_k = x_k + iy_k$ $(k = 1, 2, \cdots, q)$ be zeros of $J_G(t)$ of the graph G. Then, according to Theorem 9, we have

(1) $J_G(t) = \prod_{k=1}^{q}(t - z_k)$.

(2) Let $\sum_+ z_k$ (resp. $\sum_- z_k$) indicates the sum of those z_k, $k = 1, 2, \cdots, q$ whose real parts are positive (resp. negative). Then $\sum_+ z_k$ (resp. $\sum_- z_k$) is a positive (resp. negative) real number.

(3) $\sum z_k = q_s - p + 1 \geqslant 0$.

(4) $\prod_{k=1}^{q} z_k = 1$.

Now we start to discuss the real and non-real zeros of the Jones polynomial of a graph.

Theorem 11. *Let G be a connected bridgeless and loopless graph. Then*

(1) $J_G(t)$ has no negative real zeros.

(2) $J_G(t)$ has no rational zeros.

Proof. Theorem 11 (1) follows directly from Theorem 3. Note that the extreme coefficients of $J_G(t)$ are both ± 1, so by the Rational Root Theorem, only ± 1's may be possible rational roots. However 1 is not a zero by Lemma 10 and -1 is not a zero either by Theorem 11 (1). Thus Theorem 11 (2) holds. □

Remark 12. The results in Theorem 11 on zeros of the Jones polynomial of knots and links, i.e. the case of the Jones polynomial of a planar graph have been obtained previously. See Theorem 3.1 in [25].

Corollary 13. *Let G be a connected bridgeless and loopless graph. If the number of edges of G is odd, then $J_G(t)$ has a (positive and irrational) real zero.*

It is natural to look for a small positive ϵ such that any $z \in (0, \epsilon)$ is not a zero of the Jones polynomials of a graph. In the following we shall show such an ϵ does not exist.

Let G_n be the 3-regular planar graph shown in the upper part of Fig. 3. It is called the Gustin current graph, see [7]. Note that G_n has $2n+2$ vertices and $3n+3$ edges. We will show that for any $\epsilon > 0$, there exists an integer n such that $J_{G_n}(t)$ has a root in $(0, \epsilon)$.

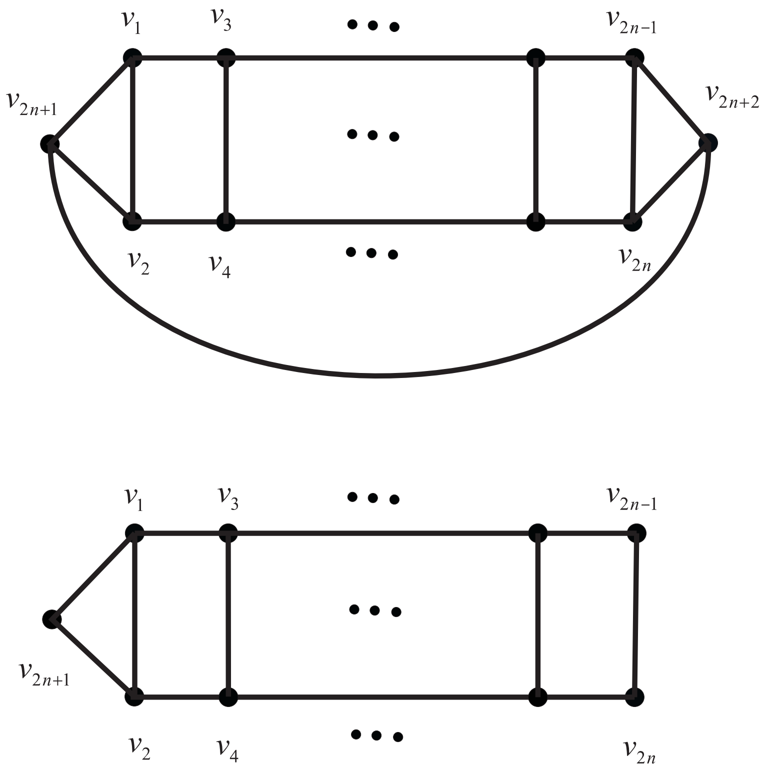

Fig. 3: The graph G_n (upper) and the graph H_n (down)

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Let $N_G(t) = (-1)^{|E|} J_G(t) = (-1)^{|E|+|V|-1} t^{|E|-|V|+1} T_G(-t, -t^{-1})$. Then $N_G(t)$ can be obtained using the following recurrence relations:

$$N_G(t) = \begin{cases} 1, & \text{if } G = K_1; \\ -tN_{G/e}(t), & \text{if } e \text{ is a bridge of } G; \\ N_{G-e}(t), & \text{if } e \text{ is a loop of } G; \\ -tN_{G-e}(t) + N_{G/e}(t), & \text{if } e \text{ is not a bridge nor a loop of } G. \end{cases}$$

Note that if G is bridgeless, $N_G(t)$ is a polynomial with constant term equal to 1, i.e., $N_G(0) = 1$. Using this property, we see that $N_G(t)$ has a root in $(0, a)$ if $N_G(a) < 0$, and $a > 0$.

Now we want to find an expression for $N_{G_n}(t)$. For this purpose, we need to make use of another two graphs H_n and Q_n. Let H_n be the graph, shown in lower part of Fig. 3, obtained from G_n by deleting the vertex v_{2n+2} and Q_n be the graph obtained from H_n by adding an edge parallel to the edge joining v_{2n-1} and v_{2n}.

Lemma 14.

$$N_{H_1}(t) = 1 + t^2 - t^3,$$

$$N_{H_2}(t) = 1 - t + t^2 - t^3 + t^4 + t^2(1-t)N_{H_1}(t),$$

and for $n \geqslant 3$, we have

$$N_{H_n}(t) = (1-t)(1+t^2)N_{H_{n-1}}(t) + t^3 N_{H_{n-2}}(t). \tag{8}$$

Proof. For convenience let $h_n = N_{H_n}(t)$ and $q_n = N_{Q_n}(t)$. Note that H_1 is actually K_3 and so we have $h_1 = 1 + t^2 - t^3$. When $n \geqslant 2$, we have

$$h_n = (t^2 - t^3)h_{n-1} + q_{n-1}$$

and

$$q_n = t^2 h_{n-1} - t h_n + q_{n-1}.$$

It is easy to check that $q_1 = 1 - t + t^2 - t^3 + t^4$. Thus $h_2 = 1 - t + t^2 - t^3 + t^4 + t^2(1-t)h_1$. For $n \geqslant 2$, we have

$$\begin{aligned} h_{n+1} - h_n &= t^2(1-t)(h_n - h_{n-1}) + (q_n - q_{n-1}) \\ &= t^2(1-t)(h_n - h_{n-1}) + t^2 h_{n-1} - t h_n, \end{aligned}$$

implying that

$$h_{n+1} = (1-t)(1+t^2)h_n + t^3 h_{n-1}.$$

□

Lemma 15. *For $n \geqslant 3$,*

$$N_{G_n}(t) = (1-t)^2 N_{G_{n-1}}(t) - t^2 N_{G_{n-2}}(t) - t^3 N_{H_n}(t) - t^5 N_{H_{n-1}}(t).$$

Proof. We write g_n for $N_{G_n}(t)$. For $i = 1, 2$, let e_i be the edge joining vertices v_{2n+2} and v_{2n-2+i}. Then for $n \geqslant 2$,

$$g_n = N_{G_n/e_1/e_2}(t) - tN_{(G_n-e_1)/e_2}(t) - tN_{(G_n/e_1)-e_2}(t) + t^2 N_{G_n-e_1-e_2}(t).$$

Note that $N_{G_n/e_1/e_2}(t) = g_{n-1}$, $N_{G_n-e_1-e_2}(t) = -tN_{H_n}(t)$ and $N_{(G_n-e_1)/e_2}(t) = N_{(G_n/e_1)-e_2}(t) = N_{T_n}(t)$, where T_n is the graph obtained from H_n by adding an edge joining v_{2n} and v_{2n+1}. Thus

$$g_n = g_{n-1} - 2tN_{T_n}(t) - t^3 h_n.$$

For $n \geqslant 2$, we also have

$$\begin{aligned} N_{T_n}(t) &= N_{T_n/e_3}(t) - tN_{T_n-e_3}(t) \\ &= g_{n-1} + t^2 N_{(T_n-e_3)/e_4}(t) \\ &= g_{n-1} + t^2(N_{T_{n-1}}(t) + t^2 h_{n-1}), \end{aligned}$$

implying that $N_{T_n}(t) - t^2 N_{T_{n-1}}(t) = g_{n-1} + t^4 h_{n-1}$, where e_3 is the edge joining v_{2n-1} and v_{2n} and e_4 is the edge joining v_{2n-1} and v_{2n-3}. Hence, for $n \geqslant 2$, we have

$$\begin{aligned}
g_{n+1} - t^2 g_n &= g_n - 2tN_{T_{n+1}}(t) - t^3 h_{n+1} - t^2(g_{n-1} - 2tN_{T_n}(t) - t^3 h_n) \\
&= g_n - t^3 h_{n+1} - t^2(g_{n-1} - t^3 h_n) - 2t(N_{T_{n+1}}(t) - t^2 N_{T_n}(t)) \\
&= g_n - t^3 h_{n+1} - t^2(g_{n-1} - t^3 h_n) - 2t(g_n + t^4 h_n),
\end{aligned}$$

implying that

$$g_{n+1} = (1-t)^2 g_n - t^2 g_{n-1} - t^3 h_{n+1} - t^5 h_n. \qquad \square$$

Lemma 14 implies that $N_{H_n}(t) > 0$ for all $n \geqslant 1$ when $t \in (0,1)$. So, by Lemma 14, for all $n \geqslant 3$,

$$N_{H_n}(t) > (1-t) N_{H_{n-1}}(t),$$

implying that

$$N_{H_n}(t) > (1-t)^{n-2} N_{H_2}(t)$$

and of course

$$N_{H_n}(t) + t^2 N_{H_{n-1}}(t) > (1-t)^{n-2} N_{H_2}(t).$$

Lemma 16. *For any $t \in (0,1)$, there exists m such that $N_{G_m}(t) < 0$.*

Proof. Let $t \in (0,1)$ be fixed. Suppose that the result fails. Thus $g_n \geqslant 0$ for all n. Then, by Lemma 15, for all $n \geqslant 3$,

$$\begin{aligned}
g_n &\leqslant (1-t)^2 g_{n-1} - t^3 h_n - t^5 h_{n-1} \\
&\leqslant (1-t)^2 g_{n-1} - t^3 (1-t)^{n-2} h_2,
\end{aligned}$$

implying that

$$\begin{aligned}
g_n &\leqslant (1-t)^{2(n-2)} g_2 - t^3 h_2 \sum_{i=1}^{n-2} (1-t)^{2(n-2-i)} (1-t)^i \\
&= (1-t)^{2(n-2)} g_2 - t^3 (1-t)^{n-2} h_2 \sum_{i=1}^{n-2} (1-t)^{n-2-i} \\
&= (1-t)^{2(n-2)} g_2 - t^2 (1-t)^{n-2} h_2 (1 - (1-t)^{n-2}) \\
&= (1-t)^{n-2} \left((1-t)^{n-2} (g_2 + t^2 h_2) - t^2 h_2 \right),
\end{aligned}$$

where the last expression will be negative when n is sufficiently large because $t^2 h_2 > 0$, a contradiction. $\square$

Theorem 17. *For any $0 < \epsilon < 1$, there is an integer $m > 0$ such that $N_{G_m}(t)$ has a zero in $(0, \epsilon)$.*

Proof. Note that $N_{G_n}(0) = 1$ for all $n \geqslant 1$. By Lemma 16, there is an integer m such that $N_{G_m}(\epsilon) < 0$. Thus the polynomial $N_{G_m}(t)$ has a zero in $(0, \epsilon)$. $\square$

In Lemma 14, if we assume the linearly recursive relation (8) holds for $n = 2$, then $N_{H_0}(t) = 1$. The characteristic equation of the linearly recursive relation (8) is

$$x^2 - (1-t)(1+t^2)x - t^3 = 0.$$

For any fixed real $t > 1$, it has two non-equal solutions L_1 and L_2:

$$\begin{aligned} L_1(t) &= \frac{1}{2}[(1-t)(t^2+1) + \sqrt{t^6 - 2t^5 + 3t^4 + 3t^2 - 2t + 1}] \\ L_2(t) &= \frac{1}{2}[(1-t)(t^2+1) - \sqrt{t^6 - 2t^5 + 3t^4 + 3t^2 - 2t + 1}]. \end{aligned}$$

Thus, there exist $A(t)$ and $B(t)$ such that for all $n \geqslant 0$,

$$N_{H_n}(t) = A(t)[L_1(t)]^n + B(t)[L_2(t)]^n.$$

By the given condition on $N_{H_0}(t)$ and $N_{H_1}(t)$, it is easy to find $A(t)$ and $B(t)$:

$$A(t) = \frac{1 + t^2 - t^3 - L_2(t)}{L_1(t) - L_2(t)}; \quad B(t) = -\frac{1 + t^2 - t^3 - L_1(t)}{L_1(t) - L_2(t)}.$$

Now we claim:

For any real $t > 1$, $A(t) > 0$, $B(t) > 0$, $L_1(t) > 0$ and $L_2(t) < 0$.

The above claim is shown. As an example, we show that $B(t) > 0$. It is clear that $L_1(t) - L_2(t) > 0$ for $t > 1$. So $B(t) > 0$ if and only if $L_1(t) - (1 + t^2 - t^3) > 0$. Observe that

$$L_1(t) - (1 + t^2 - t^3) = \frac{1}{2}\left(t^3 - t^2 - t - 1 + \sqrt{t^6 - 2t^5 + 3t^4 + 3t^2 - 2t + 1}\right)$$

and

$$\begin{aligned} & t^3 - t^2 - t - 1 + \sqrt{t^6 - 2t^5 + 3t^4 + 3t^2 - 2t + 1} \\ = \ & (t-1)^2(t+1) - 2 + \sqrt{t^6 - 2t^5 + 3t^4 + 3t^2 - 2t + 1} \\ = \ & (t-1)^2(t+1) - 2 + \sqrt{4 + (t-1)(t^5 - t^4 + 2t^3 + 2t^2 + 5t + 3)} \\ > \ & (t-1)^2(t+1) - 2 + \sqrt{4} \\ > \ & 0 \end{aligned}$$

for $t > 1$.

Since $L_1(t) + L_2(t) = (1-t)(1+t^2) < 0, L_1(t) > 0$ and $L_2(t) < 0$ for $t > 1$, we have $\frac{L_2(t)}{L_1(t)} < -1$. Hence, for any fixed $t > 1$,

$$\begin{aligned} N_{H_n}(t) &= A(t)[L_1(t)]^n + B(t)[L_2(t)]^n \\ &= [L_1(t)]^n \left[A(t) + B(t)\left(\frac{L_2(t)}{L_1(t)}\right)^n\right] \\ &< 0 \end{aligned}$$

for all sufficiently large odd n. In addition, by Lemma 14, it is easy to prove $N_{H_n}(1) = 1$ for all $n \geqslant 0$. Thus,

Theorem 18. *For any $\epsilon > 0$, there exists an integer m such that for all odd $n \geqslant m$, $N_{H_n}(t)$ has a real zero in $(1, 1+\epsilon)$.*

Remark 19. By Theorem 2, for any small ϵ, there is a real zero of a Jones polynomial in $(1-\epsilon, 1)$ and for any large $M > 0$, there is a real zero in $(M, +\infty)$.

Now we consider non-real zeros of Jones polynomials.

Theorem 20. *Let G be a connected bridgeless and loopless nontrivial graph with notations above. If*

$$tri < \binom{q_s - p + 2}{2} + \tilde{q}_s - \frac{q-1}{2q}(p-1-q_s)^2,$$

then $J_G(t)$ has a non-real zero.

Proof. Theorem 20 follows from the following theorem (see [8]): let $f(x) = \sum_{k=0}^{n} a_k x^k = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ be a polynomial of degree $n \geqslant 2$ with real coefficients, a necessary condition for all zeros of $f(x)$ to be real is $\frac{k}{n-k+1}\frac{n-k}{k+1}a_k^2 - a_{k-1}a_{k+1} \geqslant 0$ for each $k = 1, 2, \cdots, n-1$. Taking $k = n-1$, if $\frac{n-1}{2n}a_{n-1}^2 - a_n a_{n-2} < 0$ which is equivalent to the condition in Theorem 20 for $J_G(t)$ via Theorem 9, then $J_G(t)$ has a non-real zero. □

Corollary 21. *Let G be a connected bridgeless simple triangle-free non-trivial graph. Then $J_G(t)$ has a non-real zero.*

Proof. In this case the parallel reduction of G is itself, $q_s = q$ and $\tilde{q}_s = 0$. Note that $c(G) = q - p + 1 > 0$ since G is bridgeless and non-trivial. Thus,

$$\begin{aligned} & \binom{q_s - p + 2}{2} + \tilde{q}_s - \frac{q-1}{2q}(p-1-q_s)^2 \\ = & \binom{c(G)+1}{2} - \frac{q-1}{2q}c^2(G) \\ = & \frac{1}{2q}c^2(G) + \frac{c(G)}{2} \\ > & 0. \end{aligned}$$

By Theorem 20, Corollary 21 holds. □

4 Complex bounds

In this section, we consider lower and upper bounds for $r(G)$.

Theorem 22. *Let G be a connected bridgeless and loopless graph with p vertices, q edges, q_s parallel classes and $\tilde{q}_s$ nontrivial parallel classes. Let $\Delta = (q-1)^2(q_s - p + 1)^2 - 2q(q-1)\left[\binom{q_s - p + 2}{2} + \tilde{q}_s - tri\right]$, where tri is the number of triangles of the parallel reduction of G. Then*

(1) $J_G(t)$ *has a root* z *whose real part is at least* $\frac{q_s-p+1}{q}+\frac{\sqrt{\Delta}}{q(q-1)}$ *if* $\Delta \geqslant 0$*;*

(2) $J_G(t)$ *has two (not necessarily distinct) roots* z_1 *and* z_2 *such that the real part of* z_1 *is at least* $\frac{q_s-p+1}{q}$ *and the imaginary part of* z_2 *is at least* $\frac{\sqrt{-\Delta}}{q(q-1)}$ *if* $\Delta < 0$*.*

Proof. Note that in (6), $b_q = 1$,$b_{q-1} = (p-1-q_s)$ and $b_{q-2} = \left(\begin{array}{c} q_s - p + 2 \\ 2 \end{array}\right) + \tilde{q}_s - tri$. Consider the $(q-2)th$ derivative of $J_G(t)$. We obtain

$$\frac{1}{(q-2)!} J_G^{(q-2)}(t) = \frac{q(q-1)}{2} t^2 - (q-1)(q_s - p + 1)t + b_{q-2}.$$

Then $\frac{q_s-p+1}{q}+\frac{\sqrt{\Delta}}{q(q-1)}$ is one of the roots of the above quadratic. Lucas's theorem states that if f is a nonconstant polynomial, then the roots of the derivative f' lie in the convex hull of the roots of f. It follows that the theorem holds. □

As a direct consequence, we have:

Corollary 23. *Let* $G = (V, E)$ *be a connected bridgeless and loopless graph with* q_s *parallel classes. Then*

$$r(G) \geqslant \frac{q_s - |V| + 1}{|E|}. \tag{9}$$

Finally we try to find an upper bound. Let $G = (V, E)$ be a graph. Assign to each edge $e \in E$ a complex weight v_e. Let $Z_G(q, \{v_e\})$ be the Potts model partition function, i.e.

$$Z_G(q, \{v_e\}) = \sum_{F \subset E} q^{k(F)} \prod_{e \in F} v_e, \tag{10}$$

where the summation is over all subsets of E, and $k(F)$ is the number of components of the spanning subgraph of G with edge set F.

In [22], Sokal proved the following important result (Corollary 5.2) on zeros of the Potts model partition function:

Theorem 24. *Let* $G = (V, E)$ *be a loopless graph of maximum degree* $\leqslant r$*, equipped with complex edge weights* $\{v_e\}_{e\in E}$ *satisfying* $|1 + v_e| \leqslant 1$ *for all* e*. Let* $v_{\max} = \max_{e\in E} |v_e|$*. Then all the zeros of* $Z_G(q, \{v_e\})$ *lie in the disc* $|q| < C(r)v_{\max}$*, where* $C(r) < Kr$ *with* $K \leqslant 7.963907$*.*

Remark 25. In [12], the upper bound of the value of K has been improved to approximately 6.907652.

Lemma 26. *Let* $G = (V, E)$ *be a connected graph. Then*

$$J_G(t) = t^{|E|+1}(1 + t)^{-|V|-1} Z_G(t + 2 + t^{-1}, -t^{-1} - 1). \tag{11}$$

Proof. From the rank-generating form of the Tutte polynomial [2], we obtain

$$\begin{aligned} J_G(t) &= (-1)^{|V|-1}t^{|E|-|V|+1}\sum_{F\subset E}(-t-1)^{k(F)-1}(-t^{-1}-1)^{|F|-|V|+k(F)} \\ &= t^{|E|-|V|+1}(1+t)^{-1}\sum_{F\subset E}(-1)^{|F|}(t+1)^{k(F)}(t^{-1}+1)^{|F|-|V|+k(F)} \\ &= t^{|E|-|V|+1}(t+1)^{-1}\sum_{F\subset E}(-1)^{|F|}(t+2+t^{-1})^{k(F)}(t^{-1}+1)^{|F|-|V|} \\ &= t^{|E|-|V|+1}(t+1)^{-1}(t^{-1}+1)^{-|V|}\sum_{F\subset E}(t+2+t^{-1})^{k(F)}(-t^{-1}-1)^{|F|} \\ &= t^{|E|+1}(1+t)^{-|V|-1}Z_G(t+2+t^{-1},-t^{-1}-1). \end{aligned}$$

□

Theorem 27. *Let $G=(V,E)$ be a connected bridgeless and loopless graph of maximum degree Δ_G. Let z be any complex zero of $J_G(t)$. Then $|z+1|<6.907652\Delta_G$.*

Proof. By Lemma 26, it suffices to show that any zero z of $Z_G(t+2+t^{-1},-t^{-1}-1)$ satisfies $|z+1|<6.907652\Delta_G$. Let z be any fixed zero of $Z_G(t+2+t^{-1},-t^{-1}-1)$. Note that $\Delta_G\geqslant 2$, hence if $|z|\leqslant 1$, it is clear that $|z+1|<6.907652\Delta_G$. Now we suppose that $|z|>1$. Note that $|z|>1$ implies that $|-z^{-1}|\leqslant 1$. By Theorem 24 and the following Remark 25, we have $|z+2+z^{-1}|<6.907652\Delta_G|-z^{-1}-1|$, which implies $|z+1|<6.907652\Delta_G$. □

Corollary 28. *Let G be a connected bridgeless and loopless graph of maximum degree Δ_G. Then*

$$r(G)<1+6.907652\Delta_G. \tag{12}$$

We have mentioned in the introduction that if we restrict $J_G(t)$ to connected plane graphs, we obtain the Jones polynomial of the corresponding alternating links up to a factor $\pm t^k$ for some k. See Eq. (2). In this case Theorem 11 will reduce to a (partial) result of Theorem 3.1 in [25].

Let L be a connected reduced (equivalently, the corresponding checkerboard graph is bridgeless and loopless) alternating link with N crossings, $\lambda_1,\lambda_2,\cdots,\lambda_N$ be zeros of $V_L(t)$ (treated as a polynomial in the common sense such that $V_L(0)\neq 0$) and let $\parallel L\parallel=|\lambda_1|+|\lambda_2|+\cdots+|\lambda_N|$. In [17], X.-S. Lin claimed that $\parallel L\parallel\geqslant N$ and posed the following conjecture:

Conjecture 29. $\parallel L\parallel$ has an upper bound in the order of $N^{1+\epsilon}$.

Some experimental data on some families of alternating links in favor of the conjecture have been given by X.-S. Lin himself in [17] and the conjecture also holds for an alternating link whose positive checkerboard graph can be obtained from a connected plane graph by subdividing some of its edges uniformly and the times of subdivision are large enough [13]. Now we apply Corollary 28 to provide more evidence for the conjecture.

Suppose that L is a connected reduced link diagram and its corresponding positive checkerboard graph is G which is connected, bridgeless and loopless. Note that the number N of crossings of L is exactly the number of edges of G. Then

Corollary 30. *For any fixed* $\epsilon > 0$, *if* $N = |E(G)| > (1 + 7\Delta_G)^{\frac{1}{\epsilon}}$, *then* $\parallel L \parallel < N^{1+\epsilon}$.

A connected plane graph with a fixed maximum degree and edge number large enough exists widely. For example, the dual of a triangulation (i.e. maximal planar graph) is 3-regular and the number of its edges can be large enough, the cylindrical square lattice $P_m \times C_n$ whose maximum degree is 4 and edge number is $(2m-1)n$. In the case of $P_{\frac{n+1}{2}} \times C_n$, it corresponds exactly to the 'weaving' knot J_n in [17].

5 Concluding remarks

It is known (see [11, 26]) that no graph has a root of its chromatic polyno- mial in $(-\infty, 0) \cup (0, 1) \cup (1, 32/27]$, and that for any $\lambda \geqslant 32/27$, there is a graph whose chromatic polynomial has a real root arbitrary close to λ. It is natural to pose the following problem for further consideration.

Open Problem 31. Characterize other zero-free intervals of Jones polynomials of connected bridgeless and loopless graphs in the real axis or prove that real zeros of Jones polynomials of connected bridgeless and loopless graphs are dense in the positive real axis.

Open Problem 32. Does there exist a connected bridgeless and loopless graph G such that all the zeros of $J_G(t)$ are real?

By now, we have not found a connected bridgeless and loopless graph for which all the zeros of the Jones polynomial are real.

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