

On the largest component of a hyperbolic model of complex networks

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Abstract

We consider a model for complex networks that was introduced by Krioukov et al. In this model, N points are chosen randomly inside a disk on the hyperbolic plane and any two of them are joined by an edge if they are within a certain hyperbolic distance. The N points are distributed according to a *quasi-uniform* distribution, which is a distorted version of the uniform distribution. The model turns out to behave similarly to the well-known Chung-Lu model, but without the independence between the edges. Namely, it exhibits a power-law degree sequence and small distances but, unlike the Chung-Lu model and many other well-known models for complex networks, it also exhibits clustering.

The model is controlled by two parameters α and ν where, roughly speaking, α controls the exponent of the power-law and ν controls the average degree. The present paper focuses on the evolution of the component structure of the random graph. We show that **(a)** for $\alpha > 1$ and ν arbitrary, with high probability, as the number of vertices grows, the largest component of the random graph has sublinear order; **(b)** for $\alpha < 1$ and ν arbitrary with high probability there is a “giant” component of linear order, and **(c)** when $\alpha = 1$ then there is a non-trivial phase transition for the existence of a linear-sized component in terms of ν .

Keywords: random graphs on the hyperbolic plane; component structure; giant component; phase transition.

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1 Introduction

The term “*complex networks*” describes a class of large networks which exhibit the following fundamental properties:

1. they are *sparse*, that is, the number of their edges is proportional to the number of nodes;
2. they exhibit the *small world phenomenon*: almost all pairs of vertices that are in the same component are within a short distance from each other;
3. *clustering* is present: two nodes of the network that have a common neighbour are somewhat more likely to be connected with each other;
4. their degree distribution is *scale free*. This means that its tail follows a *power law*. There has been extensive experimental evidence (see for example [2]) which suggests that many networks that emerge in applications have a degree distribution whose tail follows a power law with exponent between 2 and 3.

The books of Chung and Lu [10] and of Dorogovtsev [7] are excellent references for a detailed discussion of these properties.

During the last 15 years a number of models have been developed in a series of attempts to capture these features. Among the first such models is the *preferential attachment model*. This is a class of models of randomly growing graphs whose aim is to capture a basic feature of such networks: nodes which are already popular tend to become more popular as the network grows. It was introduced by Barabási and Albert [2] and subsequently defined and studied rigorously by Bollobás, Riordan and co-authors (see for example [6], [5]).

Another extensively studied model was defined by Chung and Lu [8], [9]. Here every vertex has a weight which effectively corresponds to its expected degree and every two vertices are joined *independently* of every other pair with probability that is proportional to the product of their weights. If these weights follow a power-law distribution, then it turns out that the resulting random graph has power-law degree distribution. This model is a special case of an *inhomogeneous random graph*.

All these models have their shortcomings. Namely, none of them succeeds in incorporating *all* the above features. For example, the Chung–Lu model exhibits a power law degree distribution (provided the weights of the vertices are suitably chosen) and average distance of order $O(\log \log N)$ (when the exponent of the power law is between 2 and 3, see [8]), but it does not exhibit clustering. This is also the situation in the Barabási–Albert model.

In the case of the Chung–Lu model the absence of clustering is essentially due to the fact that pairs of vertices form edges independently. On the contrary, the presence of clustering requires the edges not to appear independently. If two edges share a common endvertex, then the probability that their other two endvertices are joined must be higher compared to that where we assume nothing about these edges. This property is naturally present in random graphs that are created over metric spaces, such as random geometric

graphs. In this context, the vertices are a random set of points in a given metric space and any two of them are adjacent if their distance is smaller than a certain distance.

Recently Krioukov et al. [19] introduced a model which naturally exhibits these typical features. In this model, a random network is created on the hyperbolic plane (we will see the detailed definition shortly). In particular, Krioukov et al. [19] determined the degree distribution showing that it is *scale free* and its tail follows a power law, whose exponent is determined by one of the parameters of the model. The exponent can take any value that is at least 2. Furthermore, they consider the clustering properties of the resulting random network. A numerical approach in [19] suggests that the (local) clustering coefficient is positive and it is determined by one of the parameters of the model. These characteristics have been verified rigorously by Gugelmann, Panagiotou and Peter [12].

The basic hypothesis of Krioukov et al. [19] was that the hyperbolic geometry underlies complex networks. In particular, the heterogeneity of the nodes, whose expression is the power law degree distribution, is in fact the expression of an underlying hyperbolic geometry. Complex networks do exhibit some sort of hierarchy in the sense that their nodes/members form groups which are further organised into subgroups etc. Thus, there is a hidden tree-like structure and the hyperbolic geometry is the natural space which can accommodate such a structure.

The aim of the present work is the study the component structure of such a random graph and more specifically the number of vertices that are contained in a largest component of the graph. One of our main findings is that for the range of the parameters of the random graph model where the exponent of the power law is larger than 3, the random graph typically consists of many relatively small components, no matter how large the average degree of the graph is. This is in sharp contrast with the classical Erdős-Rényi model (see [3]) as well as with the situation on random geometric graphs on Euclidean spaces (see [20]) where the dependence on the average degree is crucial. However, we show that the structure of the random geometric graph is significantly different when the exponent of the power law is smaller than 3. In fact, we show that in this case a *giant* component exists with high probability, that is, a component containing a positive fraction of the vertices of the random graph.

1.1 The model

We start by recalling some facts about the hyperbolic plane \mathbb{H} . The hyperbolic plane is an unbounded surface of constant negative curvature -1 . There are several ways to represent it in two dimensions, including the half-plane model, the Beltrami-Klein disk model and the Poincaré disk model. In the Poincaré disk model, we equip the unit disk $\mathbb{D} := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ with the metric determined¹ by the differential form $ds^2 = 4 \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}$. The book of Stillwell [22] covers the basic theory of hyperbolic geometry. In this paper we find it helpful to draw pictures in the *native model* of \mathbb{H} . This is obtained from the Poincaré disk model by multiplying each point $(x, y) \in \mathbb{D}$ by

¹This means that the length of a curve $\gamma : [0, 1] \rightarrow \mathbb{D}$ is given by $\int_0^1 \frac{\sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}}{1 - \gamma_1^2(t) - \gamma_2^2(t)} dt$.

a scalar that equals the ratio of the distance to the origin in the hyperbolic metric, over the distance to the origin in the euclidean metric (in the case of the origin itself we define this ratio to equal one). This produces a model of \mathbb{H} that fills all of \mathbb{R}^2 . It lacks many of the properties that make the classical models of \mathbb{H} elegant to work with, but it does allow us to see more detail in visualizations of the graph model we are about to introduce. To the best of our knowledge the native model was first introduced by Krioukov et al. [19].

Basic facts about \mathbb{H} that we will rely on heavily in the paper are that in \mathbb{H} a disk of radius r (i.e. the set of points at hyperbolic distance at most r from a given point) has area equal to $2\pi(\cosh(r) - 1)$ and circumference length equal to $2\pi \sinh(r)$. Another important fact that we will rely on in the paper is the *hyperbolic cosine rule*. It states that if A, B, C are distinct points on the hyperbolic plane, and we denote by a the distance between B, C , by b the distance between A, C , by c the distance between A, B and by γ the angle (at C) between the shortest AC - and BC -paths, then $\cosh(c) = \cosh(a) \cosh(b) - \cos(\gamma) \sinh(a) \sinh(b)$.

We are now ready to introduce the model we will be studying in this paper. We will name it the Krioukov-Papadopoulos-Kitsak-Vahdat-Boguñá-model, after its inventors. For convenience we will abbreviate this to KPKVB-model throughout the rest of the paper. The model has three parameters: the number of vertices N , which we think of as large, and $\alpha, \nu > 0$ which we think of as fixed. Given N, ν, α , we compute $R := 2 \log(N/\nu)$. We now select N points independently at random from the disk of radius R centred at the origin O , which we denote by \mathcal{D}_R , according to the following probability distribution. If the random point u has polar coordinates (r, θ) , then θ, r are independent, θ is uniformly distributed in $(0, 2\pi]$ and the probability distribution of r has density function given by:

$$\rho(r) = \begin{cases} \alpha \frac{\sinh \alpha r}{\cosh \alpha R - 1} & \text{if } 0 \leq r \leq R, \\ 0 & \text{otherwise.} \end{cases} \quad (1.1)$$

Note that when $\alpha = 1$, then this is simply the uniform distribution on \mathcal{D}_R .

An alternative way to view this distribution is as follows. If we multiply the differential form in the Poincaré disk model by a factor $1/\alpha^2$ then we obtain (a model of) the hyperbolic plane \mathbb{H}_α of curvature $-\alpha^2$. It can be seen that the above probability distribution corresponds precisely to a point taken uniformly at random from the disk of radius R around the origin in \mathbb{H}_α . (We however treat these points as points of the ordinary hyperbolic plane.) The set of N points we have thus obtained will be the vertex set of our random graph and we denote it by V_N . The KPKVB-random graph, denoted $\mathcal{G}(N; \alpha, \nu)$, is formed when we join each pair of vertices, if and only if they are within (hyperbolic) distance R . Figure 1 shows an example of such a random graph on $N = 1000$ vertices.

We should mention that Krioukov et al. in fact had an additional parameter ζ in their definition of the model. In their definition, the points were taken inside a disk of radius $R_\zeta := (2/\zeta) \log(N/\nu)$ on the hyperbolic plane \mathbb{H}_ζ of curvature $-\zeta^2$, and the points were generated according to (1.1) with R_ζ in place of R . In this case the random graph is denoted by $\mathcal{G}(N; \zeta, \alpha, \nu)$. However, it turns out that there is no need for the extra parameter ζ . The following lemma, which we prove in Appendix A, shows that we can take $\zeta = 1$ without any loss of generality. We remind the reader that a *coupling* of two

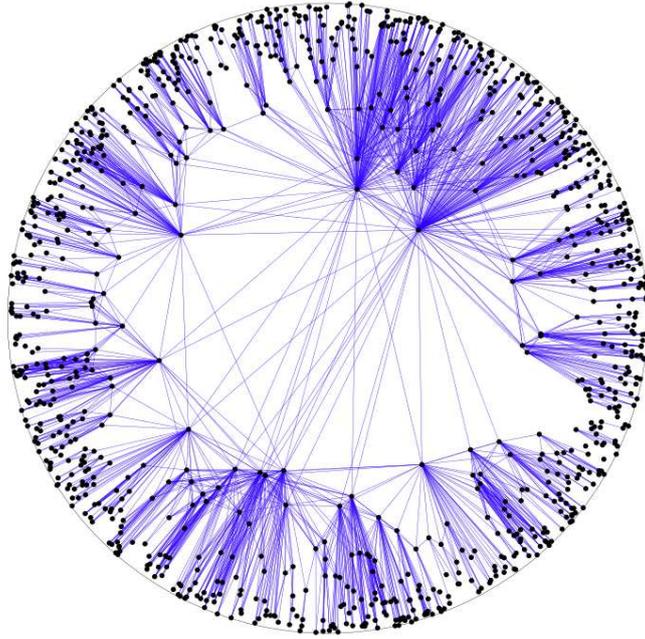


Figure 1: Simulation of the KPKVB-model with $N = 1000, \alpha = .9, \nu = 2$. (Depicted in the native model.)

random objects X, Y is a common probability space for a pair of objects (X', Y') whose marginal distributions satisfy $X' \stackrel{d}{=} X, Y' \stackrel{d}{=} Y$.

Lemma 1.1. *Let $\alpha, \alpha', \zeta, \zeta' > 0$ be such that $\zeta/\alpha = \zeta'/\alpha'$. For every ν and $N \in \mathbb{N}$, there exists a coupling between $\mathcal{G}(N; \zeta, \alpha, \nu)$ and $\mathcal{G}(N; \zeta', \alpha', \nu)$ such that $G(N; \zeta, \alpha, \nu) = G(N; \zeta', \alpha', \nu)$.*

Thus, the previous lemma states that one can define $\mathcal{G}(N; \zeta, \alpha, \nu)$ and $\mathcal{G}(N; \zeta', \alpha', \nu)$ on a common probability space in such a way that the two graphs are isomorphic (with probability one). Let us also remark that *edge-set of $\mathcal{G}(N; \alpha, \nu)$ is decreasing in α and increasing in ν* in the following precise sense.

Lemma 1.2. *Let $\alpha, \alpha', \nu, \nu' > 0$ be such that $\alpha \geq \alpha'$ and $\nu \leq \nu'$. For every $N \in \mathbb{N}$, there exists a coupling such that $G(N; \alpha, \nu)$ is a subgraph of $G(N; \alpha', \nu')$.*

The proof of Lemma 1.2 is given in Appendix B.

Krioukov et al. [19] focus on the degree distribution of $\mathcal{G}(N; \alpha, \nu)$, showing that when $\alpha > \frac{1}{2}$ this follows a power law with exponent $2\alpha + 1$. They also discuss clustering on a smooth version of the above model. Their results have been verified rigorously by Gugelmann et al. [12]. Note that when $\alpha = 1$, that is, when the N vertices are uniformly distributed in \mathcal{D}_R , the exponent of the power law is equal to 3. When $\frac{1}{2} < \alpha < 1$, the exponent is between 2 and 3, as is the case in a number of networks that emerge in applications such as computer networks, social networks and biological networks (see for

example [2]). They have also shown that the average degree of the random graph can be “tuned” through the parameter ν .

Throughout the paper, we will be using the notion of the *type* of a vertex. For a vertex $u \in V_N$, its *type* t_u is defined to be equal to $R - r_u$ where r_u is the radius of u in \mathcal{D}_R . Similarly, a point $p \in \mathcal{D}_R$ of radius r_p has type $t_p = R - r_p$. We note the following lemma, making it easier to work with the distribution of the types.

Lemma 1.3. *Uniformly for $0 \leq t < 0.99R$ we have*

$$\bar{\rho}(t) := \rho(R - t) = (1 + o_R(1))\alpha e^{-\alpha t}. \quad (1.2)$$

Proof. Using the pdf in Equation 1.1, we get

$$\begin{aligned} \rho(R - t) &= \alpha \frac{\sinh(\alpha(R - t))}{\cosh(\alpha R) - 1} = \alpha \frac{e^{\alpha(R-t)} - e^{-\alpha(R+t)}}{2} \frac{2}{e^{\alpha R} + e^{-\alpha R} - 2} \\ &= \alpha \frac{e^{\alpha(R-t)} - e^{-\alpha(R+t)}}{e^{\alpha R} + e^{-\alpha R} - 2} = \alpha \frac{(1 + o(1))e^{\alpha(R-t)}}{(1 + o(1))e^{\alpha R}} \\ &= (1 + o(1))\alpha e^{-\alpha t}. \end{aligned}$$

□

1.1.1 $\mathcal{G}(N; \alpha, \nu)$ and the Chung-Lu model

The notion of *inhomogeneous random graphs* was introduced by Söderberg [21] but was defined more generally and studied in great detail by Bollobás, Janson and Riordan in [4]. In its most general setting, there is an underlying compact metric space \mathcal{S} equipped with a measure μ on its Borel σ -algebra. This is the space of *types* of the vertices. A *kernel* κ is a bounded real-valued, non-negative function on $\mathcal{S} \times \mathcal{S}$, which is symmetric and measurable. It is assumed that the vertices of the random graph are points in \mathcal{S} . If $x, y \in \mathcal{S}$, then the corresponding vertices are joined with probability that is equal to $\frac{\kappa(x,y)}{N} \wedge 1$, where N is the total number of vertices, independently of every other pair. The points that are the vertices of the graph are approximately distributed according to μ . More specifically, the empirical measure induced by the N points converges weakly to μ as $N \rightarrow \infty$.

Of particular interest is the case where the kernel function can be factorised and can be written as $\kappa(x, y) = t(x)t(y)$ – this is called a *kernel of rank 1*. Here, the function $t(x)$ represents the weight of a vertex of type x and, in fact, it is approximately its expected degree. The special case where $t(x)$ follows a distribution that has a power law tail was considered by Chung and Lu in a series of papers [8], [9] (see also [14]).

In the random graph $\mathcal{G}(N; \alpha, \nu)$ the probability that two vertices are adjacent has this form. The proof of this fact relies on Lemma 2.1, which we will state and prove later. This provides an approximate characterization of what it means for two points u, v to have hyperbolic distance at most R in terms of their *relative angle*, which we denote by $\theta_{u,v}$.

As we shall see later in Lemma 2.1, two vertices u and v of types t_u and t_v are within distance R (essentially) if and only if $\theta_{u,v} < 2\nu e^{t_u/2} e^{t_v/2} / N$. Hence, conditional on their types the probability that u and v are adjacent is proportional to $e^{t_u/2} e^{t_v/2} / N$.

But Lemma 1.3 shows that the type of a vertex is approximately exponentially distributed. Thus, if we set $t(u) = e^{t_u/2}$, then $\mathbb{P}(t(u) \geq x) = \mathbb{P}(t_u \geq 2 \ln x) \asymp e^{-2\alpha \ln x} = 1/x^{2\alpha}$. In other words, the distribution of $t(u)$ has a power-law tail with parameter $2\alpha + 1$. Thus, the random graph $\mathcal{G}(N; \alpha, \nu)$ can be seen as a *dependent* version of the Chung-Lu model that emerges naturally from the hyperbolic geometry of the underlying space. The fact that this is a random geometric graph gives rise to the existence of local clustering, which is missing in the Chung-Lu model. There, most vertices have tree-like neighbourhoods.

In fact, it can be shown that the degree of a vertex u in $\mathcal{G}(N; \alpha, \nu)$ that has type t_u is approximately distributed as a Poisson random variable with parameter proportional to $e^{t_u/2}$.

1.2 Component structure of $\mathcal{G}(N; \alpha, \nu)$

This paper focuses on the component structure of $\mathcal{G}(N; \alpha, \nu)$ and, in particular, the size of its largest component. We denote by $|L_1|$ the size of a largest connected component of $\mathcal{G}(N; \alpha, \nu)$.

Among the key findings of Erdős and Rényi on the theory of random graphs is the emergence of a component of linear order in a uniformly chosen random graph on N vertices and M edges, usually denoted by $G(N, M)$. Erdős and Rényi [13] showed that $N/2$ is the critical number of edges for the appearance of a component whose number of vertices is proportional to N . That is, with probability $1 - o(1)$, if $\frac{M}{N} < \frac{1-\varepsilon}{2}$, for $\varepsilon > 0$, then every component of $G(N, M)$ contains at most $C \ln N$ vertices, for some $C = C(\varepsilon) > 0$, but if $\frac{M}{N} > \frac{1+\varepsilon}{2}$, then there exist constants c_1, c_2 depending on ε such that the largest component of $G(N, M)$ has at least $c_1 N$ vertices whereas every other component contains at most $c_2 \ln N$ vertices. In the latter case, the largest component is also known as the *giant component*. See [3] or [16] for a detailed exposition and analysis of the emergence of the giant component.

For (euclidean) random geometric graphs it is known that there exists a critical value c for the average degree such that the size of the biggest component is sublinear if the expected average degree is less than c and linear for expected average degree greater than c . The exact value of c remains unknown. For more information on giant components in random geometric graphs see [20].

In this contribution, we show that when α crosses 1 a “phase transition” occurs. More specifically, if $\alpha > 1$, then *asymptotically almost surely (a.a.s.)*, that is, with probability $1 - o(1)$ as $N \rightarrow \infty$, $|L_1|$ is bounded by a function which is sublinear in N , whereas if $\alpha < 1$, then $|L_1|$ is proportional to N .

Theorem 1.4. *Let α, ν be positive real numbers. The following hold:*

- if $\alpha > 1$, then a.a.s. $|L_1| < 8R^2 \log^3 R N^{1/\alpha}$.
- if $\alpha < 1$, then there exists $c = c(\alpha, \nu) > 0$ such that a.a.s. $|L_1| > cN$.

Recently, Kiwi and Mitsche [18] showed that the second largest component is in fact at most poly-logarithmic in N , and at least logarithmic in N .

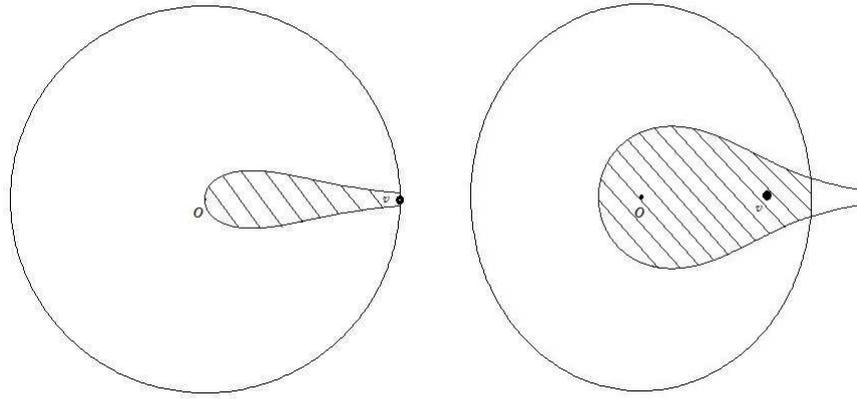


Figure 2: Sets of the points of distance R from certain points. (Depicted in the native model.)

The previous theorem shows there is a phase-change at $\alpha = 1$. The next result shows that inside this phase-change, when $\alpha = 1$, the existence or not of a giant component depends on the value of ν .

Theorem 1.5. *Assume that $\alpha = 1$. There exist constants $\frac{\pi}{8} \leq \nu_0 \leq \nu_1 \leq 20\pi$ such that the following hold:*

- *If $\nu < \nu_0$, then a.a.s. $|L_1| \leq \frac{N}{\log \log R}$.*
- *If $\nu > \nu_1$, then a.a.s. $|L_1| \geq N/610$.*

We now proceed with some auxiliary results and the proof of the above theorems.

2 Auxiliary results

We start by deriving some tools that help to approximate the probability that two points are adjacent in the graph. Let us first remark that the shape of the set of all point of \mathcal{D}_R within distance R of a given point varies greatly depending on the type of the point. See Figure 2 for a depiction.

In particular, when the type of the point is small, then the set of points of \mathcal{D}_R at distance R from it resembles a “long and very thin balloon” in the native model. The following lemma is crucial to many computations in the paper.

Lemma 2.1. *For any $\varepsilon > 0$ there exists an $N_0 > 0$ and a $c_0 > 0$ such that for any $N > N_0$ and $u, v \in \mathcal{D}_R$ with $t_u + t_v < R - c_0$ the following hold.*

- *If $\theta_{u,v} < 2(1 - \varepsilon) \exp\left(\frac{1}{2}(t_u + t_v - R)\right)$, then $d(u, v) < R$.*
- *If $\theta_{u,v} > 2(1 + \varepsilon) \exp\left(\frac{1}{2}(t_u + t_v - R)\right)$, then $d(u, v) > R$.*

Proof. We begin with the hyperbolic law of cosines:

$$\begin{aligned} \cosh(d(u, v)) &= \cosh(R - t_u) \cosh(R - t_v) \\ &\quad - \sinh(R - t_u) \sinh(R - t_v) \cos(\theta_{u,v}). \end{aligned}$$

The right-hand side of the above becomes:

$$\begin{aligned} &\cosh(R - t_u) \cosh(R - t_v) - \sinh(R - t_u) \sinh(R - t_v) \cos(\theta_{u,v}) \\ &= \frac{e^{2R-(t_u+t_v)}}{4} \left[(1 + e^{-2(R-t_u)}) (1 + e^{-2(R-t_v)}) \right. \\ &\quad \left. - (1 - e^{-2(R-t_u)}) (1 - e^{-2(R-t_v)}) \cos(\theta_{u,v}) \right] \\ &= \frac{e^{(2R-(t_u+t_v))}}{4} \left[1 - \cos(\theta_{u,v}) + (1 + \cos(\theta_{u,v})) (e^{-2(R-t_u)} + e^{-2(R-t_v)}) \right. \\ &\quad \left. + O(e^{-2(2R-(t_u+t_v))}) \right]. \end{aligned} \tag{2.1}$$

Therefore,

$$\begin{aligned} \cosh(d(u, v)) &\leq \\ &\frac{e^{(2R-(t_u+t_v))}}{4} \left[1 - \cos(\theta_{u,v}) + 2(e^{-2(R-t_u)} + e^{-2(R-t_v)}) + O(e^{-2(2R-(t_u+t_v))}) \right]. \end{aligned}$$

Since $t_u + t_v < R - c_0$, the last error term is $O(N^{-4})$. Also, it is a basic trigonometric identity that $1 - \cos(\theta_{u,v}) = 2 \sin^2\left(\frac{\theta_{u,v}}{2}\right)$. The latter is at most $\frac{\theta_{u,v}^2}{2}$. Therefore, the upper bound on $\theta_{u,v}$ yields:

$$\begin{aligned} &\cosh(d(u, v)) \\ &\leq \frac{e^{2R-(t_u+t_v)}}{4} \left(\frac{\theta_{u,v}^2}{2} + 2(e^{-2(R-t_u)} + e^{-2(R-t_v)}) + O\left(\frac{1}{N^4}\right) \right) \\ &\leq \frac{e^{(2R-(t_u+t_v))}}{4} (2(1 - \varepsilon)^2 e^{t_u+t_v-R} + 2(e^{-2(R-t_u)} + e^{-2(R-t_v)})) + O(1) \\ &= (1 - \varepsilon)^2 \frac{e^R}{2} + \frac{1}{2} (e^{t_u-t_v} + e^{t_v-t_u}) + O(1) \\ &< (1 - \varepsilon)^2 \frac{e^R}{2} + \varepsilon \frac{e^R}{2} + O(1) < \frac{e^R}{2}, \end{aligned}$$

for N sufficiently large and c_0 such that $e^{-c_0} < \frac{1}{2}\varepsilon$, since $t_u + t_v < R - c_0$ and $t_u, t_v \geq 0$. This implies that $t_u - t_v, t_v - t_u < R - c_0$ and, therefore, $\frac{1}{2}(e^{t_u-t_v} + e^{t_v-t_u}) < \frac{1}{2}(e^{R-c_0} + e^{R-c_0}) < \varepsilon \frac{e^R}{2}$. Also, since $\cosh(d(u, v)) > \frac{1}{2}e^{d(u,v)}$, it follows that $d(u, v) < R$.

To deduce the second part of the lemma, we consider a lower bound on (2.1) using the lower bound on $\theta_{u,v}$:

$$\begin{aligned} \cosh(d(u, v)) &\geq \frac{e^{2R-(t_u+t_v)}}{4} (1 - \cos(\theta_{u,v})) + O(1) \\ &\geq \frac{e^{2R-(t_u+t_v)}}{4} \left(1 - \cos\left(2(1 + \varepsilon)e^{\frac{1}{2}(t_u+t_v-R)}\right) \right) + O(1). \end{aligned} \tag{2.2}$$

Using again that $1 - \cos(\theta) = 2 \sin^2\left(\frac{\theta}{2}\right)$ we deduce that

$$1 - \cos\left(2(1 + \varepsilon)e^{\frac{1}{2}(t_u + t_v - R)}\right) = 2 \sin^2\left(\frac{1}{2} 4(1 + \varepsilon)^2 e^{t_u + t_v - R}\right).$$

Since $t_u + t_v < R - c_0$, it follows that $t_u + t_v - R < -c_0$. So the latter is

$$\sin\left(\frac{1}{2} 4(1 + \varepsilon)^2 e^{t_u + t_v - R}\right) > 2\left(1 + \frac{\varepsilon}{2}\right)^2 e^{t_u + t_v - R},$$

for N and c_0 large enough, using the Taylor's expansion of the sine function around 0. Substituting this bound into (2.2) we have

$$\cosh(d(u, v)) \geq \left(1 + \frac{\varepsilon}{2}\right)^2 \frac{e^R}{2} + O(1).$$

Thus, if $d(u, v) \leq R$, the left-hand side would be smaller than the right-hand side which would lead to a contradiction. \square

We will define approximating areas of the circle of radius R around a given point u , motivated by Lemma 2.1. We call these bounding areas *inner* and *outer tube* of the point u .

Definition 2.2. For a given point $u \in \mathcal{D}_R$ and for ε and N_0 as in Lemma 2.1 we call the sets

- $T_u^- := \{v \in \mathcal{D}_R : \theta_{u,v} \leq 2(1 - \varepsilon) \exp\left(\frac{1}{2}(t_u + t_v - R)\right)\}$ the inner tube and
- $T_u^+ := \{v \in \mathcal{D}_R : \theta_{u,v} \leq 2(1 + \varepsilon) \exp\left(\frac{1}{2}(t_u + t_v - R)\right)\}$ the outer tube

of the point u .

Although by our definition there is no unique inner and outer tube, we will talk of the inner and outer tube. These should always be for suitably chosen ε and N_0 in the given context. Lemma 2.1 shows that, for sufficiently large graphs, all points in the inner tube of a typical vertex u (that is, a vertex of low type) are of distance at most R of u and all vertices of distance at most R of u are within the outer tube of u . We will use outer and inner tubes to derive stochastic bounds on the size of a component.

During our proofs we will also use the following lemma, which states that every vertex in the neighbourhood of a given vertex u will still be connected to u when we increase t_u .

Lemma 2.3. Let u, v and w be points in \mathcal{D}_R with $\theta_{u,v} = 0$ and $t_u < t_v$. If $d(u, w) < R$ then $d(v, w) < R$.

Proof. Using basic properties of the geometry of the hyperbolic plane, it follows that the geodesic between two points of radius at most R uses only points of radius at most R . Also note that the geodesic between the origin and any point is the ray from the origin through the point. Let O be the origin, and consider vertices u, v, w as in the statement of the lemma. Consider some isometric mapping which maps w to O . As $w \in \mathcal{D}_R$, we have $d(O, w) \leq R$. By the requirements of the lemma, $d(u, w) \leq R$. So O and u are within the disk of radius R around w , and so is their geodesic. Since $\theta_{u,v} = 0$, it follows that v lies on this geodesic and therefore $d(v, w) \leq R$. \square

3 Theorem 1.4: the subcritical case

To show the first part of Theorem 1.4, we construct a process that exposes the angle that the component of a given vertex v covers. We already discussed that vertices that are close to the centre have a higher expected degree than those close to the periphery. The connectivity in a hyperbolic random geometric graph highly depends on the structure of the sub-graph that is induced by vertices of high type. When $\alpha > \frac{1}{2}$, the types of all vertices of $\mathcal{G}(N; \alpha, \nu)$ are bounded away from R a.a.s. This is made precise in the following lemma.

Lemma 3.1. *Let $\omega : \mathbb{N} \rightarrow \mathbb{N}$ be an increasing function such that $\omega(N) \rightarrow \infty$ as $N \rightarrow \infty$. With probability $1 - o(1)$ all vertices in V_N have type at most $\frac{1}{2\alpha}R + \omega(N)$.*

The proof of this lemma can be found in [11] (Corollary 2.2). Thus, it suffices to consider vertices of type no larger than this bound. Note that all vertices have type smaller than and asymptotically bounded away from $R/2$, since $\alpha > 1$. We will consider a vertex u of type $\frac{1}{2\alpha}R + \omega(N)$ and analyse a *breadth exploration process*, through which we will bound the *total angle* of the component which contains u : if $C(u)$ denotes the connected component u belongs to, we define

$$\Theta(u) := \max \{ \theta_{u,v} : v \in C(u) \}.$$

This quantity represents the “width” of the component u belongs to from the point of view of u itself. When working with this we generally need to double it as it only considers the direction of maximum extent but we need to take both into account. Let $0 \leq \theta_c(u, v) < 2\pi$ be the angle between the points u and v , in clockwise direction from the point u . We define a *bounding path*, which is a path on \mathcal{D}_R that is not crossed by any edge. The length of a bounding path essentially bounds the number of vertices of the component in which it is rooted. In particular, if a bounding path induces a partition of \mathcal{D}_R into two parts, one of which covers an angle of at most $o(1)$, then a.a.s. any component in this part will be of sublinear size.

Definition 3.2. *We call a series of points $P = (p_1, p_2, \dots, p_m)$ in \mathcal{D}_R a bounding path for $\mathcal{G}(N; \alpha, \nu)$, if the following hold:*

- (i) *The points p_1 and p_m are on the boundary of \mathcal{D}_R , i.e. their radius is R . Also, $\theta_{p_1, p_2} = \theta_{p_{m-1}, p_m} = 0$.*
- (ii) *For even i , we have $\theta_c(p_1, p_i) < \theta_c(p_1, p_{i+1})$ and $t_{p_i} = t_{p_{i+1}}$, while for odd i we have $\theta_c(p_1, p_i) = \theta_c(p_1, p_{i+1})$ and $t_{p_i} \neq t_{p_{i+1}}$.*
- (iii) *Let $A \cup B$ be the partition of \mathcal{D}_R incurred by P , using radial lines to connect vertices that only differ in type and arcs to connect vertices that only differ in angle. Let B be the part containing the origin and let A contain all points on the connections. There is no pair of adjacent vertices $a \in A$ and $b \in B$ in $\mathcal{G}(N; \alpha, \nu)$.*

Note that (ii) ensures that P does not cross itself, so it does partition the disk into two parts and (iii) makes sense. Also, for $1 < i < m$ and any vertex v with $\theta_{v,p_i} = 0$ and $t_v < t_{p_i}$, the component of $\mathcal{G}(N; \alpha, \nu)$ that v belongs to covers an angle of at most $\theta_c(p_1, p_m)$.

We will now proceed with the definition of the *breadth exploration process* that we will use to get a short bounding path. Note that $\frac{1}{2\alpha-1} < 1$ as $1 < 2\alpha - 1 \Leftrightarrow \alpha > 1$. We choose $\varepsilon > 0$ small enough so that $\frac{1}{\lambda} := 2\alpha - 1 - \varepsilon > 1$. Throughout this section we will need several small constants ε . We will assume that we choose one ε for all these and require N to be large enough to satisfy everything. Given some constant $C > 0$, let i_0 be the minimum i such that $\lambda^i \left(\frac{1}{2\alpha}R + \omega(N)\right) < C$. Note that $C_\lambda := \lambda^{i_0} \left(\frac{1}{2\alpha}R + \omega(N)\right) \geq \lambda C$. We partition the disk \mathcal{D}_R into three bands:

$$\begin{aligned}\mathcal{B}_0 &= \{v \in \mathcal{D}_R : \frac{1}{2\alpha}R + \omega(N) < t_v \leq R\} \\ \mathcal{B}_{C_\lambda} &= \{v \in \mathcal{D}_R : C_\lambda < t_v \leq \frac{1}{2\alpha}R + \omega(N)\} \\ \mathcal{B}_- &= \{v \in \mathcal{D}_R : 0 \leq t_v \leq C_\lambda\}\end{aligned}$$

By Lemma 3.1 a.a.s. \mathcal{B}_0 does not contain any vertices. We define two phases for our random process, one on \mathcal{B}_{C_λ} and one on \mathcal{B}_- . We start the process from a point $u \in \mathcal{D}_R$ with $0 < t_u \leq \frac{1}{2\alpha}R + \omega(N)$; in fact, due to Lemma 2.3 we may assume that $t_u = \frac{1}{2\alpha}R + \omega(N)$.

Phase I Letting $t_i := \lambda^i \left(\frac{1}{2\alpha}R + \omega(N)\right)$, we partition \mathcal{B}_{C_λ} into i_0 bands

$$\mathcal{B}_{C_\lambda}^{(i)} = \{v \in \mathcal{D}_R : t_i < t_v \leq t_{i-1}\}$$

We know that there exists $i_u \in \{1, \dots, i_0\}$ such that $u \in \mathcal{B}_{C_\lambda}^{(i_u)}$. We consider the *domain of attraction* around u :

$$\mathcal{A}_u := T_u^+ \cap \mathcal{B}_{C_\lambda} = \left\{ v : \theta_{u,v} \leq 2(1 + \varepsilon)\nu \frac{e^{\frac{1}{2}(t_u+t_v)}}{N}, v \in \mathcal{B}_{C_\lambda} \right\}$$

and for $i = 1, \dots, i_0$ we let $\mathcal{A}_u^{(i)}$ denote the set of points in $\mathcal{A}_u \cap \mathcal{B}_{C_\lambda}^{(i)}$ that are in the clockwise direction from u .

By Lemma 2.1, the domain of attraction $\mathcal{A}_u^{(i)}$ contains all points of the band $\mathcal{B}_{C_\lambda}^{(i)}$ that are within distance R in clockwise direction of the point u , but not every point in $\mathcal{A}_u^{(i)}$ must necessarily be within distance R of u . We define the first phase of the breadth exploration process in the clockwise direction started at u as follows. Note that the auxiliary points defined in the process do not necessarily (in fact, with probability 1 they do not) correspond to vertices of the graph.

1. $v := u$ and $\Theta' := 0$; let i_v be such that $v \in \mathcal{B}_{C_\lambda}^{(i_v)}$;

2. let j_0 be the smallest i such that $\mathcal{A}_v^{(i)}$ contains a vertex;
 if such an index does not exist, then go to Phase II;
 if $j_0 \leq i_v$, then go to Step 5; (we then say that a *backward jump* occurs)
3. let $\hat{\Theta}_1 := 2\nu(1 + \varepsilon) \frac{e^{1/2(t_v + t_{j_0-1})}}{N}$. Let w be the point of polar coordinates $(R - t_{j_0-1}, \theta_v - \hat{\Theta}_1)$.
4. go to Step 2, setting $v := w$ and $\Theta' := \Theta' + \hat{\Theta}_1$;
5. let v' be the point of polar coordinates $(R - t_0, \theta_v - 2\nu(1 + \varepsilon) \frac{e^{\frac{1}{2}(t_v + t_0)}}{N})$; set $v := v'$;

Phase II

1. let v and Θ' have their final values after the execution of Phase I;
2. let $w \in \mathcal{D}_R$ be the point of type C_λ and $\theta_{v,w} = 2\nu(1 + \varepsilon) \frac{e^{1/2(t_v + C_\lambda)}}{N}$ in the clockwise direction;
3. set $\Theta' := \Theta' + 2\nu(1 + \varepsilon) \frac{e^{1/2(t_v + C_\lambda)}}{N}$ and let T_w^+ be the half-tube containing every point u that has relative angle at most $2\nu(1 + \varepsilon) \frac{e^{1/2(t_u + C_\lambda)}}{N}$ with w in clockwise direction;
4. if T_w^+ is empty, then exit;
 else start the process again from Step 2 of Phase I with $\Theta' := \Theta' + 2\nu(1 + \varepsilon) \frac{e^{1/2(t_0 + C_\lambda)}}{N}$ and v of polar coordinates $(R - t_0, \theta_w - 2\nu(1 + \varepsilon) \frac{e^{1/2(t_0 + C_\lambda)}}{N})$.

Note that this process does not involve any points of type higher than t_0 . Indeed, this is not necessary as by Lemma 3.1 a.a.s. all vertices in V_N have types no more than t_0 .

We call a single execution of Steps 2-4 of Phase I a *round*. A maximal series of consecutive rounds is called a *cycle*. Thus, if at the end of a cycle a backward jump occurs, then Phase I proceeds to Step 5, initiating a cycle starting at a point of type t_0 . This ensures that no matter where the backward jump takes place, vertices that are within distance R from the new root will be covered.

The set of rounds up to the end of Phase II is called an *epoch*. Hence, an epoch consists of repeated cycles, whose repetitions stop with an execution of Phase II. The *breadth exploration process* starting at a vertex/point u is the process consisting of repeated epochs with the initial root v being the point of type t_0 and relative angle with respect to u that is equal to 0. (Thus, in fact, the process does not start from u but at the “image” of u that has type t_0 .)

Recall that $\Theta(u)$ is the maximum relative angle between any two vertices in the component that contains u . We prove the following lemma:

Lemma 3.3. *For any vertex $u \in V_N$ of type less than t_0 , if Θ' denotes the maximum of the angles gained during the breadth exploration process started at u in the clockwise and the anticlockwise direction, then $\Theta(u) \leq 2 \cdot \Theta'$.*

Proof. Using the breadth exploration process in the clockwise direction, we get a series of root points - these are the vertices in the beginning of Phase I. Let u_1, \dots, u_m be the part of this series that corresponds to the last cycle, i.e. there was a backwards jump before u_1 and there is no more backwards jump from there on. Let \hat{u}_i be the radial projection of the point u_i to type $t_{u_{i+1}}$. The series $u_1, \hat{u}_1, u_2, \hat{u}_2, \dots, \hat{u}_{m-1}, u_m$ thus always alters between changing the type and relative angle, as required in condition (ii). Similarly, in anticlockwise direction, we get the series $u'_1, \hat{u}'_1, u'_2, \hat{u}'_2, \dots, \hat{u}'_{\ell-1}, u'_\ell$. Letting \hat{u}_m and \hat{u}'_ℓ be the radial projections of u_m and u'_ℓ to the boundary of \mathcal{D}_R , we get the path $P = (\hat{u}'_\ell, u'_\ell, \hat{u}'_{\ell-1}, \dots, u'_1, u_1, \dots, \hat{u}_{m-1}, u_m, \hat{u}_m)$. If the breadth exploration process only uses a total angle that is $o(1)$, which is the case a.a.s., then (i) and (ii) are naturally true if $u_1 \neq u'_1$. If $u_1 = u'_1$ (implying $u_1 = u = u'_1$), almost surely (with probability 1) we can push u_1 in clockwise direction by some small amount to fix this problem without causing further problems elsewhere (i.e. all the adjacencies of u_1 stay the same).

To prove that P is a bounding path for $\mathcal{G}(N; \alpha, \nu)$ we need to show that there is no pair of vertices (v, w) such that $v \in A$, $w \in B$ and $v \sim w$. Assume for a contradiction that there is such an edge vw . Without loss of generality we only consider the series $P_1 = (u_1, \hat{u}_1, u_2, \hat{u}_2, \dots, \hat{u}_{m-1}, u_m)$. If $\theta_c(p_1, w) \leq \theta_c(p_1, u_m)$, then there are two consecutive root points, u_i and u_{i+1} such that $\theta_c(p_1, u_i) < \theta_c(p_1, w) \leq \theta_c(p_1, u_{i+1})$. By Lemma 2.3 and since $t_{u_{i+1}} < t_w$, we have $d(u_i, w) < d(u_i, u_{i+1}) \leq R$, a contradiction to the choice of u_{i+1} as the next root vertex.

Now assume that $\theta_c(p_1, w) > \theta_c(p_1, u_m)$. By Lemma 2.3 and as $v \in A$, there is an $i \geq 1$ such that $\theta_c(p_1, \hat{u}_{i-1}) < \theta_c(p_1, v) \leq \theta_c(p_1, u_i)$ (where, for convenience, $\hat{u}_0 = u'$). But as $v \in A$, we have $t_v \leq t_{u_i}$, so by Lemma 2.3 and the fact that decreasing angles decreases the distance we have that u_i is adjacent to w . By the choice of u_{i+1} in the breadth exploration process, we thus have $\theta_c(p_1, w) \leq \theta_c(p_1, u_{i+1}) \leq \theta_c(p_1, u_m)$ and $t_w < t_{u_{i+1}}$, so $w \in A$, a contradiction. Note that $i < m$ since as u_i is adjacent to w the breadth exploration process cannot have stopped at u_i .

So using the breadth exploration process twice we indeed find a bounding path. In particular, the angle gained in both direction gives a slice of the disk that contains the entire component of u . \square

We now want to bound from above the angle that can be gained during the execution of the process. Note that increasing the type of a vertex u will keep intact all the edges incident to u . Thus if u' is a vertex replacing u , of type $t_{u'} > t_u$ and with relative angle 0 to u , we have $\Theta(u') \geq \Theta(u)$. This justifies the choice of the type in the following lemma.

Lemma 3.4. *Let $u \in \mathcal{D}_R$ be a point having $t_u = t_0$. If the breadth exploration process starts at point u , then by the end of it $\Theta' \leq R^2 \log^3 RN^{1/\alpha-1}$ with probability $1 - o(N^{1/\alpha-1})$*

Proof. Let us consider the breadth exploration process started at a vertex u having type $t_0 = \frac{1}{2\alpha}R + \omega(N)$. For an $\varepsilon > 0$ we let T_ε denote the first round at the end of which $\Theta' \geq \varepsilon$ if there is such a round, otherwise $T_\varepsilon = \infty$. We also denote by $u_0(t)$ the root vertex at the beginning of the t th round and let $i_{u_0} \geq 0$ denote the index of the band this vertex belongs to. We will first bound from below the probability that the exploration process

does not backtrack during the t th round. Let $B_t^{(i_{u_0})}$ be the indicator random variable that is equal to 1 if and only if backtracking *does* occur during the t th round assuming that the root vertex is in i_{u_0} .

Claim 3.5. For $\varepsilon \in (0, 2\pi)$, let $t < T_\varepsilon$. There exists a constant $K = K(\alpha, \nu) > 0$ such that for any N that is sufficiently large we have

$$\Pr \left[B_t^{(i_{u_0})} = 0 \right] > \exp \left(-K e^{-\frac{\varepsilon}{2} t_{i_{u_0}}} \right).$$

Proof of Claim 3.5. Let us write $u_0 = u_0(t)$. For $t < T_\varepsilon$ we give a stochastic upper bound on the number of vertices that belong to $\cup_{j=0}^{i_{u_0}} \mathcal{A}_{u_0}^{(j)}$. Hence, we will be able to give a lower bound on the probability that this region is empty. In other words, we will bound from below the probability that no backtracking occurs during the t th round. Let N_t denote the number of vertices that have not been exposed at the beginning of the t th round. Using Lemma 1.3, the probability that one of them will belong to $\cup_{j=0}^{i_{u_0}} \mathcal{A}_{u_0}^{(j)}$ is bounded from above by

$$\begin{aligned} & \frac{2\nu(1+\varepsilon)\alpha}{2\pi-\Theta'} \int_{t_{i_{u_0}}}^{t_0} \frac{e^{\frac{1}{2}(t_{u_0}+s)}}{N} e^{-\alpha s} ds \leq \frac{2\nu(1+\varepsilon)\alpha}{2\pi-\varepsilon} \int_{t_{i_{u_0}}}^{t_0} \frac{e^{\frac{1}{2}(t_{u_0}+s)}}{N} e^{-\alpha s} ds \\ & \leq \frac{4\nu(1+\varepsilon)\alpha}{(2\pi-\varepsilon)(2\alpha-1)} \frac{e^{t_{u_0}/2}}{N} e^{(1/2-\alpha)t_{i_{u_0}}} \\ & \leq \frac{4\nu(1+\varepsilon)\alpha}{(2\pi-\varepsilon)(2\alpha-1)} \frac{e^{\frac{1}{2}(t_{i_{u_0}-1}+t_{i_{u_0}})-\alpha t_{i_{u_0}}}}{N} =: p_t^{(i_{u_0})}. \end{aligned}$$

Hence, the number of vertices which during round t will fall into $\cup_{j=0}^{i_{u_0}} \mathcal{A}_{u_0}^{(j)}$ is binomially distributed with parameters $N_t, p_t^{(i_{u_0})}$. In turn, this is stochastically bounded from above by a binomially distributed random variable with parameters $N, p_t^{(i_{u_0})}$. Note also that if the number of vertices that fall into $\cup_{j=0}^{i_{u_0}} \mathcal{A}_{u_0}^{(j)}$ is positive, then backtracking occurs. Hence, the probability of backtracking during round t is at least $\Pr \left[\text{Bin}(N, p_t^{(i_{u_0})}) = 0 \right]$.

Recall that $t_{i_{u_0}-1} = \frac{1}{\lambda} t_{i_{u_0}} = (2\alpha - 1 - \varepsilon) t_{i_{u_0}}$, whereby

$$\begin{aligned} \frac{1}{2}(t_{i_{u_0}-1} + t_{i_{u_0}}) &= \frac{1}{2}(2\alpha - 1 - \varepsilon + 1) t_{i_{u_0}} = \frac{1}{2}(2\alpha - 1 - \varepsilon + 1) t_{i_{u_0}} \\ &= \left(-\frac{\varepsilon}{2} + \alpha \right) t_{i_{u_0}}. \end{aligned}$$

This shows that $p_t^{(i_{u_0})} = O(\frac{1}{N})$, justifying the following exponential approximation.

Setting $K' = \frac{4\nu(1+\varepsilon)\alpha}{(2\pi-\varepsilon)(2\alpha-1)}$, we obtain an asymptotic estimate on the probability of backtracking:

$$\begin{aligned} \Pr \left[\text{Bin}(N, p_t^{(i_{u_0})}) = 0 \right] &= \left(1 - p_t^{(i_{u_0})} \right)^N \\ &= \exp \left(-K' e^{\frac{1}{2}(t_{i_{u_0}-1}+t_{i_{u_0}})-\alpha t_{i_{u_0}}} (1 + o(1)) \right). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \Pr \left[\text{Bin}(N, p_t^{(i_{u_0})}) = 0 \right] &= \left(1 - p_t^{(i_{u_0})} \right)^N = \exp \left(-(1 + o(1)) K' e^{-\frac{\varepsilon}{2} t_{i_{u_0}}} \right) \\ &> \exp \left(-2K' e^{-\frac{\varepsilon}{2} t_{i_{u_0}}} \right), \end{aligned}$$

for any N sufficiently large, uniformly over all possible values of i_{u_0} . (The latter is the case since always $t_{i_{u_0}} < R/2$.) Taking $K = 2K'$, the claim now follows. \square

Now, observe that the above claim implies that the probability of no backtracking at a certain round can become very close to 1. Indeed, note that $t_{u_0} \geq \lambda C$ and, therefore, the exponent on the right-hand side of the bound obtained in Claim 3.5 can be made as close to 0 as we want, provided we choose C large enough. Moreover, if t_{u_0} is bounded from below by a function of N that increases as $N \rightarrow \infty$, then the probability of no backtracking is in fact $1 - o(1)$. These observations are key to the deduction of the first part of the lemma.

We first show that provided that Θ' is much less than ε , the number of cycles within an epoch is essentially stochastically dominated by a geometrically distributed random variable that has probability of success $1 - \varepsilon$, provided that the parameter $C = C(\varepsilon)$ is large enough. Suppose that an epoch starts with $\Theta' \leq g(N)$ where $g(N) = o(1)$.

Recall that a cycle starts at a vertex that has type $t_0 = \frac{1}{2\alpha} R + \omega(N)$. Let T_{C_λ} denote the random variable that is the length of a cycle. We say that a cycle is *successful* if it exits to Phase II. Note that a cycle is successful, that is, no backtracking occurs, if and only if $B_t^{(i_{u_0})} = 0$, for all $t \leq T_{C_\lambda}$.

We will bound the probability that, conditional on $\Theta' \leq g(N)$ at the beginning of the epoch, the number of cycles is at least R . In particular, we will show that for every ε there exists a C such that this probability is at most ε^{R-1} .

Claim 3.6. *Let $g(N) = o(1)$. For every $\varepsilon > 0$ there exists a $C = C(\varepsilon)$ such that for any N sufficiently large, conditional on $\Theta' \leq g(N)$ at the beginning of an epoch, with probability at least $1 - \varepsilon^{R-1}$ the total angle gained during the epoch is at most $2R \log^2 RN^{1/\alpha-1}$.*

Proof. To bound this probability, we will repeatedly apply Claim 3.5. However, in order to this we need to ensure that Θ' does not exceed ε whenever at most R cycles have been executed.

Hence, we first need to give an upper bound on the angle that is gained during the execution of a cycle. If $\Theta'_{T_{C_\lambda}}$ denotes this angle, then

$$\Theta'_{T_C} < 2\nu(1 + \varepsilon) \sum_{i=0}^{i_0} \frac{e^{\frac{1}{2}(t_i+t_{i+1})}}{N}.$$

But for all i we have $t_i < \frac{1}{2\alpha} R + \omega(N)$. Hence, $t_i + t_{i+1} < \frac{1}{\alpha} R + 2\omega(N)$, whereby

$$\frac{e^{\frac{1}{2}(t_i+t_{i+1})}}{N} < \frac{e^{\frac{1}{2}R \frac{1}{\alpha} + \omega(N)}}{N} = e^{\omega(N)} N^{\frac{1}{\alpha}-1}.$$

Using this with $\omega(N) = \log \log^{\frac{1}{2}} R$, the above sum can be further bounded from above by

$$\Theta'_{T_C} < 2\nu(1 + \varepsilon)(i_0 + 1)e^{\log \log^{\frac{1}{2}} R} N^{\frac{1}{\alpha}-1} \stackrel{i_0=O(\log R)}{\leq} \log^2 R \frac{N^{1/\alpha}}{N}, \quad (3.1)$$

if N is large enough. Therefore, after $r \leq R$ cycles the angle gained will be at most $R \log^2 R \frac{N^{1/\alpha}}{N} < \varepsilon$, for any N that is sufficiently large. Note also that this quantity bounds the total angle that is gained during an epoch consisting of at most R cycles.

Hence, applying Bayes' rule repeatedly, Claim 3.5 implies that

$$\Pr \left[B_t^{(i_{u_0})} = 0, \forall t < T_{C_\lambda} \right] \geq \prod_{i=0}^{i_0} \exp \left(-K e^{-\frac{\varepsilon}{2} t_i} \right). \quad (3.2)$$

Let $a_i := e^{-\frac{\varepsilon}{2} t_i}$ and note that since $t_i = \lambda^i t_0$ we have for $i \leq i_0$

$$\begin{aligned} \frac{a_{i-1}}{a_i} &= e^{-\frac{\varepsilon}{2} t_0 (\lambda^{i-1} - \lambda^i)} = e^{-\frac{\varepsilon}{2} t_0 \lambda^i (\frac{1}{\lambda} - 1)} \leq e^{-\frac{\varepsilon}{2} t_0 \lambda^{i_0} (\frac{1}{\lambda} - 1)} \\ &\leq e^{-\frac{\varepsilon}{2} \lambda C (\frac{1}{\lambda} - 1)}. \end{aligned}$$

Thus, if C is large enough, then

$$\sum_{i=0}^{i_0} e^{-\frac{\varepsilon}{2} t_i} < e^{-\frac{\varepsilon}{2} t_{i_0}} \sum_{i=0}^{\infty} e^{-\frac{\varepsilon}{2} i \cdot C(1-\lambda)} \stackrel{t_{i_0} > \lambda C}{<} 2e^{-\frac{\varepsilon}{2} \lambda C}.$$

Substituting this bound into the right-hand side of (3.2) we obtain

$$\Pr \left[B_t^{(i_{u_0})} = 0, \forall t \leq T_{C_\lambda} \right] \geq \exp \left(-2K e^{-\frac{\varepsilon}{2} \lambda C} \right) > 1 - 2K e^{-\frac{\varepsilon}{2} \lambda C} > 1 - \varepsilon, \quad (3.3)$$

choosing C large enough so that the last inequality holds. Hence, the probability that backtracking occurs before T_{C_λ} is at most ε , for $C = C(\varepsilon)$ that is sufficiently large. In other words, the probability that the cycle is not successful conditional on $\Theta' \leq g(N)$ at the beginning of the epoch is at most ε . Therefore, the conditional probability of having R cycles during the epoch is at most ε^{R-1} .

As we pointed out above the total angle that was gained above during the execution of the R cycles is no more than $R \log^2 R N^{1/\alpha-1}$. During Phase II, the angle gained is at most

$$4\nu(1 + \varepsilon) \frac{e^{\frac{1}{2}(t_0 + C_\lambda)}}{N} \leq 4\nu(1 + \varepsilon) e^{\frac{1}{2}(C_\lambda + \omega(N))} \frac{e^{\frac{1}{2\alpha} \frac{1}{2} R}}{N} = o \left(N^{\frac{1}{\alpha}-1} \right).$$

Hence, an epoch having at most R cycles adds at most $2R \log^2 R N^{1/\alpha-1}$ to Θ' , provided that N is sufficiently large. \square

Now, we will show that as long as Θ' has not grown too much, the probability that an epoch is the final one is asymptotically bounded away from 0. To see this, we will bound from above the probability that T_v^+ , that was defined in Step 3 of Phase II, contains at least one vertex conditional on $\Theta' \leq \varepsilon$. In particular, conditional on this, the probability

that a given vertex whose exact position in \mathcal{D}_R has not been exposed yet belongs to T_v^+ is at most

$$\begin{aligned} \frac{2\nu\alpha(1+\varepsilon)}{2\pi-\varepsilon} \int_0^{t_0} \frac{e^{\frac{1}{2}(C\lambda+t)}}{N} e^{-\alpha t} dt &> \frac{2\nu\alpha(1+\varepsilon)}{2\pi-\varepsilon} \frac{e^{C\lambda/2}}{N} \int_0^\infty e^{(\frac{1}{2}-\alpha)t} dt \\ &\stackrel{1/\alpha < 2}{=} \frac{4\nu\alpha(1+\varepsilon)}{(2\alpha-1)(2\pi-\varepsilon)} \frac{e^{C\lambda/2}}{N} =: p_{II} = O\left(\frac{1}{N}\right). \end{aligned}$$

Under the above conditioning, the number of vertices that belong to T_v^+ is stochastically bounded from above by a binomially distributed random variable with parameters N, p_{II} . Hence, the probability that T_v^+ is empty conditional on $\Theta' \leq \varepsilon$ is at least

$$\begin{aligned} (1-p_{II})^N &= \exp\left(-\frac{4\nu(1+\varepsilon)}{(2\alpha-1)(2\pi-\varepsilon)} e^{C\lambda/2}(1+o(1))\right) \\ &\geq \exp\left(-\frac{4\nu(1+\varepsilon)}{\pi(2\alpha-1)} e^{C\lambda/2}\right) =: \delta, \end{aligned}$$

provided that $\varepsilon < \pi$ and N is sufficiently large.

Now, we set $E := \lfloor -\frac{1-1/\alpha}{\ln \delta} R \rfloor$. We will finish the proof by showing that the probability that less than E epochs take place each having at most R cycles is $1 - o(N^{1/\alpha-1})$. This together with Claim 3.6 imply that with this probability the total angle gained during the process is at most $2ER \log^2 RN^{1/\alpha-1}$.

Indeed, the probability of having E epochs each one having at most R cycles is at least

$$\delta^E \geq \delta^{-\frac{1-1/\alpha}{\ln \delta} R} = \exp((1/\alpha - 1) R) = O(N^{2(1/\alpha-1)}),$$

and the latter is $o(N^{1/\alpha-1})$. Also, arguing as in the proof of Claim 3.6, we deduce that the probability that there exists one among the first E cycles having more than R cycles is at most $E\varepsilon^R = o(N^{1/\alpha-1})$, provided that ε is chosen small enough. \square

The above lemma together with Lemma 3.3 imply that

Lemma 3.7. *For any $u \in V_N$ we have $\Theta(u) \leq R^2 \log^3 RN^{1/\alpha-1}$ with probability $1 - o(N^{1/\alpha-1})$.*

We will now deduce the first part of Theorem 1.4 from Lemma 3.7. Let \mathcal{B} denote the set of vertices u for which $\Theta(u) > 2R^2 \log^3 RN^{1/\alpha-1}$ - we call these vertices *bad*. Thus, Lemma 3.7 implies that

$$\mathbb{E}[|\mathcal{B}|] = o(N^{1/\alpha}).$$

Markov's inequality in turn implies that for any $\delta > 0$ we have that with probability $1 - o(1)$

$$|\mathcal{B}| < \delta N^{1/\alpha}. \tag{3.4}$$

Assume now that $\mathcal{G}(N; \alpha, \nu)$ has a component C of order greater than $8R^2 \log^3 RN^{1/\alpha}$. Hence, on the event (3.4), there is at least one (in fact, many) vertex $u \in C$ in this component that is *not* bad.

A sector of \mathcal{D}_R is the area between two radii of \mathcal{D}_R of relative angle which is less than π - we call this angle the *angle of the sector*. Hence, since u is not bad, it turns out that there is a sector of angle at most $2R^2 \log^3 RN^{1/\alpha-1}$ which contains at least $4R^2 \log^3 RN^{1/\alpha}$ vertices (in fact, our assumption implies that it contains almost twice as many vertices as this). But the next lemma shows that this is not the case with probability $1 - o(1)$ and the first part of Theorem 1.4 follows.

Lemma 3.8. *Let $\theta : \mathbb{N} \rightarrow \mathbb{R}$ be a non-negative function such that $\theta(N) = o(1)$ but $\frac{N\theta(N)}{\ln(1/\theta(N))} \rightarrow \infty$ as $N \rightarrow \infty$. Then a.a.s. there is no sector of angle $\theta(N)$ that contains at least $2N\theta(N)$ vertices.*

Proof. Consider a partition \mathcal{P} of \mathcal{D}_R into $2\pi/\theta(N)$ sectors of angle $\theta(N)$. If \mathcal{D}_R contains a sector as in the statement of the lemma, then one of the sectors in \mathcal{P} must contain at least $N\theta(N)$ vertices. Now, note that the number of vertices in a sector $\sigma \in \mathcal{P}$, which we denote by N_σ , is binomially distributed with parameters N and $\theta(N)/2\pi$. Hence, $\mathbb{E}(N_\sigma) = \frac{1}{2\pi} N\theta(N)$ and since $N\theta(N) \rightarrow \infty$ and applying a Chernoff-type bound we deduce that

$$\Pr\left[N_\sigma > N\theta(N)\right] = \exp(-\Omega(N\theta(N))).$$

Therefore, using Markov's inequality we obtain:

$$\Pr\left[\exists \sigma \in \mathcal{P} : N_\sigma > N\theta(N)\right] \leq \frac{2\pi}{\theta(N)} \exp(-\Omega(N\theta(N))) = o(1),$$

which concludes the proof of the lemma. □

4 Theorem 1.4: the supercritical case

In this section, we will show the second part of Theorem 1.4. Namely, we shall assume that $\alpha < 1$ and with $|L_1|$ denoting the size of a largest component of $\mathcal{G}(N; \alpha, \nu)$, we will show that there exists a $c = c(\alpha, \nu)$ such that a.a.s. $|L_1| > cN$.

4.1 Proof overview

We will consider a set of homocentric bands in \mathcal{D}_R . The innermost band consists of those vertices of type at least $R/2$. Note that the subgraph of $\mathcal{G}(N; \alpha, \nu)$ that is induced by the vertices which belong to this part of \mathcal{D}_R is a clique. This follows from the triangle inequality, which implies that the distance between any two vertices there is at most R . The remaining bands are determined by a sequence of numbers t_i , with $t_0 = R/2$ and t_i defined by the following recursion:

$$t_i - 2 \ln \left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_i \right) = \lambda t_{i-1}, \tag{4.1}$$

if $0 < t_i < t_{i-1}$. where now $\lambda := 2(\alpha - \frac{1}{2})$ - we assume that $\alpha > \frac{1}{2}$. The bands are now as follows:

$$\mathcal{B}_0 = \{v \in \mathcal{D}_R : R/2 < t_v \leq R\} \text{ and } \mathcal{B}_i = \{v \in \mathcal{D}_R : t_i < t_v \leq t_{i-1}\}.$$

We shall assume that $i < T$, where $T = T(\alpha, \nu, \varepsilon)$ and ε is a positive real number which we will assume to be small enough for the purposes of our calculations. We will determine T in Subsection 4.2. Observe that (4.1) implies that provided that $t_i > \nu(1 - \varepsilon)^4/(4\pi)$,

$$t_i > \lambda^i t_0 \text{ for } i > 0. \tag{4.2}$$

We denote by \mathcal{N}_i the set of vertices which belong to the i th band, for $i \geq 0$, and let N_i denote its size. Furthermore, for $i > 0$ we denote by \mathcal{N}'_i the set of vertices in \mathcal{B}_i that have at least one neighbour in \mathcal{N}'_{i-1} – here we set $\mathcal{N}'_0 = \mathcal{N}_0$. We say that these belong to the *active area* of \mathcal{B}_i . This definition together with the fact that a clique is formed in \mathcal{N}'_0 imply that the graph induced by $\bigcup_{i=0}^T \mathcal{N}'_i$ is connected and contains $\sum_{i=0}^T |\mathcal{N}'_i|$ vertices. Our aim is to show that a.s. this quantity is linear in N . We let $N'_i = |\mathcal{N}'_i|$.

More specifically, we show that the number of vertices in \mathcal{N}'_i stochastically dominates the number of vertices in a subset of \mathcal{B}_i that has arc Θ_i . This makes working with sizes a lot easier, as implications for the size of \mathcal{N}'_i can be deduced from the angle Θ_{i-1} . In particular, with probability $1 - o\left(\frac{1}{\ln N}\right)$

$$N'_{i-1} \geq N_{i-1} \frac{\Theta_{i-1}}{2\pi} (1 - \varepsilon). \tag{4.3}$$

The proof of this can be found in Section 4.4. Next we argue (cf. Section 4.4) that conditional on N'_{i-1} as above and $\Theta_{i-1} > \pi$ with high probability Θ_i is at least a certain fraction of Θ_{i-1} .

Lemma 4.1. *Conditional on $N_{i-1} \in (1 \pm \varepsilon)\mathbb{E}[N_{i-1}]$, on $\Theta_{i-1} > \pi$ as well as on N'_{i-1} satisfying (4.3) with probability $1 - o\left(\frac{1}{\ln N}\right)$ we have*

$$\Theta_i \geq \Theta_{i-1} (1 - e^{-\gamma t_i}), \tag{4.4}$$

for some constant $\gamma = \gamma(\alpha, \nu, \varepsilon) > 0$, uniformly for $i = 1, \dots, T$.

Note that we take $\Theta_0 := 2\pi$.

To derive the stochastic domination we will assume that the following conditions hold:

- any vertex of type t with $t_i \geq R/2$ is of type $R/2$;
- any vertex of type t with $t_i < t \leq t_{i-1}$ is of type t_i .

Lemma 2.3 ensures that for a vertex $v \in \mathcal{B}_{i-1}$ the area consisting of all points that belong to \mathcal{B}_i and have distance R from v becomes smaller, if the type of v within the bounds of \mathcal{B}_{i-1} decreases. Now, using the first part of Lemma 2.1, we can use the inner tubes to obtain a further lower bound on this area. In particular, we will consider only vertices that fall within the inner tube of $v \in \mathcal{N}'_{i-1}$ assuming that the type of v is t_{i-1} and deduce a stochastic lower bound on the size of \mathcal{N}'_i .

Using concentration arguments we will show that a.s. $N'_i \geq \frac{1}{2} N \frac{\Theta_i}{2\pi} (e^{-\alpha t_i} - e^{-\alpha t_{i-1}})$. Hence, if T is such that

$$\prod_{i=0}^T (1 - e^{-\gamma t_i}) > \frac{1}{2}, \tag{4.5}$$

and for some $C > 0$

$$t_T \leq C, \tag{4.6}$$

then it will follow by (4.4) that

$$\sum_{i=0}^T N'_i \geq \frac{1}{2} N (e^{-\alpha t_T} - e^{-\alpha t_0}) \prod_{i=0}^T (1 - e^{-\gamma t_i}) > cN,$$

for some $c = c(\alpha, \nu)$.

4.2 The definition of T

Firstly, we will require that $t_i > B_1$ where $B_1 = B_1(\alpha, \nu, \varepsilon) > e$ is large enough so that we have

$$2 \ln \left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_i \right) < (1-\alpha) t_i. \tag{4.7}$$

This condition implies that

$$\begin{aligned} t_i &\leq \lambda t_{i-1} + 2 \ln \left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_i \right) \stackrel{t_i \leq t_{i-1}}{<} \lambda t_{i-1} + 2 \ln \left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_{i-1} \right) \\ &\stackrel{(4.7)}{<} (\lambda + 1 - \alpha) t_{i-1} = (2\alpha - 1 + 1 - \alpha) t_{i-1} = \alpha t_{i-1} < t_{i-1}. \end{aligned} \tag{4.8}$$

We use (4.8) in order to deduce that if $t_i > B_1$, then

$$e^{-\alpha(t_{i-1}-t_i)} < e^{-t_{i-1}\alpha(1-\alpha)}. \tag{4.9}$$

Given $\varepsilon > 0$, let $C = C(\alpha, \nu, \varepsilon) > B_1$ be such that

$$\begin{aligned} e^{-C\alpha(1-\alpha)} &< \varepsilon, \\ \frac{\nu(1-\varepsilon)^4}{4\pi} &< C, \\ \text{if } \lambda C < t - 2 \ln \left(\frac{4\pi}{\nu(1-\varepsilon)^4} t \right) &=: f(t), \text{ then } f(t) < t, \\ \text{for all } t > C \text{ we have } 2 \ln \left(\frac{4\pi}{\nu(1-\varepsilon)^4} t \right) &< \frac{1-\alpha}{2} t, \\ \frac{e^{-\gamma\lambda C/2}}{1 - e^{-\gamma(1-\alpha)C}} &< \frac{1}{2}, \end{aligned} \tag{4.10}$$

where $\gamma = \gamma(\alpha, \varepsilon) > 0$ will be specified later. Let

$$T := \min\{i : t_i < C \text{ or } \Theta_i < \pi\}.$$

Thus, by (4.8) we deduce that

$$T = O(\log R). \tag{4.11}$$

Hence, (4.9) implies that for any $i < T$ we have

$$e^{-\alpha(t_{i-1}-t_i)} \stackrel{t_{i-1} \geq C}{<} \varepsilon. \quad (4.12)$$

The definition of T also implies that for $i < T$

$$t_i > \frac{\nu(1-\varepsilon)^4}{4\pi},$$

as required above. Recall that this ensures that the second term in the left-hand side of (4.1) is positive as

$$\ln\left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_i\right) > 0,$$

and thereby (4.2) holds for all $i < T$.

Secondly, we will require that

$$\prod_{i=1}^T (1 - e^{-\gamma t_i}) > \frac{1}{2}. \quad (4.13)$$

As we shall see in the next section, this will imply that

$$\Theta_i \geq \Theta_0 \prod_{j=1}^T (1 - e^{-\gamma t_j}) > \pi. \quad (4.14)$$

As $\Theta_0 = 2\pi$ we need that the product on the right-hand side of the above is at least $1/2$. This will be the case if

$$\sum_{j=1}^T e^{-\gamma t_j} < \frac{1}{2}.$$

To bound the above sum, we will give an upper bound on the difference of $t_j - t_{j-1}$. We have

$$t_j - t_{j-1} \stackrel{(4.8)}{<} (\alpha - 1)t_{j-1} \stackrel{t_{j-1} > C, \alpha < 1}{<} (\alpha - 1)C.$$

Hence, we can write

$$\sum_{j=1}^T e^{-\gamma t_j} < e^{-\gamma t_T} \sum_{j=0}^{\infty} e^{-j\gamma(1-\alpha)C}.$$

Also by the third condition in (4.10)

$$\lambda C \leq \lambda t_{T-1} = t_T - 2 \ln\left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_T\right) < t_T, \quad (4.15)$$

whereby

$$t_T > \lambda C. \quad (4.16)$$

Therefore,

$$\sum_{j=1}^T e^{-\gamma t_j} < \frac{e^{-\gamma \lambda C}}{1 - e^{-\gamma(1-\alpha)C}} \stackrel{(4.10)}{<} \frac{1}{2}.$$

4.3 Some concentration results

In this sub-section, we will show that the number of vertices that belong to each band is almost determined. Note that by Lemma 1.3 (since $t_i \leq R/2$, for all $i \geq 0$) we have uniformly for all i

$$\mathbb{E}[N_i] = (1 - o(1))N\alpha \int_{t_i}^{t_{i-1}} e^{-\alpha t} dt = (1 - o(1))N(e^{-\alpha t_i} - e^{-\alpha t_{i-1}}). \quad (4.17)$$

(Here we take $t_{-1} = R$.) We need to show that this quantity grows fast enough as a function of N . To see that this is indeed the case, we write

$$\mathbb{E}[N_i] = (1 - o(1))Ne^{-\alpha t_i} (1 - e^{-\alpha(t_{i-1}-t_i)}) \stackrel{(4.12)}{>} Ne^{-\alpha t_i}(1 - \varepsilon). \quad (4.18)$$

Hence, since $t_i \leq R/2$, it follows that

$$\mathbb{E}[N_i] > (1 - \varepsilon)\nu e^{\frac{1}{2}(1-\alpha)R} = \Omega(N^{1-\alpha}), \quad (4.19)$$

which tends to infinity as N grows, since $\alpha < 1$.

Hence, applying a standard Chernoff bound we deduce that with probability $1 - \exp(-\Omega(N^{1-\alpha}))$ we have

$$N_i = (1 \pm \varepsilon)\mathbb{E}[N_i].$$

Hence, since $T = O(\ln R)$ (cf. (4.8)), a simple first-moment argument shows that with probability $1 - \exp(-\Omega(N^{1-\alpha}))$ we have

$$N_i = (1 \pm \varepsilon)\mathbb{E}[N_i], \quad (4.20)$$

for all $0 \leq i \leq T$. In what follows, we shall condition on this event, which we denote by \mathcal{C}_N .

4.4 The inductive step

Throughout this section, we will have $\theta^{(i)} = 2(1 - \varepsilon)e^{\frac{1}{2}(t_i+t_{i-1}-R)}$, for $0 < i \leq T$ and $\theta^{(0)} = 2(1 - \varepsilon)$. Assume that there are N'_{i-1} vertices in the active area of \mathcal{B}_{i-1} . For a vertex in $v \in \mathcal{N}'_{i-1}$ let $S(v)$ denote the arc of angle $\theta^{(i)}$ around the projection of v on the circle of radius $R - t_i$ (in other words, the set of points of type t_i) - we denote this circle by \mathcal{C}_i . We call this the *shadow* of v . Let

$$S_i := \bigcup_{v \in \mathcal{N}'_{i-1}} S(v)$$

denote the union of the shadows of the vertices in \mathcal{N}'_{i-1} — this is the active area of the band \mathcal{B}_i . Let Θ_i be the total angle of S_i .

We will determine Θ_i conditional on Θ_{i-1} , assuming that we have not specified which vertices among those in \mathcal{N}_{i-1} belong to S_{i-1} . Let S'_{i-1} denote the projection of S_{i-1} on

the circle \mathcal{C}_i . Note that S'_{i-1} is the disjoint union of arcs each of them having angle which is at least $\theta^{(i-1)}$. Moreover, the total angle covered by S'_{i-1} is Θ_{i-1} as well.

Assuming that the vertices of \mathcal{N}_{i-1} have all type t_{i-1} , we expose their positions on \mathcal{C}_{i-1} and consider the shadows of those points that will fall into S_{i-1} . Recall that this number is a stochastic lower bound on N'_{i-1} , whereby

$$\mathbb{E} [N'_{i-1} \mid N_{i-1}, \Theta_{i-1}] \geq N_{i-1} \frac{\Theta_{i-1}}{2\pi}.$$

Furthermore, since $\Theta_{i-1} \geq \pi$, as $i - 1 < T$ and $N_{i-1} = \Omega(N^{1-\alpha})$, as the event \mathcal{C}_N is realised, an application of the Chernoff bound implies that with probability $1 - o\left(\frac{1}{\ln N}\right)$

$$N'_{i-1} \geq N_{i-1} \frac{\Theta_{i-1}}{2\pi} (1 - \varepsilon). \quad (4.21)$$

We will show that conditional on N'_{i-1} as above and Θ_{i-1} with high probability Θ_i is at least a certain fraction of Θ_{i-1} .

Lemma 4.2. *Conditional on N_{i-1} satisfying \mathcal{C}_N , on Θ_{i-1} as well as on N'_{i-1} satisfying (4.21), with probability $1 - o\left(\frac{1}{\ln N}\right)$ we have*

$$\Theta_i \geq \Theta_{i-1} (1 - e^{-\gamma t_i}),$$

for some constant $\gamma = \gamma(\alpha, \nu, \varepsilon) > 0$, uniformly for $i = 1, \dots, T$.

Proof. To show this statement, we divide each subinterval of S'_{i-1} into segments of angle

$$\ell_{i-1} := \frac{\Theta_{i-1}}{N_{i-1}^2}.$$

It is possible that each of these subintervals contains at least one segment of smaller angle. However, each subinterval of S'_{i-1} contains many segments of angle ℓ_{i-1} . We denote by \mathcal{P} the collection of all *those* segments. We will use a bounded-differences concentration inequality in order to show that with high probability most of them are contained in $S(v)$ for some $v \in \mathcal{N}'_{i-1}$.

Firstly, let us bound from below the size of \mathcal{P} . Recall that each subinterval of S'_{i-1} has angle at least $\theta^{(i-1)}$. Therefore, there are at most $\Theta_{i-1}/\theta^{(i-1)}$ subintervals. Each such subinterval contains at most one segment of angle less than ℓ_{i-1} . Hence,

$$|\mathcal{P}| \geq \left\lfloor \frac{\Theta_{i-1}}{\ell_{i-1}} \right\rfloor - \frac{\Theta_{i-1}}{\theta^{(i-1)}} \stackrel{\Theta_{i-1} < 2\pi}{>} N_{i-1}^2 - \frac{2\pi}{\theta^{(i-1)}}. \quad (4.22)$$

But

$$\theta^{(i-1)} = 2(1 - \varepsilon)e^{\frac{1}{2}(t_{i-1} + t_{i-2} - R)} = \stackrel{t_{i-2} > t_{i-1}}{>} 2(1 - \varepsilon)e^{\frac{1}{2}(2t_{i-1} - R)} = 2\nu(1 - \varepsilon) \frac{e^{t_{i-1}}}{N}, \quad (4.23)$$

and also since \mathcal{C}_N is realised (4.18) implies that

$$N_{i-1} \geq (1 - \varepsilon)^2 N e^{-\alpha t_{i-1}} = (1 - \varepsilon)^2 \nu e^{R/2 - \alpha t_{i-1}}.$$

Thus, in this case

$$N_{i-1}^2 \theta^{(i-1)} \geq N_{i-1} \theta^{(i-1)} = 2\nu(1-\varepsilon)^3 e^{(1-\alpha)t_{i-1}} \stackrel{t_{i-1} \geq R/2}{\geq} 2\nu(1-\varepsilon)^3 e^{(1-\alpha)R/2}.$$

This together with (4.22) imply that

$$|\mathcal{P}| \geq N_{i-1}^2 (1 - O(N^{\alpha-1})). \quad (4.24)$$

It is now immediate that the total angle covered by the union of the segments in \mathcal{P} , which we denote by $\Theta_{\mathcal{P}}$, satisfies

$$\Theta_{\mathcal{P}} \geq \Theta_{i-1} (1 - O(N^{\alpha-1})). \quad (4.25)$$

To be more precise, recall that each subinterval has angle that is at least $\theta^{(i-1)}$. Since the event \mathcal{C}_N is realised, we have

$$\ell_{i-1} = \frac{\Theta_{i-1}}{N_{i-1}^2} \stackrel{(4.18)}{\leq} \frac{2\pi}{(1-\varepsilon)^2} \frac{e^{2\alpha t_i}}{N^2}.$$

Now, (4.23) implies that

$$\begin{aligned} \frac{\ell_{i-1}}{\theta^{(i)}} &< \frac{(1-\varepsilon)\pi}{\nu} \frac{e^{(2\alpha-1)t_i}}{N} \stackrel{1 < 2\alpha, t_i < t_0 = R/2}{<} \frac{(1-\varepsilon)\pi}{\nu} \frac{e^{(2\alpha-1)R/2}}{N} \\ &= \frac{(1-\varepsilon)\pi}{\nu^2} e^{(\alpha-1)R} \stackrel{\alpha \leq 1}{\leq} o(1). \end{aligned}$$

In other words, uniformly for all $i < T$, we have $\theta^{(i-1)} > \theta^{(i)} \gg \ell_{i-1}$.

For a segment $\sigma \in \mathcal{P}$, let \mathcal{E}_{σ} denote the event that the segment σ is not covered by $S(v)$, for all $v \in \mathcal{N}'_{i-1}$. The probability that the segment is indeed covered for a certain $v \in \mathcal{N}'_{i-1}$ is at least $\frac{\theta^{(i)}}{\Theta_{i-1}} \geq \frac{\theta^{(i)}}{2\pi}$. Hence

$$\begin{aligned} \Pr[\mathcal{E}_{\sigma} \mid N'_{i-1}, \Theta_{i-1}] &\leq \left(1 - \frac{\theta^{(i)}}{2\pi}\right)^{N'_{i-1}} \leq \exp\left(-\frac{\theta^{(i)} N'_{i-1}}{2\pi}\right) \\ &\stackrel{(4.21)}{\leq} \exp\left(-(1-\varepsilon) \frac{\theta^{(i)} \Theta_{i-1} N_{i-1}}{4\pi^2}\right) \stackrel{(\Theta_{i-1} > \pi)}{\leq} \exp\left(-(1-\varepsilon) \frac{\theta^{(i)} N_{i-1}}{4\pi}\right). \end{aligned} \quad (4.26)$$

Now, on the event \mathcal{C}_N , the following holds through (4.18) and (4.20)

$$\theta^{(i)} N_{i-1} \geq 2\nu(1-\varepsilon)^3 e^{\frac{1}{2}(t_i+t_{i-1})-\alpha t_{i-1}}.$$

But by (4.1) we have

$$\frac{1}{2}(t_i + t_{i-1}) - \alpha t_{i-1} = \ln\left(\frac{4\pi}{2\nu(1-\varepsilon)^4 t_i}\right)$$

which, if substitute in (4.26) implies that

$$\Pr\left[\mathcal{E}_\sigma \mid N'_{i-1}, \Theta_{i-1}\right] \leq e^{-t_i}. \quad (4.27)$$

Let \mathcal{P}' denote the subset of segments of \mathcal{P} that are covered by $S(v)$, for some $v \in \mathcal{N}'_{i-1}$. Therefore,

$$\mu_{\mathcal{P}'} := \mathbb{E}\left[|\mathcal{P}'| \mid N'_{i-1}, \Theta_{i-1}\right] \geq |\mathcal{P}|(1 - e^{-t_i}). \quad (4.28)$$

For any vertex $v \in \mathcal{N}'_{i-1}$, $S(v)$ covers at most $2\theta^{(i)}/\ell_{i-1}$ segments by Lemma 2.1. Thus changing the position of one vertex in \mathcal{N}'_{i-1} changes the number of covered segments by at most $2\theta^{(i)}/\ell_{i-1}$. Hence, applying the Hoeffding-Azuma concentration bound (cf. [16] Theorem 2.25 p. 37) we deduce that

$$\Pr\left[|\mathcal{P}'| < (1 - e^{-(1-\alpha)t_{i-1}/8})\mu_{\mathcal{P}'} \mid N'_{i-1}, \Theta_{i-1}\right] = \exp\left(-\Omega\left(\frac{\mu_{\mathcal{P}'}^2 e^{-(1-\alpha)t_{i-1}/4} \ell_{i-1}^2}{N'_{i-1}(\theta^{(i)})^2}\right)\right).$$

We will show now that

$$\frac{\mu_{\mathcal{P}'}^2 e^{-(1-\alpha)t_{i-1}/4} \ell_{i-1}^2}{N'_{i-1}(\theta^{(i)})^2} = \Omega\left(N^{\frac{5}{4}(1-\alpha)}\right). \quad (4.29)$$

We will estimate the above quantity up to absolute multiplicative constants – we write $A \gtrsim B$ to denote that A/B is bounded from below by some constants that depend only on α, ν and ε . To derive the above lower bound we will need to deduce a stronger upper bound on t_i in terms of t_{i-1} . By (4.1) we have

$$\begin{aligned} t_i &= \lambda t_{i-1} + 2 \ln\left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_i\right) \\ &\stackrel{t_i < t_{i-1}}{\leq} \lambda t_{i-1} + 2 \ln\left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_{i-1}\right) \stackrel{t_{i-1} > C, (4.10)}{<} \left(\lambda + \frac{1-\alpha}{2}\right) t_{i-1}. \end{aligned} \quad (4.30)$$

Now, we have

$$\begin{aligned} &\frac{\mu_{\mathcal{P}'}^2 e^{-(1-\alpha)t_{i-1}/4} \ell_{i-1}^2}{N'_{i-1}(\theta^{(i)})^2} \stackrel{(4.15), (4.18), (4.24)}{\gtrsim} \frac{N_{i-1}^4 e^{-(1-\alpha)t_{i-1}/4} \ell_{i-1}^2}{N_{i-1}(\theta^{(i)})^2} \\ &\gtrsim N_{i-1}^3 e^{-(1-\alpha)t_{i-1}/4} \left(\frac{\Theta_{i-1}}{N_{i-1}^2}\right)^2 \frac{N^2}{e^{t_i+t_{i-1}}} \stackrel{(4.13)}{\gtrsim} \frac{N^2}{N_{i-1}} e^{-(5-\alpha)t_{i-1}/4-t_i} \\ &\stackrel{(4.18), t_i \leq t_{i-1}}{\gtrsim} N e^{-(5-\alpha)t_{i-1}/4-t_i+\alpha t_{i-1}} \stackrel{(4.30)}{\gtrsim} N e^{-(5-\alpha)/4-\lambda+(1-\alpha)/2+\alpha)t_{i-1}}. \end{aligned}$$

But

$$-(5-\alpha)/4 - \lambda + (1-\alpha)/2 + \alpha = -(5-\alpha)/4 - 2\alpha + 1 + (1-\alpha)/2 + \alpha = 1/4 - 5\alpha/4.$$

Hence,

$$\frac{\mu_{\mathcal{P}'}^2 e^{-(1-\alpha)t_{i-1}/4} \ell_{i-1}^2}{N'_{i-1}(\theta^{(i)})^2} \gtrsim N e^{(1-5\alpha)t_{i-1}/4} \stackrel{t_{i-1} \leq R/2, \alpha > 1/2}{\gtrsim} e^{\frac{5}{4}(1-\alpha)\frac{R}{2}} \gtrsim N^{\frac{5}{4}(1-\alpha)}. \quad (4.31)$$

Now,

$$\Theta_i \geq \ell_{i-1} |\mathcal{P}'|.$$

Hence, conditional on N'_{i-1} and Θ_{i-1} with probability $1 - o\left(\frac{1}{\ln N}\right)$ we have

$$\Theta_i \geq \ell_{i-1} \mu_{\mathcal{P}'} (1 - e^{-(\alpha - \frac{1}{2})t_{i-1}/2}).$$

We will bound the right-hand side of the above from below as follows:

$$\begin{aligned} \ell_{i-1} \mu_{\mathcal{P}'} (1 - e^{-(\alpha - \frac{1}{2})t_{i-1}/2}) &\stackrel{(4.28)}{\geq} \ell_{i-1} |\mathcal{P}| (1 - e^{-t_i}) (1 - e^{-(\alpha - \frac{1}{2})t_{i-1}/2}) \\ &\stackrel{(4.24)}{\geq} \ell_{i-1} N_{i-1}^2 (1 - O(N^{\alpha-1})) (1 - e^{-t_i}) (1 - e^{-(\alpha - \frac{1}{2})t_{i-1}/2}) \\ &= \Theta_{i-1} (1 - O(N^{\alpha-1})) (1 - e^{-t_i}) (1 - e^{-(\alpha - \frac{1}{2})t_{i-1}/2}) \\ &\geq \Theta_{i-1} (1 - O(N^{\alpha-1})) \left(1 - e^{-t_i} - e^{-(\alpha - \frac{1}{2})t_{i-1}/2}\right). \end{aligned} \tag{4.32}$$

But by (4.1) we have

$$\frac{1}{2} \left(\alpha - \frac{1}{2}\right) t_{i-1} = \frac{1}{4} t_i - \frac{1}{2} \ln \left(\frac{4\pi}{\nu(1-\varepsilon)^4} t_i\right) \stackrel{(4.10), (4.16)}{>} \frac{1}{4} (1 - \varepsilon) t_i.$$

We substitute this bound into the last expression of (4.32) and deduce the following: there exists a constant $\gamma = \gamma(\varepsilon) > 0$ such that for all N sufficiently large and for all $i = 1, \dots, T$ we have

$$\Theta_i \geq \Theta_{i-1} (1 - e^{-\gamma t_i}). \quad \square$$

4.5 Proof of Theorem 1.4

For $i = 1, \dots, T$ let \mathcal{E}_i denote the event that for all $0 \leq j \leq i$, we have

$$\Theta_j \geq \Theta_{j-1} (1 - e^{-\gamma t_j})$$

and

$$N'_j \geq N_j \frac{\Theta_j}{2\pi} (1 - \varepsilon).$$

Note that conditional on \mathcal{C}_N , the latter inequality together with (4.17) and (4.20) implies that for any N sufficiently large

$$N'_j \geq (1 - \varepsilon)^3 N (e^{-\alpha t_j} - e^{-\alpha t_{j-1}}) \frac{\Theta_j}{2\pi}.$$

Now by (4.21) and Lemma 4.2, we have

$$\Pr \left[\mathcal{E}_i \mid \mathcal{E}_{i-1}, \mathcal{C}_N \right] = 1 - o\left(\frac{1}{R}\right).$$

But as the sequence $\{t_i\}_{i=1,\dots,T}$ decreases exponentially fast (cf. (4.8)), we have $T = O(\ln R)$. Hence, since the events $\{\mathcal{E}_i\}_{i=1,\dots,T}$ form a decreasing sequence, we deduce that

$$\Pr\left[\mathcal{E}_T \mid \mathcal{C}_N\right] = 1 - o(1).$$

On the event \mathcal{E}_T , we have $\Theta_i > \pi$ for all $i = 1, \dots, T$ (cf. (4.14)). Therefore,

$$N'_i \geq \frac{1}{2} (1 - \varepsilon)^3 N (e^{-\alpha t_i} - e^{-\alpha t_{i-1}}).$$

which in turn implies that

$$\sum_{i=1}^T N'_i \geq \frac{1}{2} (1 - \varepsilon)^3 N (e^{-\alpha t_T} - e^{-\alpha t_0}) > \frac{1}{2} (1 - \varepsilon)^3 N (e^{-\alpha \lambda C/2} - o(1)).$$

5 Proof of Theorem 1.5

In the critical case, that is, when $\alpha = 1$, the probability of having a giant component turns out to depend on the value of ν . It will be convenient to work with the *Poisson model* $\mathcal{P}(N; \alpha, \nu)$ in which the vertex set is the set of points of a Poisson point process inside \mathcal{D}_R . In Lemma 5.1 below, we show that for certain graph properties, if they hold a.a.s. in $\mathcal{P}(N; \alpha, \nu)$, then this is also the case in the $\mathcal{G}(N; \alpha, \nu)$ model.

5.1 Poissonisation

Sometimes it is easier to work under the setting where instead of N (fixed) random points, our vertex set consists of $\text{Po}(N)$ points on \mathcal{D}_R – of course still sampled independently according to (1.1). We denote the resulting graph by $\mathcal{P}(N; \alpha, \nu)$. The benefit is that in this way our vertex set consists of a Poisson point process (see for example [17]). In particular, the numbers of points in any finite collection of pairwise disjoint measurable subsets of \mathcal{D}_R are independent Poisson-distributed random variables. The term *asymptotically almost surely (a.a.s.)* has the same meaning in this context as before. We prove the following lemma that allows us to transfer results from the Poisson model into the $\mathcal{G}(N; \alpha, \nu)$ model. Let \mathcal{A}_n denote a set of graphs on $V_n := \{1, \dots, n\}$ that is closed under automorphisms. We call a family $\mathcal{A} = \{\mathcal{A}_n\}_{n \in \mathbb{N}}$ of graphs (*vertex-*) *non-decreasing*, if $G - v \in \mathcal{A}_{n-1}$ for any $v \in V(G)$ implies $G \in \mathcal{A}_n$. Similarly, we call the family (*vertex-*) *non-increasing*, if $G - v \notin \mathcal{A}_{n-1}$ for any $v \in V(G)$ implies $G \notin \mathcal{A}_n$.

Lemma 5.1. *Assume that $\alpha > 0$ is fixed. Let \mathcal{A} be a (*vertex-*) *non-decreasing* family of graphs. For N large enough we have $\mathbb{P}(\mathcal{G}(N; \alpha, \nu) \notin \mathcal{A}) < 4\mathbb{P}(\mathcal{P}(N; \alpha, \nu) \notin \mathcal{A})$. The same holds if \mathcal{A} is (*vertex-*) *non-increasing*.*

Proof. Denote by E_{Po} and E the events that $\mathcal{P}(N; \alpha, \nu) \notin \mathcal{A}$ and $\mathcal{G}(N; \alpha, \nu) \notin \mathcal{A}$, respectively. We write

$$\begin{aligned} \mathbb{P}(E_{\text{Po}}) &= \sum_{N'=0}^{\infty} \mathbb{P}(E_{\text{Po}} | \text{Po}(N) = N') \cdot \mathbb{P}(\text{Po}(N) = N') \\ &\geq \sum_{N'=N}^{\infty} \mathbb{P}(E_{\text{Po}} | \text{Po}(N) = N') \cdot \mathbb{P}(\text{Po}(N) = N') \\ &\geq \sum_{N'=N}^{\infty} \mathbb{P}(E_{\text{Po}} | \text{Po}(N) = N) \cdot \mathbb{P}(\text{Po}(N) = N'), \end{aligned}$$

where we have used in the last line that, since \mathcal{A} is non-decreasing, we have $\mathbb{P}(E_{\text{Po}} | \text{Po}(N) = N') \geq \mathbb{P}(E_{\text{Po}} | \text{Po}(N) = N)$ for $N' \geq N$. Let us also note that $\mathbb{P}(E_{\text{Po}} | \text{Po}(N) = N) = \mathbb{P}(E)$. Thus,

$$\begin{aligned} \mathbb{P}(E_{\text{Po}}) &\geq \sum_{N'=N}^{\infty} \mathbb{P}(E) \cdot \mathbb{P}(\text{Po}(N) = N') \\ &= \mathbb{P}(E) \cdot \mathbb{P}(\text{Po}(N) \geq N) \\ &> \frac{1}{4} \cdot \mathbb{P}(E), \end{aligned}$$

where the last line holds N large enough (by an application of, say, the central limit theorem). The second part of the lemma follows similarly, bounding the sum by taking only the terms where $N' \geq N$. \square

This implies that if $\mathbb{P}(\mathcal{P}(N; \alpha, \nu) \notin \mathcal{A}) = o(1)$, then $\mathbb{P}(\mathcal{G}(N; \alpha, \nu) \notin \mathcal{A}) = o(1)$. The families we consider will be $\mathcal{A} = \{G : G \text{ has a component of order at least } C\}$ or $\mathcal{B} = \{G : G \text{ has no component of order at least } C\}$, where C does not depend on the number of vertices in G . In particular, in the second part of Theorem 1.5, having chosen the parameter N , we consider the family $\{G : G \text{ has a component of order at least } N/65\}$, where $|G| = N$ is not required.

5.2 The subcritical case

We prove the first part of Theorem 1.5 for $\mathcal{P}(N; \alpha, \nu)$ by contradiction. Assuming we have a component of size $\frac{N}{\log \log R}$, then at least $\frac{N}{2 \log \log R}$ vertices of type at most $T = \log \log R$ must be contained in that component as a.a.s. at most $\frac{N}{2 \log \log R}$ vertices have type larger than T .

We will use a *smooth breadth exploration process* starting at a point v_0 of type at most $\log \log R$, a continuous-type version of the process defined in Section 3:

- Let $T_0 = t_{v_0}$.

- Recursively for $i \geq 1$, if v_{i-1} has a neighbour in clockwise direction, let v be the neighbour of v_{i-1} of highest type in clockwise direction and let T_i be the type of v . Let θ_i be the maximum relative angle between v_{i-1} and any adjacent point of type T_i . Define v_i as the point of type T_i with relative angle θ_i from v_{i-1} in the clockwise direction. We call this the i th root vertex.
- There are three stopping conditions for the process:
 - (i) v_i has no neighbour in the clockwise direction;
 - (ii) $T_i > \log R$;
 - (iii) $i = \log^2 \log R$ or the angle between v_i and v_0 exceeds π .

We define stopping times τ_1 and τ_2 as the stopping times that correspond to the first and the second stopping conditions, respectively. Because of the third stopping condition, we know that $0 \leq \tau_i \leq \log \log R$ or $\tau_i = \infty$ for $i = 1, 2$.

The following two lemmas show that the process stops quickly for a suitable choice of ν .

Lemma 5.2. *For $\nu < \frac{\pi}{8}$, starting at a vertex of type at most $\log \log R$, we have*

$$\mathbb{P}(\tau_1 < \infty) = 1 - o\left(\left(\frac{\log \log R}{\log R}\right)^c\right) \text{ for some } c > 0.$$

Lemma 5.3. *For $\nu < \frac{\pi}{8}$, starting at a vertex of type at most $\log \log R$, we have*

$$\mathbb{P}(\tau_2 < \infty) = o\left(\frac{\log^2 \log R}{R^c}\right) \text{ for some } 0 < c < 1.$$

The above process is repeated in the clockwise as well as in the anticlockwise direction. Parametrising with respect to the initial vertex v , we denote the above stopping times by $\tau_1^+(v)$ ($\tau_1^-(v)$, respectively) and $\tau_2^+(v)$ ($\tau_2^-(v)$, resp.). Let also $\tau^+(v)$ and $\tau_-(v)$ denote the stopping times of the two processes.

Using the two lemmas we can prove the first part of Theorem 1.5.

Proof of the first part of Theorem 1.5. In what follows, we assume that N is large enough for all our estimates to hold. For simplicity, we shall denote the set of vertices by V_N - note that this is now a random set of vertices as described above. Let $V_{high} = \{v \in V_N : t_v \geq \log \log R\}$ and $V_{low} = V_N \setminus V_{high}$. In other words, we partition the set of vertices into two parts: that of vertices of low type and that of vertices of high type. Now, we set

$$V_{small} = \{v \in V_{low} : \tau_1^+(v) < \infty, \tau_2^+(v) = \infty \text{ and } \tau_1^-(v) < \infty, \tau_2^-(v) = \infty\},$$

and $V_{large} = V_{low} \setminus V_{small}$. The smooth breadth exploration process started at a vertex $v \in V_{small}$ terminates after at most $\log^2 \log R$ steps and has only had root vertices of type at most $\log R$. In this case, by Lemma 2.1 the angle gained at every single step is at most $2.5 \exp(\log R - R/2)$. Hence, the total angle gained during the process in both directions

is at most $5 \exp(\log R - R/2) \log^2 \log R$. (This justifies the name V_{small} : the total angle of the component of v is bounded by this expression which is a decaying function of N .)

Now, Lemmas 5.2 and 5.3 imply that there exists a positive constant c such that for a vertex $v \in V_{low}$ we have $\mathbb{P}(v \in V_{large} \mid v \in V_{low}) = o\left(\left(\frac{\log \log R}{\log R}\right)^c\right)$. Thereby,

$$\mathbb{E}(|V_{large}|) = o\left(N \left(\frac{\log \log R}{\log R}\right)^c\right).$$

Hence, by Markov's inequality, a.a.s.

$$|V_{large}| \leq \frac{N}{2 \log \log R}.$$

Also, using Lemma 1.3 and the concentration of the Poisson distribution, we can deduce that a.a.s.

$$|V_{high}| \leq \frac{N}{2 \log \log R}.$$

Thus, a.a.s.

$$|V_{large} \cup V_{high}| \leq \frac{N}{\log \log N}.$$

Now, by Lemma 3.8, a.a.s. every vertex in V_{small} is contained in a component with at most $10N \exp(\log R - R/2) \log^2 \log R < 10\nu R \log^2 \log R$ vertices. Hence, any component of large size must be induced by vertices in $V_{large} \cup V_{high}$, whereby $|L_1| \leq \frac{N}{\log \log R}$. \square

To prove the lemmas, we simplify the process in a way that allows for any real types, dropping the $0 \leq t \leq R$ requirement. We use the following cdf and pdf for the type T_i of v_i given the type t_{i-1} of v_{i-1} :

$$F_{T_i}(t) = \exp\left(-4\nu \frac{1.01}{\pi} e^{\frac{1}{2}(t_{i-1}-t)}\right) \tag{5.1}$$

$$f_{T_i}(t) = 2\nu \frac{1.01}{\pi} e^{\frac{1}{2}(t_{i-1}-t)} \exp\left(-4\nu \frac{1.01}{\pi} e^{\frac{1}{2}(t_{i-1}-t)}\right). \tag{5.2}$$

Claim 5.4. *Lemmas 5.2 and 5.3 hold if they hold for the extended and simplified distribution of types in Equation (5.1), where not finding a next neighbour in the original distribution corresponds to a negative type in the extended one.*

Proof. We prove the result by showing that the given cdf is a lower bound on the actual cdf at any point. This means there is a coupling in which any vertex of the actual distribution is coupled with a vertex of higher or equal type and same angle in the simplified distribution. We later prove that the distance we change the type by does not depend on the type of the active vertex for the vertices that we consider, so a higher type cannot have a negative influence on the result we want to prove. If the type of a vertex is less than $\log R$, we can use Lemma 2.1 with $\varepsilon = 0.009$ to get a bound on the relative angle for

possibly adjacent vertices up to type $R - 2 \log R < R - \log R - c_0(0.009)$ for sufficiently large N . Note that the expected number of vertices of type larger than $R - 2 \log R$ is

$$N \frac{\cosh(2 \log R) - 1}{\cosh(R) - 1} = o(1).$$

Thus the probability of having such a vertex is $o(1)$ and we can condition on no such vertex existing. We use outer tubes to estimate the expected value of $N_{t_0}^t$, the number of neighbours of type at least t of a vertex of type $t_0 \leq \log R$, using the outer tubes for $\varepsilon = 0.009$ as an upper bound, taking N large enough and using Lemma 1.3:

$$\begin{aligned} \mathbb{E}N_{t_0}^t &\lesssim N \int_t^{R-2 \log R} 2 \frac{1.009}{\pi} e^{\frac{1}{2}(t_0+t'-R)} e^{-t'} dt' \\ &\leq \nu e^{\frac{1}{2}R} \int_t^{R-2 \log R} 2 \frac{1.01}{\pi} e^{\frac{1}{2}(t_0+t'-R)} e^{-t'} dt' \\ &= 2\nu \frac{1.01}{\pi} e^{\frac{t_0}{2}} \int_t^{R-2 \log R} e^{-\frac{1}{2}t'} dt' \\ &\leq 2\nu \frac{1.01}{\pi} e^{\frac{t_0}{2}} \int_t^\infty e^{-\frac{1}{2}t'} dt' \\ &= 4\nu \frac{1.01}{\pi} e^{\frac{1}{2}(t_0-t)}. \end{aligned}$$

With this, as we are using the Poisson distribution, we get the following cdf for the distribution of the next type, given we are at step i with type t_0 :

$$\begin{aligned} F_{T_i}(t) &= \mathbb{P}(T_i \leq t | T_{i-1} = t_0) = \mathbb{P}(|N_{t_0}^t| = 0) \\ &= \exp(-\mathbb{E}(N_{t_0}^t)) \geq \exp\left(-4\nu \frac{1.01}{\pi} e^{\frac{1}{2}(t_0-t)}\right). \end{aligned}$$

□

To prove the two lemmas we need to introduce new notation. Instead of looking at the types of the vertices at some step, we analyse the jump $J_i = T_i - T_{i-1}$ in some step, the difference in types from one step to the next. This makes sense as $F_{T_i}(t)$ only depends on the difference of T_i and t , each jump is distributed as

$$\begin{aligned} F_J(j) &= \exp\left(-4\nu \frac{1.01}{\pi} e^{-\frac{j}{2}}\right) \\ f_J(j) &= 2\nu \frac{1.01}{\pi} e^{-\frac{j}{2}} \exp\left(-4\nu \frac{1.01}{\pi} e^{-\frac{j}{2}}\right). \end{aligned}$$

Starting at a vertex of type $T_0 (= \log \log R)$, we write

$$T_i = T_0 + \sum_{k=1}^i J_k,$$

where we couple with a sequence of independent random variables having as their cdf the function F_J . The type T_i is thus coupled with the sum of independent copies of the jump.

We are now ready to prove Lemma 5.3.

Proof of Lemma 5.3. We first calculate the following expectation: for $s > 0$ we have

$$\begin{aligned}\mathbb{E}e^{sJ_k} &= \int_{-\infty}^{\infty} e^{sx} 2\nu \frac{1.01}{\pi} e^{-\frac{x}{2}} \exp\left(-4\nu \frac{1.01}{\pi} e^{-\frac{x}{2}}\right) dx \\ &= 2\nu \frac{1.01}{\pi} \int_{-\infty}^{\infty} e^{(s-\frac{1}{2})x} \exp\left(-4\nu \frac{1.01}{\pi} e^{-\frac{x}{2}}\right) dx.\end{aligned}$$

Changing variables $y = 4\nu \frac{1.01}{\pi} e^{-\frac{x}{2}}$, $dy = -2\nu \frac{1.01}{\pi} e^{-\frac{x}{2}} dx$ we get $e^{sx} = \left(4\nu \frac{1.01}{\pi y}\right)^{2s}$ and

$$\begin{aligned}\mathbb{E}e^{sJ_k} &= - \int_{\infty}^0 \left(4\nu \frac{1.01}{\pi y}\right)^{2s} e^{-y} dy = \left(4\nu \frac{1.01}{\pi}\right)^{2s} \int_0^{\infty} y^{-2s} e^{-y} dy \\ &= \left(4\nu \frac{1.01}{\pi}\right)^{2s} \int_0^{\infty} y^{(1-2s)-1} e^{-y} dy = \left(4\nu \frac{1.01}{\pi}\right)^{2s} \Gamma(1-2s).\end{aligned}$$

Given a starting vertex of type $\log \log R$, we calculate, for large N , the probability of reaching type at least $\log R$ in any step $i < \log \log R$. For $s > 0$ arbitrary we have

$$\begin{aligned}\mathbb{P}(T_i \geq \log R) &= \mathbb{P}(T_0 + \sum_{k=1}^i J_k \geq \log R) \leq \mathbb{P}\left(\sum_{k=1}^i J_k \geq \frac{1}{2} \log R\right) \\ &= \mathbb{P}\left(e^s \sum_{k=1}^i J_k \geq e^{\frac{s}{2}} \log R\right) \\ &\leq \mathbb{E}e^{s \sum_{k=1}^i J_k} e^{-\frac{s}{2} \log R} \\ &= e^{-\frac{s}{2} \log R} \prod_{k=1}^i \mathbb{E}e^{sJ_k},\end{aligned}$$

using Markov's inequality. Choosing $0 < s < \frac{1}{2}$ arbitrarily, we get some constant $C > 1$ such that $\mathbb{E}e^{sJ_k} = C$. Thus

$$\mathbb{P}(T_i \geq \log R) \leq e^{-\frac{s}{2} \log R} C^i \leq e^{-\frac{s}{2} \log R} C^{\log^2 \log R} = o(R^{-c}),$$

for some $0 < c = c(s) < 1$. With this, we can use the union bound to bound the probability that the smooth breadth exploration process has type at least $\log R$ at one of the first $\log^2 \log R$ steps:

$$\mathbb{P}(\exists 0 < i \leq \log^2 \log R : T_i \geq \log R) \leq \sum_{i=1}^{\log^2 \log R} \mathbb{P}(T_i \geq \log R) = o\left(\frac{\log^2 \log R}{R^c}\right). \quad \square$$

We use the same technique to prove Lemma 5.2

Proof of Lemma 5.2. We would like to determine the probability that the process $(T_i)_{i \geq 0}$ crosses 0 by step $M := \log^2 \log R$. We have

$$\begin{aligned} \mathbb{P}(T_M \geq 0) &= \mathbb{P}(T_0 + \sum_{k=1}^M J_k \geq 0) \leq \mathbb{P}\left(\sum_{k=1}^M J_k \geq -\log \log R\right) \\ &\stackrel{s > 0}{\leq} \mathbb{P}(e^s \sum_{k=1}^M J_k \geq e^{-s \log \log R}) \\ &\leq (\log R)^s \prod_{k=1}^M \mathbb{E} e^{s J_k}, \end{aligned}$$

again using Markov's inequality. As $\nu < \frac{\pi}{8}$ and using the calculations from the proof of Lemma 5.3, we can find $0 < s < \frac{1}{2}$ and $0 < \hat{c} < 1$ such that for $1 \leq k \leq M$

$$\mathbb{E} e^{s J_k} \leq \left(4\nu \frac{1.01}{\pi}\right)^{2s} \Gamma(1 - 2s) = \hat{c} < 1.$$

With this we have

$$\mathbb{P}(T_M \geq 0) \leq (\log R)^s \hat{c}^{\log^2 \log R} = o\left(\left(\frac{\log \log R}{\log R}\right)^c\right),$$

for some $c > 0$. Thus $\mathbb{P}(T_M < 0) = 1 - o\left(\left(\frac{\log \log R}{\log R}\right)^c\right)$. □

5.3 The supercritical case

For the second part of Theorem 1.5, we split the disk \mathcal{D}_R into cells so that the expected number of vertices in each cell is constant. Furthermore, the cells are defined so that if two neighbouring cells contain at least one vertex each, then these vertices are adjacent. With some rules for adjacencies of the cells we can then explore possible components and estimate the number of vertices in them by the number of cells that correspond to them.

For the discretisation, we define the sequence $t_i = i \log 2$ for $0 \leq i \leq \left\lceil \frac{R}{4 \log 2} \right\rceil =: T$. Using this, we define the bands $B_i = \{p \in \mathcal{D}_R : t_{i-1} \leq t_p < t_i\}$, for $1 \leq i \leq T$. Let $M = 2^T \left\lceil \frac{2\pi}{2^{T-1} 0.95 e^{-R/2}} \right\rceil$. For each i , we split band B_i into $M/2^{i-1}$ cells $C_j^{(i)}$, starting at angle 0 with $j = 1$. Note that the definition of M implies that the projection of a cell in B_i is exactly split into two in B_{i-1} . See Figure 3 for an example of the discretisation. As $2^T = o(e^{\frac{R}{2}})$, it follows that $M = 2^T \frac{2\pi}{2^{T-1} 0.95 e^{-R/2}} (1 + o(1))$, whereby the number of cells in band i is $(1 + o(1)) \frac{2\pi e^{R/2}}{2^{i-1} 0.95}$. We claim the following property:

Lemma 5.5. *Any vertex in cell $C_j^{(i)}$ is adjacent to any vertex in any of the cells $C_{j-1}^{(i)}$, $C_{j+1}^{(i)}$, $C_{2j-1}^{(i-1)}$, $C_{2j}^{(i-1)}$ and $C_{\lceil j/2 \rceil}^{(i+1)}$, where we let $C_0^{(i)} = C_{M/2^{i-1}}^{(i)}$ and $C_{M/2^{i-1}+1}^{(i)} = C_1^{(i)}$.*

Proof. This configuration of cells is illustrated on the left part of Figure 4. As we consider vertices of type at most $R/4 + \log 2$, Lemma 2.1 implies that any two of these vertices

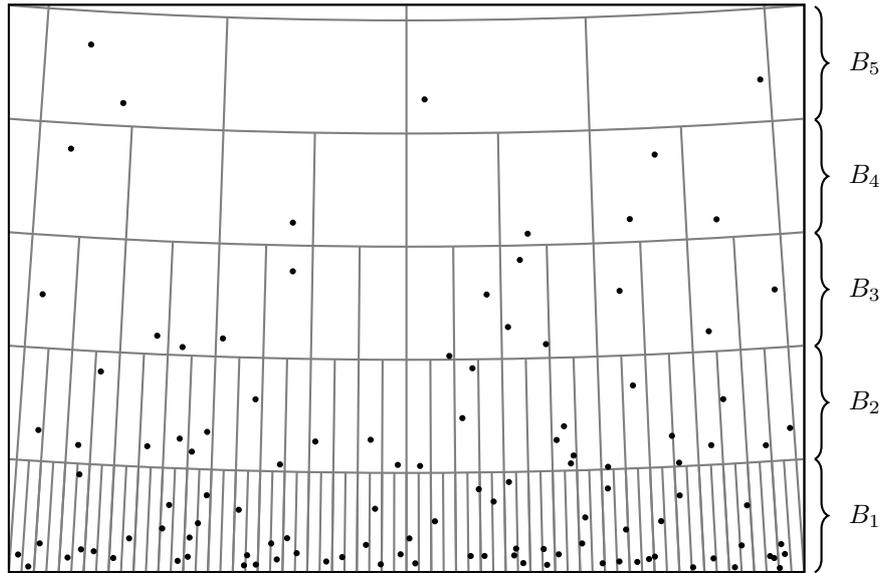


Figure 3: The discretisation of \mathcal{D}_R .

are connected if their relative angle is at most $1.92e^{1/2(t_u+t_v-R)}$, provided that N is large enough.

The vertices in $C_j^{(i)}$ have type at least $t_{i-1} = (i-1)\log 2$ and the relative angle of a vertex v in $C_{j-1}^{(i)}$ and u in $C_j^{(i)}$ is at most $2^{i-1} \cdot 2 \cdot 0.96 e^{-R/2} = 1.92e^{-R/2+(i-1)\log 2} \leq 1.92e^{1/2(t_u+t_v-R)}$, so v and u are adjacent. The same bounds on the maximum relative angle and minimum type hold for vertices u' in $C_j^{(i)}$ and v' in $C_{\lfloor j/2 \rfloor}^{(i+1)}$, so u' and v' are adjacent. By simple change of variables we get all the desired adjacencies. \square

We define two auxiliary graphs, a blue graph G_b and a red graph G_r . The vertices of the blue graph will be those cells that contain at least one vertex, whereas the vertices of the red graph are those that contain no vertex. Two vertices of G_b are connected if the corresponding cells share an edge. This means that vertices of $\mathcal{P}(N; \alpha, \nu)$ in the cells corresponding to adjacent vertices in G_b are themselves adjacent by Lemma 5.5. Two vertices of G_r are connected if their corresponding cells share at least a point. This corresponds to the same adjacency as in G_b but with added “diagonal” edges. These adjacencies are illustrated in Figure 4. We denote by G_c the union of the two graphs. For the graph resulting from the example in Figure 3, see Figure 5. Whenever necessary we will refer to the vertices of the graphs as cells. A *blue component* is a component of G_c that consists of blue vertices. A *red path* is a path in G_c that consists of red vertices. These are the main structures we use, but we also use the red/blue notation for other structures. Because of the adjacency rules, a blue component is always surrounded by a red path or a collection of red paths, the periphery and the inside of the disk. We are now interested in the probability of a vertex being blue or red. Note that

$$M = 2^T \left[\frac{2\pi}{2^T 0.95e^{-R/2}} \right] \leq \frac{2\pi}{0.95e^{-R/2}} + 2^T \leq \frac{2\pi + 2^T e^{-R/2}}{0.95e^{-R/2}}. \quad (5.3)$$

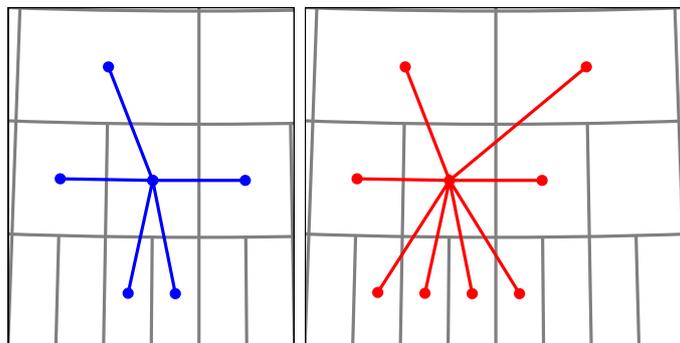


Figure 4: Blue(left) and red(right) neighbourhood.

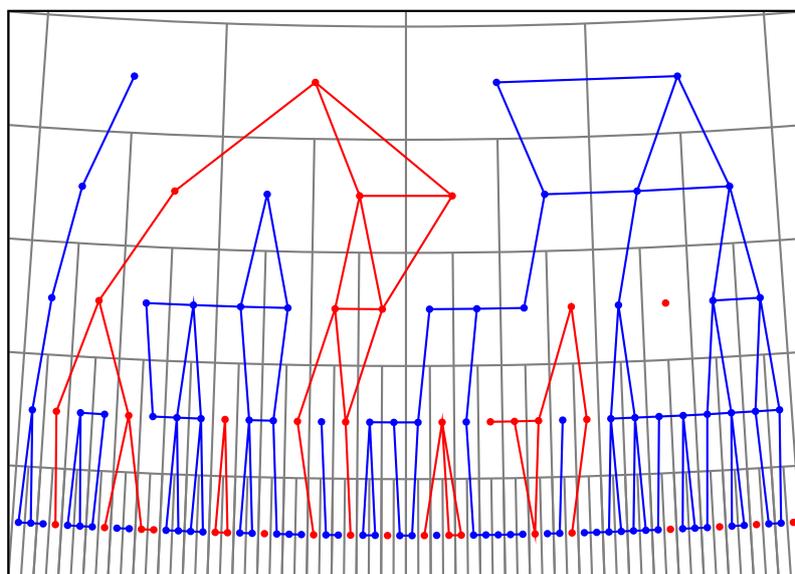


Figure 5: G_c resulting from the Graph in Figure 3.

Claim 5.6. *The probability of a vertex in G_c being red is at most $e^{-\frac{\nu}{5\pi}}$, the probability of being blue is at least $1 - e^{-\frac{\nu}{5\pi}}$.*

Proof. Let $v_{i,j}$ be a vertex of G_c that corresponds to $C_i^{(j)}$. Let $P_i^{(j)}$ be the number of vertices in $C_i^{(j)}$. Using Lemma 1.3 and the above upper bound on M (5.3), we have for large N

$$\begin{aligned} \mathbb{E}(P_i^{(j)}) &= N \int_{t_{i-1}}^{t_i} \frac{2^{i-1}}{M} (1 + o(1)) e^{-t} dt \\ &\geq 0.94\nu e^{R/2} \frac{2^{i-1} e^{-R/2}}{2\pi + 2^T e^{-R/2}} \int_{t_{i-1}}^{t_i} e^{-t} dt \\ &\geq 0.94\nu \frac{2^{i-1}}{2\pi + 2^{\frac{R}{4\log 2} + 1} e^{-R/2}} (e^{-t_{i-1}} - e^{-t_i}) \\ &= 0.94\nu \frac{2^{i-1}}{2\pi + 2e^{-R/4}} (e^{-(i-1)\log 2} - e^{-i\log 2}) \\ &= 0.94\nu \frac{2^{i-1}}{2\pi + 2e^{-R/4}} 2^{-i} (2 - 1) \\ &\geq \frac{\nu}{5\pi}, \end{aligned}$$

for N large enough. Since the number of vertices in $C_i^{(j)}$ follows the Poisson distribution with parameter $\mathbb{E}(P_i^{(j)})$, we have

$$\begin{aligned} \mathbb{P}(v_{i,j} \text{ red}) &= \mathbb{P}(P_i^{(j)} = 0) = e^{-\mathbb{E}(P_i^{(j)})} \leq e^{-\frac{\nu}{5\pi}} \\ \mathbb{P}(v_{i,j} \text{ blue}) &= \mathbb{P}(P_i^{(j)} > 0) = 1 - e^{-\mathbb{E}(P_i^{(j)})} \geq 1 - e^{-\frac{\nu}{5\pi}} \end{aligned}$$

□

Our aim is now to show that there is a blue component of εN cells for some $\varepsilon > 0$. Note that this implies that a giant component in $\mathcal{P}(N; \alpha, \nu)$ exists as each of the εN cells contain at least one vertex and the adjacency rules of the vertices of G_c imply that these vertices induce a connected component in $\mathcal{P}(N; \alpha, \nu)$. Our main tool for proving this is the following lemma that bounds the maximum length of a red path, using an argument similar to Peierls' argument from percolation theory (see [15] [20]).

Lemma 5.7. *Let $\nu > 20\pi$. A.a.s. all red paths have length at most $L := T - 1 = \left\lceil \frac{R}{4\log 2} \right\rceil - 1$.*

Proof. Note that any cell in the red graph has at most 8 neighbours. The number of cells is at most $M(1 + 1/2 + 1/4 + \dots) = 2M \stackrel{(5.3)}{<} \frac{4\pi}{0.94} e^{R/2} < 5\pi e^{R/2}$. As $\nu > 20\pi$, this implies that the number of cells is at most N . Let $p = \mathbb{P}(v_{i,j} \text{ blue}) \geq 1 - e^{-\frac{\nu}{5\pi}} \geq 1 - e^{-\frac{20\pi}{5\pi}} > 0.98$ by Claim 5.6. Thus $8(1 - p) < 0.2$. Let $P_c(\ell)$ be the number of red paths starting at cell c and having length ℓ . We have

$$\mathbb{E}(P_c(\ell)) \leq 8^\ell (1 - p)^\ell = (8(1 - p))^\ell < 0.2^\ell.$$

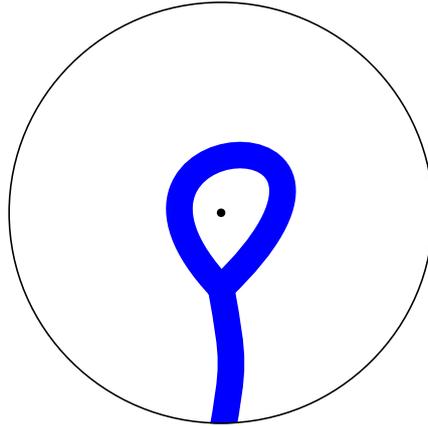


Figure 6: Blue lollipop.

Since the number of cells is at most N we thus can bound the expected number of red paths of length ℓ by $0.2^\ell N$. Let c be such that $1 + c = \frac{\log 5}{2 \log 2} \simeq 1.16$. The expected number of paths of length at least L is

$$\begin{aligned} N \sum_{\ell \geq L} 0.2^\ell &= O(N 0.2^L) = O\left(e^{R/2} 0.2^{\frac{R}{4 \log 2}}\right) \\ &= O\left(e^{R/2} e^{-\log 5 \left(\frac{R}{4 \log 2}\right)}\right) \\ &= O\left(e^{R/2} e^{-(1+c)\frac{R}{2}}\right) = O\left(\nu e^{-c\frac{R}{2}}\right) = o(1). \end{aligned}$$

This means the probability of having a red path of length at least L is $o(1)$. □

With this we are able to make statements on the structure of G_c . We prove that a.a.s. there is a *blue lollipop* L_b : a blue cycle surrounding the origin of \mathcal{D}_R that contains a cell in B_1 or is connected to such a cell by a blue path (see Figure 6). We call the relevant cell of B_1 the *base* of the lollipop.

Claim 5.8. *Let $\nu > 20\pi$. A.a.s. G_c contains a blue lollipop.*

Proof. By Lemma 5.7, a.a.s. there is no red path of length L . The number of bands is $T > L$, so a.a.s. there is no red path connecting B_1 and B_T . This implies that there must be a blue cycle C surrounding the origin of \mathcal{D}_R . Now, the number of cells in any band is at least

$$M/2^T \geq \frac{2\pi}{2^T e^{-R/2}} \geq \frac{2\pi}{2^{\frac{R}{4 \log 2} + 1} e^{-R/2}} = \pi e^{R/4} \geq L$$

for N sufficiently large, as $L = O(R)$. Thus any red cycle surrounding C would have length at least L . But by Lemma 5.7 a.a.s. there is no such cycle. This implies that C must either contain a cell of B_1 or there must be a blue path P connecting C and some cell in B_1 . In either case we have a blue lollipop. □

Starting at the base c_1 of the lollipop we will consider the following process traversing the band B_1 in clockwise direction:

- Starting with $i = 1$, do the following steps until c_1 is reached again.
- Let c'_i be the first red cell in B_1 in clockwise direction from c_i . Let S_i be the number of blue cells from c_i to c'_i , including c_i .
- Let r_i be the red cell on B_1 that is farthest away from c'_i in clockwise direction and that is connected to c'_i via a red path - if such a cell does not exist, then $r_i = c'_i$. Denote by R_i the number of cells of B_1 between c'_i and r_i in clockwise direction, including these two. Let c_{i+1} be the cell succeeding r_i .

The process will end at some index $i = K$. Assume that S_K is the number of blue cells between c_K and c_1 in clockwise direction, including c_K if $c_K \neq c_1$ but excluding c_1 . Note that the number of cells in B_1 is $M = 2^T \lceil \frac{2\pi}{2^{T0.95}e^{-R/2}} \rceil \geq \frac{2\pi}{\nu}N$. This means we have $S_K + \sum_{i=1}^{K-1} (S_i + R_i) = M \geq \frac{2\pi}{\nu}N$. We will prove that a.a.s. K is linear in N . We begin with the following properties:

Claim 5.9.

1. Any two cells c_i and c_j are connected by a blue path for $1 \leq i, j \leq K$.
2. If the red path connecting c'_i and r_i has length ℓ , then $R_i < 2^{\ell/2+2}$.

Proof.

1. Let \hat{c} be the blue cell preceding c'_1 . If there was no blue path connecting \hat{c} to c_2 then there would be a red path originating at a red cell between c'_1 and c_2 and ending in a red cell \hat{r}_1 that is farther in the anticlockwise direction from c'_1 than c_2 , thus also farther than r_1 . But this means this red path must meet the path from c'_1 to r_1 , creating a path from c'_1 to \hat{r}_1 — a contradiction. This means that \hat{c} is connected to c_2 via a blue path. But all the cells between c_1 and \hat{c} are blue, so there is a blue path from c_1 to c_2 . Similarly, we can show that c_i is connected to c_{i+1} .
2. We fix ℓ and want to find the path of length ℓ between c'_i and r_i that “encloses” most cells in B_1 . Note that the cells in a band become half as many when we increase the index of the band by one. We claim that the optimal path for this choice increases the band by one with each of its first $\lfloor \ell/2 \rfloor$ edges, stays in the same band for one edge if ℓ is odd, and then decreases the band by one for the remaining $\lfloor \ell/2 \rfloor$ edges, such as the one in Figure 7. Assume this is not the case, then at least one of the following must be true:
 - There is an edge between two cells at the same band that is not of the highest index among all cells in the path. Then, taking away this edge and instead inserting one at the highest index, thus shifting the remaining path, means that at least one more cell is covered with a path of the same length, a contradiction.

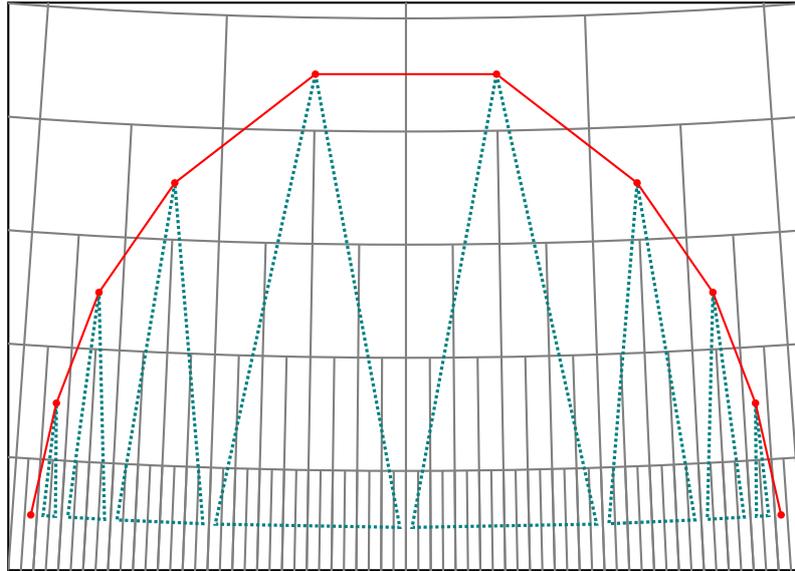


Figure 7: Red path of length 9 covering the maximum number of cells.

- There is an index i such that there are two cells c_1 and c_2 , both in the band B_i , that are connected on the path via a subpath of length at least 2 using only cells in the bands $B_1 - B_i$. In this case we can create a new subpath of the same length that just uses cells in the band B_{i+1} as inner vertices and leading to a new cell c'_2 . Because the number of cells doubles in each band this new subpath must cover more cells than the old one and thus we can create a path of length ℓ that covers more cells, a contradiction.

So the optimal path has the desired form, as shown in Figure 7. Note that each cell in band i covers 2^{i-1} cells in band B_1 . If ℓ is odd, this yields $\frac{\ell-1}{2}$ upward/downward edges each and one edge (2 cells) staying at the same level, yielding $\sum_{i=1}^{(\ell-1)/2} 2^{i-1} + 2 \cdot 2^{(\ell-1)/2} = 3 \cdot 2^{(\ell-1)/2} - 1 < 2^{\ell/2+2}$ covered cells. If ℓ is even, the path uses $\frac{\ell}{2}$ upward/downward edges each and no edge (one cells) staying at the same level, yielding $\sum_{i=1}^{\ell/2} 2^{i-1} + 2^{\ell/2} = 2^{\ell/2+1} - 1 < 2^{\ell/2+2}$ covered cells. \square

Let ℓ_i be the length of the red path connecting c'_i and r_i . We define independent random variables K_i distributed as $\text{Geom}(1 - 8e^{-\nu/5\pi})$. Note that this independence can only make the bound larger as no two of the corresponding paths can meet, so a path cannot use anything that has been exposed already by a different path. Also, the number of available next cells in any step can only go down if we consider the dependent case. Because every red cell has at most 8 neighbours and every cell is red with probability at most $e^{-\nu/5\pi}$, we have

$$\mathbb{P}(\ell_i \geq k_i) < (8e^{-\nu/5\pi})^{k_i}.$$

In other words, ℓ_i is stochastically bounded from above by K_i . Also, by Claim 5.9(ii) R_i is stochastically bounded by $2^{\ell_i/2+2}$, which in turn is stochastically bounded from above by $2^{K_i/2+2}$. We denote the latter by Y_i .

We define independent random variables X_i and Y_i , where $X_i = \text{Geom}(e^{-\nu/5\pi})$ and $Y_i = 2^{K_i/2+2}$. With this S_i is stochastically bounded by X_i and R_i is stochastically bounded by Y_i . Define

$$L_t = \sum_{i=1}^t (X_i + Y_i)$$

and let $\mathcal{T} := \max\{t : L_t < \frac{2\pi}{\nu}N\}$. As the number of cells in B_1 is at least $\frac{2\pi}{\nu}N$ and with Claim 5.9, \mathcal{T} is a stochastic lower bound on the number of steps we need to take to cover B_1 with S_i and R_i , which we denoted by K . But the latter is also a bound on the vertices of $\mathcal{P}(N; \alpha, \nu)$ which belong to the component that contains the vertices induced by the lollipop. With $\mu_1 = \mathbb{E}(X_1) = e^{\nu/5\pi}$ and $\mu_2 = \mathbb{E}(Y_1)$ we have $\mathbb{E}(L_t) = t(\mu_1 + \mu_2)$. Let $\hat{t} = \frac{1.9\pi}{\nu(\mu_1 + \mu_2)}N$. Note that $\mu_1 + \mu_2 > 1$ as $\mu_2 \geq 0$ and $\mu_1 = e^{\nu/5\pi} > 1$. Hence, $\mathbb{E}(L_{\hat{t}}) \leq \frac{1.9\pi}{\nu}N$. If $\mathbb{P}(L_{\hat{t}} > \frac{2\pi}{\nu}N) = o(1)$, it follows that a.a.s. $\mathcal{T} > \hat{t}$. Hence, by the stochastic domination we deduce that a.a.s. $K > \hat{t}$.

Claim 5.10. *Let $\nu > 20\pi$. We have*

$$\mathbb{P}\left(L_{\hat{t}} > \frac{2\pi}{\nu}N\right) = o(1).$$

Proof. We have

$$\begin{aligned} \text{Var}(L_t) &= t(\text{Var}(X_1) + \text{Var}(Y_1)), \\ \text{Var}(X_1) &= \text{Var}(\text{Geom}(e^{-\nu/5\pi})) = \frac{1 - e^{-\nu/5\pi}}{e^{-2\nu/5\pi}} < e^{\frac{\nu}{2\pi}} \text{ and} \\ \text{Var}(Y_1) &= \mathbb{E}(Y_1^2) - \mathbb{E}^2(Y_1). \end{aligned}$$

But

$$\mathbb{E}(Y_1^2) = \sum_{k=1}^{\infty} 2^{k+4} (8e^{-\nu/5\pi})^k \leq 16 \sum_{k=1}^{\infty} (16e^{-\nu/5\pi})^k \stackrel{\nu > 20\pi}{<} 16 \sum_{k=1}^{\infty} (2/e)^k < \frac{32}{e-2} = 64.$$

Thus $\text{Var}(Y_1) < 64$ and $\text{Var}(L_{\hat{t}}) < (64 + e^{\frac{\nu}{2\pi}})\hat{t} = O(N)$. By Chebyshev's inequality,

$$\mathbb{P}\left(L_{\hat{t}} > \frac{2\pi}{\nu}N\right) = O(1/N). \quad \square$$

Note that $\mathbb{E}(X_1) = e^{\frac{\nu}{5\pi}}$ and

$$\mathbb{E}(Y_1) \leq 1 + \sum_{k=1}^{\infty} (8e^{-\nu/5\pi})^k 2^{k/2+2} \stackrel{\nu > 20\pi}{<} 1 + 4 \sum_{k=1}^{\infty} (8e^{-4})^k 2^{k/2} < 1 + 4 \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k < 3.$$

So $\hat{t} > \frac{1.9\pi}{\nu(e^{\nu/5\pi}+3)}N$. We can now prove the second part of the main theorem.

Proof of the second part of Theorem 1.5. Let $\nu > 20\pi$ and let G_c be defined as above. By Claim 5.8 there is a blue lollipop a.a.s. By Claims 5.9.1 and 5.10 a.a.s. the blue lollipop extends into a blue component of order at least $\hat{t} > \frac{1.9\pi}{\nu(e^{\nu/5\pi}+3)}N$. Setting $\nu = 20\pi$, this quantity is at least $\frac{1.9\pi}{20\pi(e^4+3)}N > N/610$. Note that increasing ν can only stochastically increase the order of the largest connected component. Thus, for any such $\nu > 20\pi$, the order of the largest component is a.a.s. at least $N/610$. \square

6 Conclusion

In this paper we have studied the component structure of the KPKVB-model, a geometric model of random graphs on the hyperbolic plane. This can be viewed as a dependent version of the well-known Chung-Lu model of inhomogeneous random graphs. The model was recently introduced by Krioukov et al. [19], whose aim is to develop a geometric framework for the study of complex networks. We determine the critical parameters which control the typical component structure of such a random graph. Namely, we determine the critical parameter which controls the emergence of a giant component. We show that in the regime where the random graph exhibits power law degree distribution with exponent greater than 3, all components are sub-linear. On the other hand, when the exponent of the power law is less than 3, then a component that contains a positive fraction of the vertices exists with high probability.

However, our results are not precise as far as the size of the giant component is concerned. Showing, for example, a law of large numbers seems to be a challenging problem, mainly due to the dependencies that are present in this model. Also, for the case $\alpha = 1$, we conjecture that there is a critical value for ν above which the giant component emerges.

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A Proof of Lemma 1.1

Proof of Lemma 1.1. Note that $R = (2/\zeta) \log(N/\nu)$ and $R' := (2/\zeta') \log(N/\nu)$ are chosen such that $N = \nu e^{\zeta R/2} = \nu e^{\zeta' R'/2}$.

The desired coupling is constructed as follows. We pick $\theta_1, \dots, \theta_N$ i.i.d. uniform on $[0, 2\pi)$ and we pick U_1, \dots, U_N i.i.d. uniform on $[0, 1]$.

We now let ρ_1, \dots, ρ_N and ρ'_1, \dots, ρ'_N be defined by the equations:

$$F_{\alpha,R}(\rho_i) = F_{\alpha',R'}(\rho'_i) = U_i \quad (\text{for } i = 1, \dots, N) \quad (\text{A.1})$$

where $F_{\alpha,R}$ is the cdf that goes with the pdf (1.1). That is:

$$F_{\alpha;R}(r) = \begin{cases} 0 & \text{if } r < 0, \\ \frac{\cosh(\alpha r) - 1}{\cosh(\alpha R) - 1} & \text{if } 0 \leq r \leq R; \\ 1 & \text{otherwise.} \end{cases} \quad (\text{A.2})$$

(Note that in this way the ρ_i s have exactly the distribution with pdf (1.1) and the ρ'_i s have the same pdf but with α', R' in place of α, R .) The points used in the construction of $G(N; \zeta, \alpha, \nu)$ will be $(\theta_1, \rho_1), \dots, (\theta_N, \rho_N)$ while the points used in the construction of $G(N; \zeta', \alpha', \nu)$ will be $(\theta_1, \rho'_1), \dots, (\theta_N, \rho'_N)$.

It remains to be seen that this way we get two isomorphic graphs.

Claim A.1. *We have $\rho'_i = (\alpha/\alpha')\rho_i$ for all i .*

Proof. Observe that

$$\alpha'R' = \alpha' \cdot (\zeta/\zeta')R = \alpha' \cdot (\alpha/\alpha')R = \alpha R.$$

Thus, the equation (A.1) defining ρ_i and ρ'_i yields:

$$\cosh(\alpha\rho_i) = \cosh(\alpha'\rho'_i).$$

Since $\cosh(x)$ is strictly increasing for $x \geq 0$, it follows that we must have $\alpha\rho_i = \alpha'\rho'_i$. \square

Let us write d_{ij} for the distance between (θ_i, ρ_i) and (θ_j, ρ_j) in the curvature- ζ -surface, and let d'_{ij} be defined analogously.

Claim A.2. *For all i, j we have $d'_{ij} = (\alpha/\alpha')d_{ij}$.*

Proof. By the *hyperbolic cosine rule* we have that

$$\cosh(\zeta d_{ij}) = \cosh(\zeta\rho_i) \cosh(\zeta\rho_j) - \sinh(\zeta\rho_i) \sinh(\zeta\rho_j) \cos(|\theta_i - \theta_j|),$$

and

$$\cosh(\zeta' d'_{ij}) = \cosh(\zeta'\rho'_i) \cosh(\zeta'\rho'_j) - \sinh(\zeta'\rho'_i) \sinh(\zeta'\rho'_j) \cos(|\theta_i - \theta_j|).$$

Now observe that

$$\zeta\rho_i = \alpha \cdot (\zeta/\alpha) \cdot \rho_i = \alpha \cdot (\zeta'/\alpha')\rho_i = \zeta'\rho'_i,$$

using Claim A.1, and similarly $\zeta\rho_j = \zeta'\rho'_j$. It follows that

$$\cosh(\zeta' d'_{ij}) = \cosh(\zeta d_{ij}).$$

Again using that $\cosh(x)$ is strictly increasing for $x \geq 0$ (and the distances d_{ij}, d'_{ij} are nonnegative), we see that $d'_{ij} = (\zeta/\zeta')d_{ij} = (\alpha/\alpha')d_{ij}$. \square

Since $R' = (\zeta/\zeta')R = (\alpha/\alpha')R$, we see that

$$d_{ij} \leq R \text{ if and only if } d'_{ij} \leq R',$$

which proves the lemma. \square

B The proof of Lemma 1.2

Very similarly to the proof of Lemma 1.1, the coupling is constructed as follows. We pick $\theta_1, \dots, \theta_N$ i.i.d. uniform on $[0, 2\pi)$ and we pick U_1, \dots, U_N i.i.d. uniform on $[0, 1]$. We now let ρ_1, \dots, ρ_N and ρ'_1, \dots, ρ'_N be defined by the equations:

$$F_{\alpha,R}(\rho_i) = F_{\alpha',R'}(\rho'_i) = U_i \quad (\text{for } i = 1, \dots, N.) \quad (\text{B.1})$$

($F_{\alpha,R}$ is as defined in the proof of Lemma 1.1, and $R := 2 \log(N/\nu)$, $R' := 2 \log(N/\nu')$.) Again, we note that in this way the ρ_i s have exactly the distribution with cdf $F_{\alpha,R}$ and the ρ'_i s have cdf $F_{\alpha',R'}$. The points used in the construction of $G(N; \alpha, \nu)$ will be $(\rho_1, \theta_1), \dots, (\rho_N, \theta_N)$ while the points used in the construction of $G(N; \alpha', \nu')$ will be $(\rho'_1, \theta_1), \dots, (\rho'_N, \theta_N)$.

We need the following geometric fact.

Lemma B.1. *Suppose that $p = (r, \theta), q = (s, \vartheta)$ are two points in the hyperbolic plane satisfying $\text{dist}_{\mathbb{H}}(p, O), \text{dist}_{\mathbb{H}}(q, O), \text{dist}_{\mathbb{H}}(p, q) \leq R$ and let $p' = (r', \theta), q' = (s', \vartheta)$ with $r' \leq r, s' \leq s$. Then $\text{dist}_{\mathbb{H}}(p', q') \leq R$.*

Before giving the proof of this lemma, let us remind the reader that disks are convex, also in the hyperbolic plane. This means that if D is a disk in the hyperbolic plane and $x, y \in D$ then the geodesic between x, y is contained in D . One way to see this is by noting that every disk can be isometrically mapped to a disk with origin O , and that in the projective disk model of the hyperbolic plane (a.k.a. the Beltrami-Klein model) a hyperbolic disk with origin O looks like a Euclidean disk, while geodesics are just line segments in the projective disk model. (See for instance Section 4.8 of [22] for a description of the projective disk model.)

Proof of Lemma B.1: It is enough to consider the case when $r' < r$ and $s' = s$. (Another application of this case will then give the full result.) Observe that the geodesic between O and p is just the line segment between them. So in particular, p' lies on the geodesic between O and p . Since $O, p \in B(q; R)$ it follows that also $p' \in B(q; R)$, as required. ■

We also need the following observation, which can be rephrased as stating that the radius under the (α, R) -quasi uniform distribution stochastically dominates the radius under the (α', R) -quasi uniform distribution if $\alpha > \alpha'$.

Lemma B.2. *If $\alpha \geq \alpha'$ and $\nu = \nu'$ then $F_{\alpha,R}(r) \leq F_{\alpha',R}(r)$ for every $r \in \mathbb{R}$.*

Proof. Note that $\nu = \nu'$ implies that also $R = R'$. Let us thus fix $R > 0$ and $0 < r < R$, and define $\varphi(\alpha) := F_{\alpha,R}(r)$ for every $\alpha > 0$. Our aim will be to show that $\frac{d\varphi}{d\alpha}$ is non-positive for every $\alpha > 0$, which will clearly yield the result.

We obtain:

$$\frac{d\varphi}{d\alpha} = \frac{r \sinh(\alpha r)(\cosh(\alpha R) - 1) - R \sinh(\alpha R)(\cosh(\alpha r) - 1)}{(\cosh(\alpha R) - 1)^2}.$$

Observe that this is non-positive if and only if

$$\frac{\alpha R \sinh(\alpha R)}{\cosh(\alpha R) - 1} \geq \frac{\alpha r \sinh(\alpha r)}{\cosh(\alpha r) - 1}.$$

We claim this is the case for all $0 \leq r \leq R$. To see this, it suffices to show that $(x \sinh x)/(\cosh x - 1)$ is nondecreasing for $x \geq 0$. Let us thus compute

$$\begin{aligned} \left[\frac{x \sinh x}{\cosh x - 1} \right]' &= \frac{(\sinh x + x \cosh x)(\cosh x - 1) - x \sinh^2 x}{(\cosh x - 1)^2} \\ &= \frac{\sinh x \cosh x + x \cosh^2 x - \sinh x - x \cosh x - x \sinh^2 x}{(\cosh x - 1)^2} \\ &= \frac{\sinh x \cosh x + x(\cosh^2 x - \sinh^2 x) - \sinh x - x \cosh x}{(\cosh x - 1)^2} \\ &= \frac{\sinh x \cosh x + x - \sinh x - x \cosh x}{(\cosh x - 1)^2} \\ &= \frac{(\sinh x - x)(\cosh x - 1)}{(\cosh x - 1)^2} \\ &\geq 0. \end{aligned}$$

So our claim holds, and we see that indeed $\frac{d\varphi}{d\alpha} \leq 0$ for all $\alpha > 0$. This proves the lemma. \square

Lemma B.3. *If $\alpha = \alpha'$ and $\nu \leq \nu'$, then $F_{\alpha,R}(r) \leq F_{\alpha',R'}(r)$ for every $r \in \mathbb{R}$.*

Proof. Observe that $\nu \leq \nu'$ implies that $R \geq R'$. This also gives $\cosh(\alpha R) - 1 \geq \cosh(\alpha R') - 1$, and hence the lemma. \square

Combining the last two lemmas gives:

Corollary B.4. *If $\alpha \geq \alpha'$ and $\nu \leq \nu'$, then $F_{\alpha,R}(r) \leq F_{\alpha',R'}(r)$ for every $r \in \mathbb{R}$.*

Together with the definition of ρ_i, ρ'_i this immediately gives:

Corollary B.5. *If $\alpha \geq \alpha'$ and $\nu \leq \nu'$, then, in the coupling described above, we have that $\rho_i \geq \rho'_i$ for all $1 \leq i \leq N$.*

Corollary B.5 together with Lemma B.1 yield Lemma 1.2.

C Corrigendum added 29 Dec 2018

The proof of Lemma 1.2 in the earlier version of the paper contained a slight oversight. Here we provide an corrected statement with a full proof.

Lemma C.1. *Let $\alpha, \alpha', \nu, \nu' > 0$ be such that $\alpha \geq \alpha'$ and $\nu \leq \nu'$. Then there exists a coupling such that, with probability tending to one as $N \rightarrow \infty$, the graph $G(N; \alpha, \nu)$ is a subgraph of $G(N; \alpha', \nu')$.*

The only change of the statement compared to that of Lemma 1.2 is the addition of the phrase “with probability tending to one as $N \rightarrow \infty$ ”. This means that for practically all applications of the old lemma, certainly the ones we have in mind, the new lemma serves just as well.

We use the same coupling as in the original paper, but for completeness we specify it again. We pick $\vartheta_1, \dots, \vartheta_N$ i.i.d. uniform on $[0, 2\pi)$ and we pick U_1, \dots, U_N i.i.d. uniform on $[0, 1]$. We now let ρ_1, \dots, ρ_N be defined by the equations:

$$F_{\alpha,R}(\rho_i) = U_i \quad (\text{for } i = 1, \dots, N.) \quad (\text{C.1})$$

where $F_{\alpha,R}(\cdot)$ denotes the cdf of the radius coordinate under the α -quasi uniform distribution. That is,

$$F_{\alpha,R}(r) = \begin{cases} 0 & \text{if } r < 0, \\ \frac{\cosh(\alpha r) - 1}{\cosh(\alpha R) - 1} & \text{if } 0 \leq r \leq R; \\ 1 & \text{otherwise.} \end{cases} \quad (\text{C.2})$$

Note that the ρ_i s have exactly the distribution with cdf $F_{\alpha,R}$. The points used in the construction of $G(N; \alpha, \nu)$ will be $(\rho_1, \vartheta_1), \dots, (\rho_N, \vartheta_N)$ where we take $R = 2 \log(N/\nu)$.

(This coupling defines $G(N; \alpha, \nu)$ simultaneously for all $\alpha, \nu > 0$.)

Let us also recall that, by the hyperbolic cosine rule, provided $\rho_i + \rho_j > R$, there is an edge between vertex i and vertex j if and only if

$$\cos(\vartheta_{ij}) \geq \frac{\cosh \rho_i \cosh \rho_j - \cosh R}{\sinh \rho_i \sinh \rho_j}, \quad (\text{C.3})$$

where $\vartheta_{ij} = \min(|\vartheta_i - \vartheta_j|, 2\pi - |\vartheta_i - \vartheta_j|)$ is the angle between the i -th and the j -th point. If $\rho_i + \rho_j \leq R$ then by the triangle inequality there is always an edge between vertex i and vertex j .

We will split the proof of Lemma C.1 into two parts, as follows.

Lemma C.2. *If $\alpha \geq \alpha'$ and $\nu = \nu'$ then, under the coupling described above, we have that $G(N; \alpha, \nu)$ is a subgraph of $G(N; \alpha', \nu')$.*

Lemma C.3. *If $\alpha = \alpha'$ and $\nu \leq \nu'$ then under the coupling described above, with probability tending to one as $N \rightarrow \infty$, we have that $G(N; \alpha, \nu)$ is a subgraph of $G(N; \alpha', \nu')$.*

Together these lemmas clearly prove Lemma C.1. Lemma C.2 relies on Lemma B.2 above, but we include a proof here as well for completeness. The proof of Lemma C.3 needs an additional argument that unfortunately was omitted in the original version of our paper.

The proof of Lemma C.2

We will imagine the situation where $\vartheta_1, \dots, \vartheta_N$ and U_1, \dots, U_N and $R = 2 \log(N/\nu)$ remain fixed as we increase α , and ρ_1, \dots, ρ_N are defined via (C.1). It is enough to show that, as we increase α , no new edges will be introduced.

We need the following observation, which can be rephrased as stating that the radius under the (α, R) -quasi uniform distribution stochastically dominates the radius under the (α', R) -quasi uniform distribution if $\alpha > \alpha'$.

Lemma C.4. *If $\alpha \geq \alpha'$ and $\nu = \nu'$ then $F_{\alpha,R}(r) \leq F_{\alpha',R}(r)$ for every $r \in \mathbb{R}$.*

Proof. Note that $\nu = \nu'$ implies that also $R = R'$. Let us thus fix $R > 0$ and $0 < r < R$, and define $\varphi(\alpha) := F_{\alpha,R}(r)$ for every $\alpha > 0$. Our aim will be to show that $\frac{d\varphi}{d\alpha}$ is non-positive for every $\alpha > 0$, which will clearly yield the result.

We obtain:

$$\frac{d\varphi}{d\alpha} = \frac{r \sinh(\alpha r)(\cosh(\alpha R) - 1) - R \sinh(\alpha R)(\cosh(\alpha r) - 1)}{(\cosh(\alpha R) - 1)^2}.$$

Observe that this is non-positive if and only if

$$\frac{\alpha R \sinh(\alpha R)}{\cosh(\alpha R) - 1} \geq \frac{\alpha r \sinh(\alpha r)}{\cosh(\alpha r) - 1}.$$

We claim this is the case for all $0 \leq r \leq R$. To see this, it suffices to show that $(x \sinh x)/(\cosh x - 1)$ is nondecreasing for $x \geq 0$. Let us thus compute

$$\begin{aligned} \left[\frac{x \sinh x}{\cosh x - 1} \right]' &= \frac{(\sinh x + x \cosh x)(\cosh x - 1) - x \sinh^2 x}{(\cosh x - 1)^2} \\ &= \frac{\sinh x \cosh x + x \cosh^2 x - \sinh x - x \cosh x - x \sinh^2 x}{(\cosh x - 1)^2} \\ &= \frac{\sinh x \cosh x + x(\cosh^2 x - \sinh^2 x) - \sinh x - x \cosh x}{(\cosh x - 1)^2} \\ &= \frac{\sinh x \cosh x + x - \sinh x - x \cosh x}{(\cosh x - 1)^2} \\ &= \frac{(\sinh x - x)(\cosh x - 1)}{(\cosh x - 1)^2} \\ &\geq 0. \end{aligned}$$

So our claim holds, and we see that indeed $\frac{d\varphi}{d\alpha} \leq 0$ for all $\alpha > 0$. This proves the lemma. \square

Note that Lemma C.4 implies that ρ_i is non-decreasing in α for $i = 1, \dots, N$. To complete the proof of Lemma C.2, it therefore suffices to show the right-hand side of (C.3)

is non-decreasing in ρ_i, ρ_j . To see this we differentiate:

$$\begin{aligned} \frac{\partial}{\partial \rho_i} \left[\frac{\cosh \rho_i \cosh \rho_j - \cosh R}{\sinh \rho_i \sinh \rho_j} \right] &= \frac{\sinh \rho_i \cosh \rho_j}{\sinh \rho_i \sinh \rho_j} - \frac{\cosh \rho_i (\cosh \rho_i \cosh \rho_j - \cosh R)}{\sinh^2 \rho_i \sinh \rho_j} \\ &= \frac{\cosh \rho_i \cosh R + (\sinh^2 \rho_i - \cosh^2 \rho_i) \cosh \rho_j}{\sinh^2 \rho_i \sinh \rho_j} \\ &= \frac{\cosh \rho_i \cosh R - \cosh \rho_j}{\sinh^2 \rho_i \sinh \rho_j} \\ &\geq 0, \end{aligned}$$

using that $\cosh^2 x - \sinh^2 x = 1$ for the penultimate line and that $\rho_j \leq R$ for the last line. By symmetry, the same of course holds for the derivative with respect to ρ_j .

The proof of Lemma C.3

We begin by noting that $R := 2 \log(N/\nu)$ is monotone decreasing in ν .

Similarly to the proof of Lemma C.2, we will imagine the situation where $\vartheta_1, \dots, \vartheta_N$ and U_1, \dots, U_N remain fixed and we vary R , and $\rho_1 = \rho_1(R), \dots, \rho_N = \rho_N(R)$ are defined via (C.1) above.

We first consider the derivatives of $\rho_i(R)$ and $\rho_j(R)$. Since we keep U_i constant, differentiation of (C.1) with respect to R gives that

$$\begin{aligned} \alpha \sinh(\alpha \rho_i) \cdot \frac{d\rho_i}{dR} &= \frac{d}{dR} \left[\cosh(\alpha \rho_i) - 1 \right] \\ &= U_i \cdot \frac{d}{dR} \left[\cosh(\alpha R) - 1 \right] \\ &= U_i \cdot \alpha \sinh(\alpha R) \\ &\stackrel{(C.1)}{=} \frac{\cosh(\alpha \rho_i) - 1}{\cosh(\alpha R) - 1} \cdot \alpha \sinh(\alpha R). \end{aligned}$$

In other words,

$$\frac{d\rho_i}{dR} = \frac{\varphi(\alpha \rho_i)}{\varphi(\alpha R)}, \tag{C.4}$$

where

$$\varphi(x) := \frac{\cosh x - 1}{\sinh x}.$$

Of course the corresponding statement holds for $\frac{d\rho_j}{dR}$.

The following observations about φ will turn out to be useful:

Lemma C.5. *The function φ is positive and strictly increasing on $(0, \infty)$, and*

$$\varphi(x) = \Theta(x), \tag{C.5}$$

as $x \searrow 0$; and, as $x \rightarrow \infty$, for every fixed $k \in \mathbb{N}$ we have

$$\varphi(x) = 1 + 2 \sum_{i=1}^k (-1)^i e^{-ix} + O(e^{-(k+1)x}). \quad (\text{C.6})$$

Proof of Lemma C.5: That φ is positive follows immediately from the definition. The derivative of φ is:

$$\varphi'(x) = 1 - \frac{(\cosh x - 1) \cosh x}{\sinh^2 x} = \frac{\sinh^2 x - \cosh^2 x + \cosh x}{\sinh^2 x} = \frac{\cosh x - 1}{\sinh^2 x} > 0.$$

(Using that $\cosh^2 x - \sinh^2 x = 1$.)

To see (C.5) we note that

$$\begin{aligned} \varphi(x) &= \frac{\frac{1}{2}(e^x + e^{-x}) - 1}{\frac{1}{2}(e^x - e^{-x})} = \frac{e^x + e^{-x} - 2}{e^x - e^{-x}} = \frac{2 + 2x^2/2 + 2x^4/4! + \dots - 2}{2x + 2x^3/3! + 2x^5/5! + \dots} \\ &= \frac{x^2/2 + x^4/4! + \dots}{x + x^3/3! + \dots} = \frac{x}{2} \cdot \frac{1 + 2x^2/4! + 2x^4/6! + \dots}{1 + x^2/3! + x^4/5! + \dots} \\ &= \Theta(x). \end{aligned}$$

To see (C.6) we note that

$$\varphi(x) = \frac{e^x + e^{-x} - 2}{e^x - e^{-x}} = 1 + \frac{2e^{-x} - 2}{e^x - e^{-x}} = 1 - 2e^{-x} \frac{1 - e^{-x}}{1 - e^{-2x}}.$$

Now we note that, for $|z| < 1$:

$$\frac{z(1-z)}{1-z^2} = z(1-z)(1+z^2+z^4+\dots) = z - z^2 + z^3 - z^4 + \dots.$$

Hence, taking $z = e^{-x}$, we see

$$\varphi(x) = 1 + 2 \sum_{i=1}^{\infty} (-1)^i e^{-ix} = 1 + 2 \sum_{i=1}^k (-1)^i e^{-ix} + O(e^{-(k+1)x}).$$

■

We need the following straightforward observations

Lemma C.6. *Let $\alpha, \nu > 0$ be fixed.*

1. *A.a.s., $G(N; \alpha, \nu)$ does not contain any point of radius $\rho_i \leq e^{-R/4}$;*
2. *If $\alpha > \frac{1}{2}$ then, a.a.s., $G(N; \alpha, \nu)$ does not contain any point with radius $\rho_i \leq 1$;*

Proof of Lemma C.6: This is straightforward application of the first moment method. The expected number of points with radius at most 1 is

$$N \cdot \frac{\cosh(\alpha) - 1}{\cosh(\alpha R) - 1} = O(1) \cdot e^{R/2 - \alpha R} = O(1) \cdot e^{R(\frac{1}{2} - \alpha)},$$

which is $o(1)$, if $\alpha > \frac{1}{2}$. Note that $\cosh(x) - 1 = O(x^2)$, when $x \rightarrow 0$. So, a similar calculation now yields the expected number of points of radius at most $e^{-R/4}$:

$$N \cdot \frac{\cosh(\alpha e^{-R/4}) - 1}{\cosh(\alpha R) - 1} = O(1) \cdot e^{R/2 - R/2 - \alpha R} = O(1) \cdot e^{-\alpha R} = o(1).$$

■

Let us remark that, under our coupling, the conclusions of Lemma C.6 in fact hold simultaneously for all $G(N; \alpha, \nu')$ with $\nu' \leq \nu$. (This is because the radii ρ_1, \dots, ρ_N and R are both non-decreasing in ν . I.e., if we decrease ν then the radii can only get larger while $e^{-R/4}$ will decrease.)

The derivative of the right-hand side of (C.3) with respect to R is

$$\begin{aligned} & \left(\frac{-\cosh \rho_j + \cosh R \cosh \rho_i}{\sinh^2 \rho_i \sinh \rho_j} \right) \frac{d\rho_i}{dR} + \left(\frac{-\cosh \rho_i + \cosh R \cosh \rho_j}{\sinh \rho_i \sinh^2 \rho_j} \right) \frac{d\rho_j}{dR} - \frac{\sinh R}{\sinh \rho_i \sinh \rho_j} \\ &= \frac{1}{\sinh \rho_i \sinh \rho_j} \left(\frac{\cosh R \cosh \rho_i - \cosh \rho_j}{\sinh \rho_i} \frac{d\rho_i}{dR} + \frac{\cosh R \cosh \rho_j - \cosh \rho_i}{\sinh \rho_j} \frac{d\rho_j}{dR} - \sinh R \right). \end{aligned} \tag{C.7}$$

We fix a small $\varepsilon > 0$, to be made precise later, and we first assume that ρ_i, ρ_j are both at least ε . Since $\rho_i + \rho_j > R$, we can assume without loss of generality that $\rho_j \geq R/2$ and $\rho_i \geq \varepsilon$. We remark that

$$\begin{aligned} \text{(C.7)} & \geq \frac{1}{\sinh \rho_i \sinh \rho_j} \left(\left(\frac{\cosh \rho_i - 1}{\sinh \rho_i} \cdot \frac{d\rho_i}{dR} + \frac{\cosh \rho_j - 1}{\sinh \rho_j} \cdot \frac{d\rho_j}{dR} \right) \cosh R - \sinh R \right) \\ &= \frac{1}{\sinh \rho_i \sinh \rho_j} \left(\frac{\varphi(\rho_i)\varphi(\alpha\rho_i) + \varphi(\rho_j)\varphi(\alpha\rho_j)}{\varphi(\alpha R)} \cosh R - \sinh R \right). \end{aligned}$$

Hence, under the assumption that $\rho_i \geq \varepsilon, \rho_j \geq R/2$, the factor in front of $\cosh R$ in (C.7) can be lower bounded as follows.

$$\frac{\varphi(\rho_i)\varphi(\alpha\rho_i) + \varphi(\rho_j)\varphi(\alpha\rho_j)}{\varphi(\alpha R)} \geq \frac{\varphi(\varepsilon)\varphi(\alpha\varepsilon) + 1 - o_R(1)}{1 - o_R(1)} = 1 + \varphi(\varepsilon)\varphi(\alpha\varepsilon) + o_R(1).$$

For sufficiently large R , this expression is larger than 1, which implies the derivative of the right-hand side of (C.3) is positive. (Using that $\varphi(\varepsilon), \varphi(\alpha\varepsilon) > 0$.) It thus remains to consider only the case when one of the two radii is less than ε . Let us also note that, by part 2 of Lemma C.6, a.a.s., there are no points with radius less than ε if $\alpha > 1/2$. Hence we can further assume in the sequel that $\alpha \leq 1/2$.

Let us thus consider a pair i, j with $\rho_i \leq \varepsilon$ and $\rho_i + \rho_j > R$ (or in other words $\rho_j > R - \rho_i$). By part 1 of Lemma C.6 we can and do further assume that $\rho_i \geq e^{-R/4}$. We now use the more detailed lower bound

$$\begin{aligned}
 \text{(C.7)} &\geq \frac{1}{\sinh \rho_i \sinh \rho_j} \left(\left(\frac{\cosh \rho_i - \frac{\cosh(\rho_j)}{\cosh(R)} \frac{d\rho_i}{dR} + \frac{\cosh \rho_j - 1}{\sinh \rho_j} \frac{d\rho_j}{dR} \right) \cosh R - \sinh R \right) \\
 &= \frac{1}{\sinh \rho_i \sinh \rho_j} \left(\left(\frac{\cosh \rho_i - \frac{\cosh(\rho_j)}{\cosh(R)} \frac{\varphi(\alpha\rho_i)}{\varphi(\alpha R)} + \frac{\varphi(\rho_j)\varphi(\alpha\rho_j)}{\varphi(\alpha R)} \right) \cosh R - \sinh R \right) \\
 &\geq \frac{1}{\sinh \rho_i \sinh \rho_j} \left(\left(\frac{\cosh \rho_i - \frac{\cosh(R-\rho_i)}{\cosh(R)} \frac{\varphi(\alpha\rho_i)}{\varphi(\alpha R)} + \frac{\varphi(R-\rho_i)\varphi(\alpha(R-\rho_i))}{\varphi(\alpha R)} \right) \cosh R - \sinh R \right) \\
 &=: \frac{1}{\sinh \rho_i \sinh \rho_j} ((A + B) \cosh R - \sinh R).
 \end{aligned}$$

For convenience, we write $r := \rho_i$ from now on. We let $k \in \mathbb{N}$ be such that $k\alpha > 1$, and note that

$$\begin{aligned}
 B &= \frac{(1 - O(e^{-R}))(1 - 2 \sum_{i=1}^k e^{i\alpha(r-R)} + O(e^{-R}))}{1 - 2 \sum_{i=1}^k e^{-i\alpha R} + O(e^{-R})} \\
 &= (1 - O(e^{-R})) \left(1 + 2 \frac{\sum_{i=1}^k e^{-i\alpha R} - \sum_{i=1}^k e^{i\alpha(r-R)} + O(e^{-R})}{1 - 2 \sum_{i=1}^k e^{-i\alpha R} + O(e^{-R})} \right) \\
 &= (1 - O(e^{-R})) \left(1 - 2 \frac{\sum_{i=1}^k (e^{i\alpha r} - 1)e^{-i\alpha R}}{1 - O(e^{-\alpha R})} \right) = (1 - O(e^{-R})) \left(1 - \frac{O(re^{-\alpha R})}{1 - O(e^{-\alpha R})} \right) \\
 &= 1 - O(re^{-\alpha R}) \\
 &= 1 - o(r).
 \end{aligned} \tag{C.8}$$

(using $\alpha < 1/2$ and $r \geq e^{-R/4}$ for the penultimate line.) Next we consider

$$\begin{aligned}
 A &= \frac{\frac{1}{2}(e^r + e^{-r}) - \frac{e^{R-r} + e^{r-R}}{e^R + e^{-R}}}{\sinh r} \cdot \frac{\Omega(r)}{1 - O(e^{-\alpha R})} = \frac{\frac{1}{2}(e^r + e^{-r}) - e^{-r} - O(e^{-2R})}{\sinh r} \cdot \frac{\Omega(r)}{1 - O(e^{-\alpha R})} \\
 &= \frac{\sinh r - O(e^{-2R})}{\sinh r} \cdot \frac{\Omega(r)}{1 - O(e^{-\alpha R})} = (1 - O(e^{-2R}/r)) \cdot \frac{\Omega(r)}{1 - O(e^{-\alpha R})} \\
 &\geq (1 - O(e^{-7R/4})) \cdot \Omega(r) \\
 &= \Omega(r).
 \end{aligned} \tag{C.9}$$

Combining (C.8) and (C.9), we see that $A + B > 1$ (provided we have chosen the constant ε appropriately small, and R is sufficiently large), which in turn implies the derivative of the right hand side of (C.3) with respect to R is nonnegative as was to be shown.

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