On edge-transitive graphs of square-free order

Cai Heng Li*
School of Mathematics and Statistics
The University of Western Australia
Crawley, WA 6009, Australia
cai.heng.li@uwa.edu.au

Zai Ping Lu†
Center for Combinatorics
LPMC-TJKLC, Nankai University
Tianjin 300071, P. R. China
lu@nankai.edu.cn

Gai Xia Wang‡
Department of Applied Mathematics
Anhui University of Technology
Maanshan 243002, P. R. China
wgx075@163.com

Submitted: Aug 4, 2014; Accepted: Aug 7, 2015; Published: Aug 14, 2015
Mathematics Subject Classifications: 05C25, 20B25

Abstract

We study the class of edge-transitive graphs of square-free order and valency at most k. It is shown that, except for a few special families of graphs, only finitely many members in this class are *basic* (namely, not a normal multicover of another member). Using this result, we determine the automorphism groups of locally primitive arc-transitive graphs with square-free order.

Keywords: edge-transitive graph; arc-transitive graph; stabilizer; quasiprimitive permutation group; almost simple group

1 Introduction

For a graph $\Gamma = (V, E)$, the number of vertices $|V|$ is called the *order* of Γ. A graph $\Gamma = (V, E)$ is called *edge-transitive* if its automorphism group $\mathsf{Aut}\Gamma$ acts transitively on the edge set E. For convenience, denote by $\mathsf{ETSQF}(k)$ the class of connected edge-transitive graphs with square-free order and valency at most k.

The study of special subclasses of $\mathsf{ETSQF}(k)$ has a long history, see for example [1, 4, 5, 17, 18, 21, 22, 23] for those graphs of order being a prime or a product of two primes.

*Supported by ARC Grant DP1096525.
†Supported by National Natural Science Foundation of China (11271267, 11371204).
‡Supported by Anhui Provincial Natural Science Foundation(1408085MA04).

Recently, several classification results about the class $\mathsf{ETSQF}(k)$ were given. Feng and Li [9] gave a classification of one-regular graphs of square-free order and prime valency. By Li et al. [12, 14], one may obtain a classification of vertex-transitive and edge-transitive tetravalent graphs of square-free order. By Li et al. [13] and Liu and Lu [16], one may deduce an explicitly classification of $\mathsf{ETSQF}(3)$. In this paper, we give a characterization about the class $\mathsf{ETSQF}(k)$.

A typical method for analyzing edge-transitive graphs is to take *normal quotient*. Let $\Gamma = (V, E)$ be a connected graph such that a subgroup $G \leqslant \mathsf{Aut}\Gamma$ acts transitively on E. Let N be a normal subgroup of G, denoted by $N \lhd G$. Then either N is transitive on V, or each N-orbit is an independent set of Γ. Let V_N be the set of all N-orbits on V. The *normal quotient* Γ_N (with respect to G and N) is defined as the graph with vertex set V_N such that distinct vertices $B, B' \in V_N$ are adjacent in Γ_N if and only if some $\alpha \in B$ and some $\alpha' \in B'$ are adjacent in Γ. We call Γ_N non-trivial if $N \neq 1$ and $|V_N| \geqslant 3$. It is well-known and easily shown that Γ_N is an edge-transitive graph. Moreover, if all N-orbits have the same length (which is obvious if G is transitive on V), then Γ_N is a regular graph of valency a divisor of the valency of Γ; in this case, Γ is called a *normal multicover* of Γ_N.

A member in $\mathsf{ETSQF}(k)$ is called *basic* if it has no non-trivial normal quotients. Then every member in $\mathsf{ETSQF}(k)$ is a multicover of some basic member, or has a non-regular normal quotient (which might occur for vertex-intransitive graphs). Thus, to a great extent, basic members play an important role in characterizing the graphs in $\mathsf{ETSQF}(k)$. The first result of this paper shows that, except for a few special families of graphs, there are only finitely many basic members in $\mathsf{ETSQF}(k)$.

Theorem 1. *Let $\Gamma = (V, E)$ be a connected graph of square-free order and valency $k \geqslant 3$. Assume that $G \leqslant \mathsf{Aut}\Gamma$ acts transitively on E and that each non-trivial normal subgroup of G has at most 2 orbits on V. Then one of the following holds:*

(1) *Γ is a complete bipartite graph, and G is described in (1) and (5) of Lemma 13;*

(2) *G is one of the Frobenius groups $\mathbb{Z}_p{:}\mathbb{Z}_k$ and $\mathbb{Z}_p{:}\mathbb{Z}_{2k}$, where p is a prime;*

(3) $\mathsf{soc}(G) = \mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}$ *or* J_1*;*

(4) *$G = \mathrm{A}_n$ or S_n with $n < 3k$;*

(5) *$G = \mathrm{PSL}(2, p)$ or $\mathrm{PGL}(2, p)$;*

(6) *$\mathsf{soc}(G) = \mathrm{PSL}(2, p^f)$ with $f \geqslant 2$ and $p^f > 9$, and either k is divisible by p^{f-1} or $f = 2$ and k is divisible by $p + 1$;*

(7) *$\mathsf{soc}(G) = \mathrm{Sz}(2^f)$ and k is divisible by 2^{2f-1};*

(8) *G is of Lie type defined over $\mathrm{GF}(p^f)$ with $p \leqslant k$, and either*

  (i) *$[\frac{d}{2}]f < k$, and G is a d-dimensional classical group with $d \geqslant 3$; or*

(ii) $2f < k$, *and* $\mathsf{soc}(G) = \mathrm{G}_2(p^f)$, ${}^3\mathrm{D}_4(p^f)$, $\mathrm{F}_4(p^f)$, ${}^2\mathrm{E}_6(p^f)$, *or* $\mathrm{E}_7(p^f)$.

Remark 2 (Remarks on Theorem 1). For a finite group G, the *socle* $\mathsf{soc}(G)$ of G is the subgroup generated by all minimal normal subgroups of G. A finite group is called *almost simple* if $\mathsf{soc}(G)$ is a non-abelian simple group.

(a) The groups G in case (1) are known except for G being almost simple.

(b) The vertex-transitive graphs in case (5) are characterized in Theorem 27.

(c) Some properties about the graphs in cases (6)-(7) are given in Lemmas 14 and 15, respectively.

It would be interest to give further characterization for some special cases.

Problem 3. (i) Characterize edge-transitive graphs of square-free order which admits a group with socle $\mathrm{PSL}(2,q)$, $\mathrm{Sz}(q)$, A_n or a sporadic simple group.

(ii) Classify edge-transitive graphs of square-free order of small valencies.

For a graph $\Gamma = (V, E)$ and $G \leqslant \mathsf{Aut}\Gamma$, the graph Γ is called *G-locally primitive* if, for each $\alpha \in V$, the stabilizer of α in G induces a primitive permutation group on the neighbors of α in Γ. The second result of this paper determines, on the basis of Theorem 1, the automorphism groups of locally primitive arc-transitive graphs of square-free order.

Theorem 4. *Let $\Gamma = (V, E)$ be a connected G-locally primitive graph of square-free order and valency $k \geqslant 3$. Assume that G is transitive on V and that Γ is not a complete bipartite graph. Then one of the following statements is true.*

(1) *$G = \mathrm{D}_{2n}{:}\mathbb{Z}_k$, $2nk$ is square-free, k is the smallest prime divisor of nk, and Γ is a bipartite Cayley graph of the dihedral group D_{2n};*

(2) *$G = M{:}X$, where M is of square-free order, X is almost simple with socle T descried as in (3)-(6) and (8) of Theorem 1 such that $MT = M \times T$, T has at most two orbits on V and Γ is T-edge-transitive; in particular, if $T = \mathrm{PSL}(2, p)$, then M, T_α and k are listed in Table 3, where $\alpha \in V$.*

2 Preliminaries

Let $\Gamma = (V, E)$ be a graph without isolated vertices, and let $G \leqslant \mathsf{Aut}\Gamma$. The graph Γ is said to be *G-vertex-transitive* or *G-edge-transitive* if G acts transitively on V or E, respectively. Recall that an *arc* in Γ is an ordered pair of adjacent vertices. The graph Γ is called *G-arc-transitive* if G acts transitively on the set of arcs of Γ. For a vertex $\alpha \in V$, we denote by $\Gamma(\alpha)$ the set of neighbors of α in Γ, and by G_α the stabilizer of α in G. Then it is easily shown that Γ is G-arc-transitive if and only if Γ is G-vertex-transitive and, for $\alpha \in V$, the vertex-stabilizer G_α acts transitively on $\Gamma(\alpha)$.

Let $\Gamma = (V, E)$ be a connected G-edge-transitive graph. Note that each edge of Γ gives two arcs. Then either Γ is G-arc-transitive or G has exactly two orbits (of the same size $|E|$) on the arc set of Γ. If Γ is not G-vertex-transitive then Γ is a bipartite graph and, for $\alpha \in V$, the stabilizer G_α acts transitively on $\Gamma(\alpha)$. If Γ is G-arc-transitive, then there exists $g \in G \setminus G_\alpha$ such that $(\alpha, \beta)^g = (\beta, \alpha)$ and, since Γ is connected, $\langle g, G_\alpha \rangle = G$; obviously, this g can be chosen as a 2-element in $\mathbf{N}_G(G_{\alpha\beta})$ with $g^2 \in G_{\alpha\beta}$, where $G_{\alpha\beta} = G_\alpha \cap G_\beta$. Suppose that Γ is G-vertex-transitive but not G-arc-transitive. Then the arc set of Γ is partitioned into two G-orbits Δ and Δ^*, where $\Delta^* = \{(\alpha, \beta) \mid (\beta, \alpha) \in \Delta\}$. Thus, for $\alpha \in V$, the set $\Gamma(\alpha)$ is partitioned into two G_α-orbits $\Delta(\alpha) = \{\beta \mid (\alpha, \beta) \in \Delta\}$ and $\Delta^*(\alpha) = \{\beta \mid (\beta, \alpha) \in \Delta\}$, which have equal size. Then we have the next lemma.

Lemma 5. *Let $\Gamma = (V, E)$ be a connected G-edge-transitive graph, and $\{\alpha, \beta\} \in E$. Then one of the following holds.*

(1) *The stabilizer G_α is transitive on $\Gamma(\alpha)$, $|\Gamma(\alpha)| = |G_\alpha : G_{\alpha\beta}|$, and either*

  (i) *G is intransitive on V; or*

  (ii) *$G = \langle g, G_\alpha \rangle$ for a 2-element $g \in \mathbf{N}_G(G_{\alpha\beta}) \setminus G_\alpha$ with $(\alpha, \beta)^g = (\beta, \alpha)$ and $g^2 \in G_{\alpha\beta}$.*

(2) *Γ is G-vertex-transitive, G_α has exactly two orbits on $\Gamma(\alpha)$ of the same size $|G_\alpha : G_{\alpha\beta}|$; in particular, $|\Gamma(\alpha)| = 2|G_\alpha : G_{\alpha\beta}|$.*

Let $\Gamma = (V, E)$ be a regular graph and $G \leqslant \mathsf{Aut}\Gamma$. For $\alpha \in V$, the stabilizer G_α induces a permutation group $G_\alpha^{\Gamma(\alpha)}$ (on $\Gamma(\alpha)$). Let $G_\alpha^{[1]}$ be the kernel of this action. Then $G_\alpha^{\Gamma(\alpha)} \cong G_\alpha / G_\alpha^{[1]}$. Considering the actions of Sylow subgroups of $G_\alpha^{[1]}$ on V, it is easily shown that the next lemma holds, see [7] for example.

Lemma 6. *Let $\Gamma = (V, E)$ be a connected regular graph, $G \leqslant \mathsf{Aut}\Gamma$ and $\alpha \in V$. Assume that $G_\alpha \neq 1$. Let p be a prime divisor of $|G_\alpha|$. Then $p \leqslant |\Gamma(\alpha)|$. If further Γ is G-vertex-transitive, then p divides $|G_\alpha^{\Gamma(\alpha)}|$ and, for $\beta \in \Gamma(\alpha)$, each prime divisor of $|G_{\alpha\beta}|$ is less than $|\Gamma(\alpha)|$.*

A permutation group G on a set Ω is *semiregular* if $G_\alpha = 1$ for each $\alpha \in \Omega$. A transitive permutation group is *regular* if further it is semiregular.

Lemma 7. *Let Γ be a connected G-vertex-transitive graph, $N \lhd G \leqslant \mathsf{Aut}\Gamma$ and $\alpha \in V$. Assume that $N_\alpha^{\Gamma(\alpha)}$ is semiregular on $\Gamma(\alpha)$. Then $N_\alpha^{[1]} = 1$.*

Proof. Let $\beta \in \Gamma(\alpha)$. Then $\beta = \alpha^x$ for some $x \in G$, and hence $N_\beta = N_{\alpha^x} = N \cap G_{\alpha^x} = (N \cap G_\alpha)^x = (N_\alpha)^x$. It follows that $N_\beta^{\Gamma(\beta)}$ and $N_\alpha^{\Gamma(\alpha)}$ are permutation isomorphic; in particular, $N_\beta^{\Gamma(\beta)}$ is semiregular on $\Gamma(\beta)$. Thus $N_\alpha^{[1]}$ acts trivially on $\Gamma(\beta)$, and so $N_\alpha^{[1]} = N_\beta^{[1]}$. Since Γ is connected, $N_\alpha^{[1]}$ fixes each vertex of Γ, hence $N_\alpha^{[1]} = 1$. $\square$

Lemma 8. *Let $\Gamma = (V, E)$ be a connected graph, $N \lhd G \leqslant \mathsf{Aut}\Gamma$ and $\alpha \in V$. Assume that either N is regular on V, or Γ is a bipartite graph such that N is regular on both the bipartition subsets of Γ. Then $G_\alpha^{[1]} = 1$.*

Proof. Set $X = NG_\alpha^{[1]}$. Then $X_\alpha = G_\alpha^{[1]}$ and $X_\alpha^{[1]} = G_\alpha^{[1]}$, and hence $X_\alpha^{\Gamma(\alpha)} = 1$.

Assume first that N is regular on V. Then $G = NG_\alpha$. It follows that X is normal in G. Thus our results follows from Lemma 7.

Now assume that Γ is a bipartite graph with bipartition subsets U and W, and that N is regular on both U and W. Without loss of generality, we assume that $\alpha \in U$. Then $\Gamma(\alpha) \subseteq W$, and $X_\alpha = X_\beta$ for $\beta \in \Gamma(\alpha)$. Let $\gamma \in \Gamma(\beta)$. Then $\gamma \in U$. Set $E_0 = \{\{\gamma, \beta\}^x \mid x \in X\}$. Then $\Sigma = (V, E_0)$ is a spanning subgraph of Γ, and X acts transitively on E_0. Thus Σ is a regular graph, and X_α is transitive on $\Sigma(\alpha)$. Noting $\Sigma(\alpha) \subseteq \Gamma(\alpha)$, it follows that $|\Sigma(\alpha)| = 1$, and hence Σ is a matching. In particular, $X_\beta = X_\gamma$. It follows that $G_\alpha^{[1]} = X_\alpha = X_\beta = X_\gamma$. Since all vertices in U are equivalent under X, we have X_γ acts trivially on $\Gamma(\gamma)$. Then a similar argument as above leads to $G_\alpha^{[1]} = X_\gamma = X_\delta = X_\theta$ for any $\delta \in \Gamma(\gamma)$ and $\theta \in \Gamma(\delta)$. Then, by the connectedness, we conclude that $G_\alpha^{[1]}$ fixes each vertex of Γ. Thus $G_\alpha^{[1]} = 1$. □

We end this section by quoting a known result.

Lemma 9 ([12]). *Let $\Gamma = (V, E)$ be a connected G-edge-transitive graph, $N \lhd G \leqslant \mathsf{Aut}\Gamma$ and $\alpha \in V$. Then all N_α-orbits on $\Gamma(\alpha)$ have the same length.*

3 Complete bipartite graphs

We first list a well-known result in number theory. For integers $a > 0$ and $n > 0$, a prime divisor of $a^n - 1$ is called *primitive* if it does not divide $a^i - 1$ for any $0 < i < n$.

Theorem 10 (Zsigmondy). *For integers $a, n \geqslant 2$, if $a^n - 1$ does not have primitive prime divisors, then either $(a, n) = (2, 6)$, or $n = 2$ and $a + 1$ is a power of 2.*

Let G be a permutation group on V, and let x be a permutation on V which centralizes G. If x fixes some point $\alpha \in V$, then x fixes α^g for each $g \in G$. Thus the next simple result follows.

Lemma 11. *Let G be a permutation group on V. Assume that N is a normal transitive subgroup of G. Then the centralizer $\mathbf{C}_G(N)$ is semiregular on V, and $\mathbf{C}_G(N) = N$ if further N is abelian.*

Recall that a transitive permutation group G is *quasiprimitive* if each non-trivial normal subgroup of G is transitive. Let G be a quasiprimitive permutation group on V, and let $\mathcal{B}$ be a G-invariant partition on V. Then G induces a permutation group $G^{\mathcal{B}}$ on $\mathcal{B}$. Assume that $|\mathcal{B}| \geqslant 2$. Since G is quasiprimitive, G acts faithfully on $\mathcal{B}$. Then $G^{\mathcal{B}} \cong G$, and so $\mathsf{soc}(G^{\mathcal{B}}) \cong \mathsf{soc}(G)$.

Lemma 12. *Let G be a quasiprimitive permutation group of square-free degree. Then $\mathsf{soc}(G)$ is simple, so either G is almost simple or $G \leqslant \mathrm{AGL}(1, p)$ for a prime p.*

Proof. Let G be a quasiprimitive permutation group on V of square-free degree. Let $\mathcal{B}$ be a G-invariant partition on V such that $|\mathcal{B}| \geqslant 2$ and $G^{\mathcal{B}}$ is primitive. Noting that $|\mathcal{B}|$ is square-free, by [15], $\mathsf{soc}(G^{\mathcal{B}})$ is simple. Thus $\mathsf{soc}(G) \cong \mathsf{soc}(G^{\mathcal{B}})$ is simple, and the result follows. □

Let G be a permutation group on V. For a subset $U \subseteq V$, denote by G_U and $G_{(U)}$ the subgroups of G fixing U set-wise and point-wise, respectively. For $X \leqslant G$ and an X-invariant subset U of V, denote by X^U the restriction of X on U. Then $X^U \cong X/X_{(U)}$.

We now prove a reduction lemma for Theorem 1.

Lemma 13. *Let $\Gamma = (V, E)$ be a connected G-edge-transitive graph of square-free order and valency $k \geqslant 3$, where $G \leqslant \mathsf{Aut}\Gamma$. Assume that each minimal normal subgroup of G has at most two orbits on V. Then one of the following holds:*

(1) *$\Gamma \cong \mathsf{K}_{k,k}$, k is an odd prime, $G \cong (\mathbb{Z}_k^2{:}\mathbb{Z}_l).\mathbb{Z}_2$ and Γ is G-vertex-transitive, where l is a divisor of $k-1$;*

(2) *$|V| = p$ with $p \geqslant 3$ prime, k is even, $G \cong \mathbb{Z}_p{:}\mathbb{Z}_k$ and Γ is G-vertex-transitive;*

(3) *$|V| = 2p$ with $p \geqslant 3$ prime, and G is isomorphic to one of $\mathbb{Z}_p{:}\mathbb{Z}_k$ and $\mathbb{Z}_p{:}\mathbb{Z}_{2k}$;*

(4) *G is almost simple;*

(5) *$\Gamma \cong \mathsf{K}_{k,k}$, Γ is G-vertex-transitive, $\mathsf{soc}(G)$ is the unique minimal normal subgroup of G, $\mathsf{soc}(G) \cong T^2$ for a nonabelian simple group T and, for $\alpha \in V$, either*

   (i) *$\mathsf{soc}(G)_\alpha \cong H \times T$ for a subgroup H of T with $k = |T : H|$; or*

   (ii) *$k = 105$, $T \cong \mathrm{A}_7$ and $\mathsf{soc}(G)_\alpha \cong \mathrm{A}_6 \times \mathrm{PSL}(3, 2)$.*

Proof. Let N be a minimal normal subgroup of G. Then N is a directed product of isomorphic simple groups. Since Γ has valency $k \geqslant 3$, we know that $|V| > 3$. Since $|V|$ is square-free and N has at most two orbits on V, we conclude that N is not an elementary abelian 2-group. In particular, N has no a subgroup of index 2.

Case 1. Assume first that G has two distinct minimal normal subgroups N and M. Then $N \cap M = 1$, and hence $NM = N \times M$.

Suppose that both N and M are transitive on V. By Lemma 11, N and M are regular on V; in particular, $|N| = |M| = |V|$. Thus N and M are soluble, it implies that $N \cong M \cong \mathbb{Z}_p$ for an odd prime p. Again by Lemma 11, $N = M$, a contradiction.

Without loss of generality, we assume that N is intransitive on V. Then Γ is a bipartite graph, whose bipartition subsets are N-orbits, say U and $V \setminus U$. A similar argument as above paragraph yields that M has no subgroups of index 2. It follows that M fixes both U and $V \setminus U$ set-wise, and hence U and $V \setminus U$ are two M-orbits on V.

Let $X = NM$ and $\Delta = U$ or $V \setminus U$. By Lemma 11, both N^Δ and M^Δ are regular subgroups of X^Δ. Set $N \cong T^i$, where T is a simple group. Then $N_{(\Delta)} \cong T^j$ for some $j < i$, and so $N^\Delta \cong N/N_{(\Delta)} \cong T^{i-j}$. It follows that $|\Delta| = |N^\Delta| = |T|^{i-j}$. Since T is simple and $|\Delta|$ is square-free, $i - j = 1$ and $N^\Delta \cong T \cong \mathbb{Z}_p$, where $p = |\Delta|$ is an odd prime. Similarly, $M^\Delta \cong \mathbb{Z}_p$, and so M is abelian. In particular, $X = N \times M$ is abelian and $|X|$ is a power of p. It implies that $X^\Delta \cong \mathbb{Z}_p$. Then, by Lemma 11, $N^\Delta = M^\Delta = X^\Delta$. Thus

$N \times M = X \leqslant X^{\Delta} \times X^{V\setminus\Delta} \cong \mathbb{Z}_p^2$. Then $X \cong \mathbb{Z}_p^2$, and hence $N \cong M \cong \mathbb{Z}_p$. Moreover, $X_{(\Delta)} \cong \mathbb{Z}_p$.

Let $\alpha \in \Delta$. Then $G_\alpha \geqslant X_{(\Delta)}$. By Lemma 6, $k = |\Gamma(\alpha)| \geqslant p$, and so $\Gamma \cong \mathsf{K}_{p,p}$. Noting that N is regular on Δ and $V \setminus \Delta$, by Lemma 8, G_α acts faithfully on $\Gamma(\alpha)$, and so G_α is isomorphic to a subgroup of the symmetric group S_p. Noting that G_α has a normal subgroup $X_{(\Delta)} \cong \mathbb{Z}_p$, it follows that G_α is isomorphic to a subgroup of the Frobenius group $\mathbb{Z}_p{:}\mathbb{Z}_{p-1}$. Write $G_\alpha \cong \mathbb{Z}_p{:}\mathbb{Z}_l$, where l is a divisor of $p-1$. Then $G_\Delta = NG_\alpha \cong \mathbb{Z}_p^2{:}\mathbb{Z}_l$.

Clearly, $X_{(\Delta)}$ has at least $p+1$ orbits on V. Then, by the assumptions of this lemma, $X_{(\Delta)}$ is not normal in G. On the other hand, $(X_{(\Delta)})^g = (X^g)_{(\Delta^g)} = X_{(\Delta)}$ for each $g \in G_\Delta$, yielding $X_{(\Delta)} \lhd G_\Delta$. It follows that $G \neq G_\Delta$, and hence G is transitive on V. Note that $|G : G_\Delta| \leqslant 2$. Then part (1) of this lemma follows.

Case 2. Assume that $N := \mathsf{soc}(G)$ is the unique minimal normal subgroup of G.

Assume that N is simple. If N is nonabelian then (4) occurs. Assume that $N \cong \mathbb{Z}_p$ for some odd prime p. Then N is regular on each N-orbit on V. Thus G_α is faithful on $\Gamma(\alpha)$ by Lemma 8, where $\alpha \in V$. Noting that $\mathbf{C}_G(N)$ is normal in G, we conclude that $\mathbf{C}_G(N) = N$. Thus $G/N = \mathbf{N}_G(N)/\mathbf{C}_G(N) \lesssim \mathsf{Aut}(N) \cong \mathbb{Z}_{p-1}$, and so $G \lesssim \mathrm{AGL}(1,p)$. Set $G \cong \mathbb{Z}_p{:}\mathbb{Z}_m$, where m is a divisor of $p-1$. Let $\alpha \in U$. Then $G_\alpha \cong NG_\alpha/N \leqslant G/N \cong \mathbb{Z}_m$; in particular, G_α is cyclic. Recalling that G_α is faithful on $\Gamma(\alpha)$, it implies that $G_\alpha \cong \mathbb{Z}_k$. Thus one of (2) and (3) occurs by noting that $|G : (NG_\alpha)| \leqslant 2$.

In the following we assume that $N \cong T^l$ for an integer $l \geqslant 2$ and a simple group T. If N is transitive on V then G is quasiprimitive on V, and hence $\mathsf{soc}(G) = N$ is simple by Lemma 12, a contradiction. If G is intransitive on V, then G is faithful on each of its orbits, and then N is simple by Lemma 12, again a contradiction. Thus, in the following, we assume further that Γ is G-vertex-transitive and N has two orbits U and W on V. Note that $|U| = |W| = \frac{|V|}{2}$ is odd and square-free.

Since Γ is G-vertex-transitive, $|G : G_U| = 2$. Let $x \in G \setminus G_U$. Then $G = G_U\langle x\rangle$, $x^2 \in G_U$, $U^x = W$ and $W^x = U$. Let $\mathcal{B}$ be a G_U-invariant partition of U such that $(G_U)^{\mathcal{B}}$ is primitive. Set $\mathcal{C} = \{B^x \mid B \in \mathcal{B}\}$. Then $(G_U)^{\mathcal{C}}$ is also primitive. By [15], both $\mathsf{soc}((G_U)^{\mathcal{B}})$ and $\mathsf{soc}((G_U)^{\mathcal{C}})$ are simple. Then $\mathsf{soc}((G_U)^{\mathcal{B}}) \cong \mathsf{soc}((G_U)^{\mathcal{C}}) \cong T$. Let K be the kernel of G_U acting on $\mathcal{B}$. Then K^x is the kernel of G_U acting on $\mathcal{C}$, and $K^{x^2} = K$. Since $K, K^x \lhd G_U$, we have $K \cap K^x \lhd G_U$. Noting that $(K \cap K^x)^x = K \cap K^x$, it follows that $K \cap K^x \lhd G$. Since $K \cap K^x$ has at least $2|\mathcal{B}| > 2$ orbits on V, we have $K \cap K^x = 1$. Then $G_U \lesssim G_U/K \times G_U/K^x \cong (G_U)^{\mathcal{B}} \times (G_U)^{\mathcal{C}}$, yielding $N \cong T^2$.

We claim that T is a nonabelian simple group. Suppose that $T \cong \mathbb{Z}_p$ for some (odd) prime p. Then $(G_U)^{\mathcal{B}} \cong (G_U)^{\mathcal{C}} \lesssim \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$, and so $G = G_U.\mathbb{Z}_2 \lesssim ((\mathbb{Z}_p{:}\mathbb{Z}_{p-1}) \times (\mathbb{Z}_p{:}\mathbb{Z}_{p-1})).\mathbb{Z}_2$. Let H be a p'-Hall subgroup of G with $x \in H$. Then $G = N{:}H$, $H \lesssim (\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}).\mathbb{Z}_2$. Moreover, H_U is p'-Hall subgroup of G_U, $H = H_U\langle x\rangle$ and $H_U \lesssim \mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$. Note that N is the unique minimal normal subgroup of G. Then H is maximal in G, and thus G can be viewed as a primitive subgroup of the affine group $\mathrm{AGL}(2,p)$. Since H_U is an abelian normal subgroup of H, by [19, 2.5.10], H_U is cyclic. It follows that $H_U \lesssim \mathbb{Z}_{p-1}$. Since H_U has index 2 in H, by [19, 2.5.7], H_U is an irreducible subgroup of $\mathrm{GL}(2,p)$. Then, by [19, 2.3.2], $|H_U|$ is not a divisor of $p-1$, a contradiction. Therefore, T is a nonabelian simple group.

Set $N = T_1 \times T_2$, where $T_1 \cong T_2 \cong T$. Since T_1 and T_2 are isomorphic nonabelian simple groups, T_1 and T_2 are the only non-trivial normal subgroups of N. Thus $N_{(U)} \in \{1, T_1, T_2\}$. For $g \in G_U$, we have $(N_{(U)})^g = (N^g)_{(U^g)} = N_{(U)}$. Thus $N_{(U)} \lhd G_U$. Let $x \in G \setminus G_U$. Then $U^x = W$ and $W^x = U$, yielding $(N_{(U)})^x = N_{(W)}$ and $(N_{(W)})^x = N_{(U)}$. It follows that either $\{N_{(U)}, N_{(W)}\} = \{T_1, T_2\}$ or N is faithful on both U and W. The former case yields that $N_{(U)}$ acts transitively on W, and so (i) of part (5) follows.

Assume that N is faithful on both U and W. Then neither T_1 nor T_2 is transitive on U. Let $\mathcal{O}$ be the set of T_1-orbits on U, and let $O \in \mathcal{O}$. Then T_2 is transitive on $\mathcal{O}$. Thus T has two transitive permutation representations of degrees $|O|$ and $|\mathcal{O}|$, respectively. Then T has two primitive permutation representations of degrees n_1 and n_2, where $n_1 > 1$ is a divisor of $|O|$ and $n_2 > 1$ is a divisor of $|\mathcal{O}|$. Since $|V| = 2|U| = 2|O||\mathcal{O}|$ is square-free, n_1 and n_2 are odd, square-free and coprime. Inspecting [15, Tables 1-4], we conclude that T is either an alternating group or a classical group of Lie type.

Suppose that $T = \mathrm{PSL}(d, q)$ with $d \geqslant 3$. By the Atlas [8], neither $\mathrm{PSL}(3, 2)$ nor $\mathrm{PSL}(4, 2)$ has maximal subgroups of coprime indices. Thus we assume that $(d, q) \neq (3, 2)$ or $(4, 2)$. Then, by [15, Table 3],

$$\{n_1, n_2\} \subseteq \left\{ \frac{\prod_{j=0}^{i-1}(q^{m-j}-1)}{\prod_{j=1}^{i}(q^j-1)} \mid 1 \leqslant i < d \right\} \cup \left\{ \frac{\prod_{j=0}^{2i-1}(q^{m-j}-1)}{(\prod_{j=1}^{i}(q^j-1))^2} \mid 1 \leqslant i < \frac{d}{2} \right\}.$$

If $q^d - 1$ has a primitive prime divisor r, then both n_1 and n_2 are divisible by r, which is not possible. Thus $q^d - 1$ has no primitive prime divisor, and so $(q, d) = (2, 6)$ by Theorem 10. Computation of n_1 and n_2 shows that this is not the case.

Similarly, we exclude other classes of classical groups of Lie type except for $\mathrm{PSL}(2, p^f)$, where p is a prime. By the Atlas [8], we exclude $\mathrm{PSL}(2, p^f)$ while $p^f \leqslant 31$. Suppose that $T = \mathrm{PSL}(2, p^f)$ with $p^f \geqslant 32$. By [15, Table 3], one of n_1 and n_2 is $p^f + 1$ and the other one is divisible by p. This is not possible since one of $p^f + 1$ and p is even.

Now let $T = \mathrm{A}_c$ for some $c \geqslant 5$. By the above argument, we may assume that A_c is not isomorphic to a classical simple group of Lie type. Then $c \neq 5$, 6 or 8. Note that for $c \geqslant 5$ and $a < b < \frac{c}{2}$, the binomial coefficient $\binom{c}{b} = \binom{c}{a}\binom{c-a}{b-a}/\binom{b}{b-a}$. It is easily shown that $\binom{c}{a} > \binom{b}{b-a} = \binom{b}{a}$; in particular, $\binom{c}{a}$ is not a divisor of $\binom{b}{b-a}$. Thus $\binom{c}{a}$ and $\binom{c}{b}$ are not comprime, and so at most one of n_1 and n_2 equals to a binomial coefficient. Checking the actions listed in [15, Table 1] implies that either $c = 7$, or $c = 2a$ for $a \in \{6, 9, 10, 12, 36\}$. Suppose the later case occurs. Then one of n_1 and n_2 is $\frac{1}{2}\binom{2a}{a}$ and the other one is a binomial coefficient, say $\binom{2a}{b}$. But computation shows that such two integers are not coprime, a contradiction. Therefore, $T = \mathrm{A}_7$.

Checking the subgroups of A_7, we conclude that $\{n_1, n_2\} = \{|O|, |\mathcal{O}|\} = \{7, 15\}$. Take $\alpha \in O$. Recall that Γ is G-vertex-transitive. Then there is an element $x \in G \setminus G_U$ such that $\{\alpha, \alpha^x\} \in E$, $U^x = W$ and $W^x = U$. Since $N = T_1 \times T_2$ is the unique minimal normal subgroup of G, we know that $T_1^x = T_2$ and $T_2^x = T_1$. It follows O^x is a T_2-orbit on W, and so $\mathcal{O}^x := \{O^{hx} \mid h \in G_U\}$ is the set of T_2-orbits on W. Moreover, T_1 acts transitively on $\mathcal{O}^x$. Note that $|O| = |O^x|$ and $|\mathcal{O}| = |\mathcal{O}^x|$. Thus, without loss of generality, we may assume that $|O| = 7$ and $|\mathcal{O}| = 15$. Then $(T_2)_O \cong \mathrm{PSL}(3, 2)$ and $(T_1)_\alpha \cong \mathrm{A}_6$, where $\alpha \in O$. Recall that T_2 is intransitive on V. Since $T_2 \lhd N$ and N is transitive on U, we conclude that each T_2-orbit on U has size 15. It follows that $(T_2)_O = (T_2)_\alpha$. Then $N_\alpha \geqslant (T_1)_\alpha \times (T_2)_\alpha$, and so $N_\alpha = (T_1)_\alpha \times (T_2)_\alpha \cong \mathrm{A}_6 \times \mathrm{PSL}(3,2)$ as $|N : N_\alpha| = |U| = |O||\mathcal{O}| = 105$. Note that $N_{\alpha^x} = (N_\alpha)^x = ((T_1)_\alpha \times (T_2)_\alpha)^x = (T_2)_{\alpha^x} \times (T_1)_{\alpha^x}$. Then it is easily shown that $N_\alpha \cap N_{\alpha^x} = ((T_1)_\alpha \cap (T_1)_{\alpha^x}) \times ((T_2)_\alpha \cap (T_2)_{\alpha^x}) \cong \mathrm{S}_4 \times \mathrm{S}_4$. By the choice of x, we conclude that $|\Gamma(\alpha)| \geqslant |N_\alpha : (N_\alpha \cap N_{\alpha^x})| \geqslant 105$. Thus $\Gamma = \mathsf{K}_{105,105}$, and hence (ii) of part (5) occurs. □

4 Graphs associated with $\mathrm{PSL}(2, p^f)$ and $\mathrm{Sz}(2^f)$

Let $\Gamma = (V, E)$ be a connected graph of square-free order and valency k. Assume that $G \leqslant \mathsf{Aut}\Gamma$ is almost simple with socle T. Assume further that G is transitive on E and that T has at most two orbits on V. Let $\{\alpha, \beta\} \in E$. Then $|T_\alpha| = |T_\beta|$ as Γ is a regular graph. Then $|T_\beta : T_{\alpha\beta}| = |T_\alpha : T_{\alpha\beta}|$ and, by Lemma 9, $|T_\alpha : T_{\alpha\beta}|$ is a divisor of $k = |\Gamma(\alpha)|$. Moreover, since $|V|$ is square-free, it is easily shown that $T_\alpha \neq T_\beta$.

Lemma 14. *Let $\Gamma = (V, E)$ be a connected G-edge-transitive graph of square-free order and valency k. Assume that $\mathsf{soc}(G) = \mathrm{PSL}(2, p^f)$ with $f \geqslant 2$ and $p^f > 9$, and that $\mathsf{soc}(G)$ has at most two orbits on V. Then one of the following statements holds:*

(i) *$f = 2$, $T_\alpha = \mathrm{PGL}(2, p)$ or $\mathrm{PSL}(2, p)$, and k is divisible by p or $p + 1$;*

(ii) *$T_\alpha = \mathbb{Z}_p^{f-1}{:}\mathbb{Z}_l$ for a divisor l of $p - 1$, and k is divisible by p^{f-1}; further, if Γ is G-locally primitive then $k = p^{f-1}$;*

(iii) *$T_\alpha = \mathbb{Z}_p^f{:}\mathbb{Z}_l$ for a divisor l of $p^f - 1$, and k is divisible by p^f; further, if Γ is G-locally primitive then $k = p^f$.*

Proof. Let $T = \mathsf{soc}(G)$. Take $\alpha \in V$ and a maximal subgroup M of T with $T_\alpha \leqslant M$. Then both $|T : M|$ and $|M : T_\alpha|$ are square-free as $|T : T_\alpha|$ is square-free. By [15], either $M = \mathbb{Z}_p^f{:}\mathbb{Z}_{\frac{p^f-1}{(2,p-1)}}$ and $|T : M| = p^f + 1$, or $f = 2$, $M = \mathrm{PGL}(2, p)$ and $|T : M| = \frac{p(p^2+1)}{2}$.

Assume that T_α is insoluble. Then $f = 2$ and $T_\alpha = \mathrm{PGL}(2, p)$ or $\mathrm{PSL}(2, p)$. Let $\beta \in \Gamma(\alpha)$. Recall that $T_\alpha \neq T_\beta$ and $|T_\beta : T_{\alpha\beta}| = |T_\alpha : T_{\alpha\beta}|$ is a divisor of k. If $T_\alpha = \mathrm{PSL}(2, p)$ then, by [11, II.8.27], $|T_\alpha : T_{\alpha\beta}|$ is divisible by p or $p + 1$. Suppose that $T_\alpha = \mathrm{PGL}(2, p)$. Then T_α is maximal in T, and so $T = \langle T_\alpha, T_\beta\rangle$. Thus $|T_\beta : T_{\alpha\beta}| > 2$ as T is simple; in particular, $\mathrm{PSL}(2, p) \neq T_{\alpha\beta}$. Checking the subgroups of T_α which do not contain $\mathrm{PSL}(2, p)$ (refer to [3]), we conclude that $|T_\alpha : T_{\alpha\beta}|$ is divisible by p or $p+1$. Thus part (i) occurs.

In the following, we assume that T_α is soluble. Since p^2 is not a divisor of $|T : T_\alpha|$, each Sylow p-subgroup of T_α has p^f or p^{f-1}. Then, inspecting the subgroups of T, we conclude that $T_\alpha \cong T_\beta$ for $\beta \in \Gamma(\alpha)$, and that T_α has a unique Sylow p-subgroup.

Let Q be a Sylow p-subgroup of $T_{\alpha\beta}$. Then Q is normal in $T_{\alpha\beta}$. Suppose that $Q \neq 1$. Let P_1 and P_2 be the Sylow p-subgroups of T_α and T_β, respectively. Then $P_1 \cap P_2 = Q \neq 1$. By [11, II.8.5], any two distinct Sylow p-subgroups of T intersect trivially. It follows P_1

and P_2 are contained the same Sylow p-subgroup, say P of T. In particular, $P_1 = P_\alpha$ and $P_2 = P_\beta$. For $\gamma \in \Gamma(\beta)$, since Γ is G-edge-transitive, we have $|T_{\alpha\beta}| = |T_{\beta\gamma}|$. A similar argument implies that P_γ is the Sylow p-subgroup of T_γ. It follows from the connectedness of Γ that P_δ is the Sylow p-subgroup of T_δ for any $\delta \in V$. Thus P contains a normal subgroup $\langle P_\delta \mid \delta \in V\rangle \neq 1$ of G, a contradiction. Thus, $T_{\alpha\beta}$ is of order coprime to p, and so $|T_\alpha : T_{\alpha\beta}|$ is divisible by $|P_1| = p^{f-1}$ or p^f. Thus, by Lemma 9, k is divisible by p^{f-1} or p^f, respectively.

If $M = \mathrm{PGL}(2, p)$ then, inspecting the subgroups of M, we conclude that $T_\alpha = \mathbb{Z}_p{:}\mathbb{Z}_l$, where l is a divisor of $p-1$ and divisible by 4. Assume that $M = \mathbb{Z}_p^f{:}\mathbb{Z}_{\frac{p^f-1}{(2,p-1)}}$. Then $T_\alpha = \mathbb{Z}_p^f{:}\mathbb{Z}_l$ or $\mathbb{Z}_p^{f-1}{:}\mathbb{Z}_l$ with l dividing $\frac{p^f-1}{(2,p-1)}$. Suppose that $T_\alpha = \mathbb{Z}_p^{f-1}{:}\mathbb{Z}_l$. Noting that M is a Frobenius group, T_α is also a Frobenius group. It follows that l is a divisor of $p^{f-1} - 1$, and so l divides $p-1$.

Assume further that Γ is G-locally primitive. Then $T_\alpha^{\Gamma(\alpha)}$ is a normal transitive soluble subgroup of the primitive permutation group $G_\alpha^{\Gamma(\alpha)}$ of degree k. Since k is divisible by $|P_1|$, we have $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)}) \cong \mathbb{Z}_p^t$ for some integer $t \geqslant 1$ such that $k = p^t \geqslant |P_1|$. It follows $T_\alpha^{\Gamma(\alpha)} \cong \mathbb{Z}_p^t{:}\mathbb{Z}_{l'}$, where l' is a divisor of l. Since P_1 is the Sylow p-subgroup of T_α, we have $p^t \leqslant |P_1|$. Then $k = |P_1| = p^{f-1}$ or p^f. Thus one of (ii) and (iii) follows. □

The following lemma gives a characterization of graphs admitting Suzuki groups.

Lemma 15. *Let $\Gamma = (V, E)$ be a connected G-edge-transitive graph of square-free order and valency k. Assume that $\mathsf{soc}(G) = \mathrm{Sz}(2^f)$ with odd $f \geqslant 3$, and that $\mathsf{soc}(G)$ has at most two orbits on V. Then k is divisible by 2^{2f-1} and Γ is not G-locally primitive.*

Proof. Let $\alpha \in V$ and $\beta \in \Gamma(\alpha)$. Since $|T : T_\alpha|$ is square-free, 4 does not divide $|T : T_\alpha|$, and hence 2^{2f-1} divides $|T_\alpha|$. Then, inspecting the subgroups of T (see [20]), we get $T_\alpha = [2^n]{:}\mathbb{Z}_l$, where $n = 2f$ or $2f-1$, and l is a divisor of $2^f - 1$. So T_α has a unique Sylow 2-subgroup. By [20], for a Sylow 2-subgroup Q of T, all involutions of Q are contained in the center of Q. Noting that any two distinct conjugations of Q generate T, it follows any two distinct Sylow 2-subgroups of T intersect trivially. Thus, by a similar argument as in the above lemma, we know that $T_{\alpha\beta}$ has odd order. Thus $k = |\Gamma(\alpha)|$ is divisible by $n = 2^{2f}$ or 2^{2f-1}.

Finally, suppose that $G_\alpha^{\Gamma(\alpha)}$ is a primitive group. Let Q_1 be the Sylow 2-subgroup of T_α, and Q be a Sylow 2-subgroup of $T = \mathrm{Sz}(2^f)$ with $Q \geqslant Q_1$. Then $Q = Q_1$ or $Q_1.\mathbb{Z}_2$. By a similar argument as in the above lemma, we conclude that Q_1 is isomorphic to $\mathsf{soc}(G_\alpha^{\Gamma(\alpha)})$. It follows that Q_1 is an elementary abelian 2-group. By [20], Q_1 lies in the center of Q, and so Q is abelian, which is impossible. Then this lemma follows. □

5 Proof of Theorem 1

Let $\Gamma = (V, E)$ be a connected graph of square-free order and valency k. Assume that a subgroup $G \leqslant \mathsf{Aut}\Gamma$ acts transitively on E and that each non-trivial normal subgroup of G has at most 2 orbits on V. By Lemma 13, to complete the proof of the theorem, we may assume that G is almost simple. Let $T = \mathsf{soc}(G)$ and $\alpha \in V$. Then T is transitive or has exactly two orbits on V, and every prime divisor of $|T_\alpha|$ is at most k.

Let U be a T-orbit, and let $\mathcal{B}$ be a T-invariant partition on U such that $|\mathcal{B}| \geqslant 2$ and $T^{\mathcal{B}}$ is primitive. Noting that $|\mathcal{B}|$ is square-free, T is listed in [15, Tables 1-4]. In particular, if T is one of sporadic simple groups then part (3) of Theorem 1 follows.

Assume that $T = \mathrm{A}_n$, where $n \geqslant 5$. Suppose that $n \geqslant 3k$. By [15], there exists a prime p such that $k < p < 3k/2$, and thus p^2 divides $|T|$, and p divides $|T_\alpha|$. So $p \leqslant k$, which is a contradiction. Therefore, $n < 3k$, as in part (4) of Theorem 1.

We next deal with the classical groups and the exceptional groups of Lie type. If $T = \mathrm{PSL}(2, p^f)$ or $\mathrm{Sz}(2^f)$ then, by Lemmas 14 and 15, one of parts (5), (6) and (7) of Theorem 1 follows. Thus the following two lemmas will fulfill the proof of Theorem 1.

Lemma 16. *Let T be a d-dimensional classical simple group of Lie type over $\mathrm{GF}(p^f)$, where p is a prime. Then either $T = \mathrm{PSL}(2,p)$, or $p \leqslant k$ and one of the following holds:*

(i) *$T = \mathrm{PSL}(2, p^f)$ with $f \geqslant 2$;*

(ii) *$[\frac{d}{2}]f < k$; if further $T = \mathrm{PSU}(d, p^f)$ then $2[\frac{d}{2}]f < k$ and $[\frac{d}{2}]$ is odd.*

Proof. Let $\alpha \in V$. Then $|T : T_\alpha|$ is square-free and, by Lemma 6, each prime divisor of $|T_\alpha|$ is at most k. Assume that $T \neq \mathrm{PSL}(2,p)$. Let P be a Sylow p-subgroup of T. Then p^2 divides $|P|$. Since $|T : T_\alpha|$ is square-free, p divides $|T_\alpha|$, and so $p \leqslant k$.

Assume that $d \geqslant 3$. Let $d_0 = [\frac{d}{2}]$, the largest integer no more than $\frac{d}{2}$. Check the orders of classical simple groups of Lie type, see [2, Section 47] for example. We conclude that either

(1) $(p^{2d_0 f} - 1)(p^{d_0 f} - 1)$ divides $(d, p^f - 1)|T|$; or

(2) $T = \mathrm{PSU}(d, p^f)$ with d_0 odd, and $(p^{2d_0 f} - 1)(p^{d_0 f} + 1)$ divides $(d, p^f + 1)|T|$.

Consider part (1) first. Suppose that $p^{d_0 f} - 1$ has a primitive prime divisor r. Then $r > d_0 f$, and hence either $r = d = 3$ and $f = 1$, or r^2 divides $|T|$. For the former, $T = \mathrm{PSL}(3,p)$, and so $[\frac{d}{2}]f = 1 < k$. For the latter, r divides $|T_\alpha|$, and so $d_0 f < r \leqslant k$. Suppose that $p^{d_0 f} - 1$ has no primitive prime divisor. By Theorem 10, either $d_0 f = 2$ and $p + 1$ is a power of 2, or $(p, d_0 f) = (2, 6)$. For the former, $[\frac{d}{2}]f = d_0 f = 2 < k$. Assume that $(p, d_0 f) = (2, 6)$. Then $(d_0, f) = (1, 6)$, $(2, 3)$, $(3, 2)$, or $(6, 1)$. It follows that $(d, f) = (3, 6)$, $(4, 3)$, $(5, 3)$, $(6, 2)$, $(7, 2)$, $(12, 1)$ or $(13, 1)$. Thus $|T|$ is divisible by 7^2, and so $|T_\alpha|$ is divisible by 7. Then $[\frac{d}{2}]f = d_0 f = 6 < 7 \leqslant k$ by Lemma 6.

Now assume that $T = \mathrm{PSU}(d, p^f)$ with $d_0 = [\frac{d}{2}]$ odd. Then $(p^{2d_0 f} - 1)(p^{d_0 f} + 1)$ divides $(d, p^f + 1)|T|$. A similar argument shows that either $p^{2d_0 f} - 1$ has no primitive prime divisor, or $2d_0 f < k$. Assume that $p^{2d_0 f} - 1$ has no primitive prime divisor. Then either $2d_0 f = 2$, or $(p, 2d_0 f) = (2, 6)$. For the former, $2d_0 f = 2 < k$. Suppose that $(p, 2d_0 f) = (2, 6)$. Then $d_0 f = 3$, and so $(d, p^f) = (3, 2^3)$, $(6, 2)$ or $(7, 2)$. Thus $|T|$ is divisible by 7^2, and so $2d_0 f = 6 < 7 \leqslant k$. □

Finally we consider the exceptional simple groups of Lie type.

Lemma 17. *Let T be an exceptional simple group of Lie type defined over* $\mathrm{GF}(p^f)$ *with p prime. Then $p \leqslant k$, and one of the following holds:*

(i) $T = \mathrm{Sz}(2^f)$;

(ii) $T = \mathrm{G}_2(p^f)$ *or* ${}^3\mathrm{D}_4(p^f)$, $p^f \neq 2^3$ *and* $2f < k$;

(iii) $T = \mathrm{F}_4(p^f)$, ${}^2\mathrm{E}_6(p^f)$ *or* $\mathrm{E}_7(p^f)$, $p^f \neq 2$ *and* $6f < k$.

Proof. Note that T has order divisible by p^2. Then p divides $|T_\alpha|$, and so $p \leqslant k$. By [15, Table 4], T is one of $\mathrm{Sz}(2^f)$, $\mathrm{G}_2(p^f)$, ${}^3\mathrm{D}_4(p^f)$, $\mathrm{F}_4(p^f)$, ${}^2\mathrm{E}_6(p^f)$ and $\mathrm{E}_7(p^f)$.

For $T = \mathrm{G}_2(p^f)$ or ${}^3\mathrm{D}_4(p^f)$, the order $|T|$ is divisible by $(p^f+1)^2$ and $|T : T_\alpha|$ is divisible by p^f+1. If $p^{2f}-1$ has a primitive prime divisor r, then $|T|$ is divisible by r^2, and $|T_\alpha|$ is divisible by r, hence $2f < r \leqslant m$. Assume that $p^{2f}-1$ has no primitive prime divisor. Then either $f = 1$ and $2f = 2 < k$, or $(p, 2f) = (2, 6)$. For the latter, $T = \mathrm{G}_2(8)$ or ${}^3\mathrm{D}_4(8)$, and so 9 is a divisor of $|T : T_\alpha|$, which contradicts that $|T : T_\alpha|$ is square-free. Thus T is described as in part (ii) of this lemma.

Assume that T is one of $\mathrm{F}_4(p^f)$, ${}^2\mathrm{E}_6(p^f)$ and $\mathrm{E}_7(p^f)$. Then $|T|$ is divisible by $(p^{6f}-1)^2$ and $|T : T_\alpha|$ is divisible by $\frac{p^{6f}-1}{p^f-1}$. If $p^{6f}-1$ has a primitive divisor r say, then r divides $|T_\alpha|$, and hence $6f < r \leqslant k$. If $p^{6f}-1$ has no primitive prime divisor, then $p = 2$ and $f = 1$, and so $|T : T_\alpha|$ is not square-free as it is divisible by 9, and hence T is described as in part(iii) of this lemma. □

6 Graphs associated with $\mathrm{PSL}(2, p)$

In this section, we investigate vertex- and edge-transitive graphs associated with $\mathrm{PSL}(2, p)$, and then give a characterization for such graphs.

6.1 Examples

It is well-known that vertex- and edge-transitive graphs can be described as coset graphs. Let G be a finite group and H be a core-free subgroup of G, where core-free means that $\cap_{g\in G} H^g = 1$. Let $[G : H] = \{Hx \mid x \in G\}$, the set of right cosets of H in G. For an element $g \in G \setminus H$, define the *coset graph* $\Gamma := \mathsf{Cos}(G, H, H\{g, g^{-1}\}H)$ on $[G : H]$ such that (Hx, Hy) is an arc of Γ if and only if $yx^{-1} \in H\{g, g^{-1}\}H$. Then Γ is a well-defined regular graph, and G induces a subgroup of $\mathsf{Aut}\Gamma$ acting on $[G : H]$ by right multiplication. The next lemma collects several basic facts on coset graphs.

Lemma 18. *Let G be a finite group and H a core-free subgroup of G. Take $g \in G \setminus H$ and set $\Gamma = \mathsf{Cos}(G, H, H\{g, g^{-1}\}H)$. Then Γ is G-vertex-transitive and G-edge-transitive. Moreover,*

(1) *Γ is G-arc-transitive if and only if $H\{g, g^{-1}\}H = HxH$ for some 2-element $x \in \mathbf{N}_G(H \cap H^g) \setminus H$ with $x^2 \in H \cap H^g$;*

(2) *Γ is connected if and only if $\langle H, g\rangle = G$.*

Now we construct several examples.

Example 19. Let $T = \mathrm{PSL}(2,p)$, $\mathbb{Z}_p{:}\mathbb{Z}_l \cong H < T$ and $\mathbb{Z}_l \cong K < H$, where l is an even divisor of $\frac{p-1}{2}$ with $\frac{p-1}{2l}$ odd. Then $\mathbf{N}_T(K) \cong \mathrm{D}_{p-1}$. Set $\mathbf{N}_T(K) = \langle a\rangle{:}\langle b\rangle$. It is easily shown that $\langle b, H\rangle = T$. Then $\mathsf{Cos}(T,H,HbH)$ is a connected T-arc-transitive graph of valency p.

Example 20. Let $T = \mathrm{PSL}(2,p)$ and H a dihedral subgroup of T.

(1) Let $\mathbb{Z}_2 \cong K < H \cong \mathrm{D}_{2r}$ for an odd prime r such that $|T : H|$ is square-free. Let $\epsilon = \pm 1$ such that 4 divides $p+\epsilon$. Then $\mathbf{N}_T(K) = K \times \langle a\rangle{:}\langle b\rangle \cong \mathbb{Z}_2 \times \mathrm{D}_{\frac{p+\epsilon}{2}} \cong \mathrm{D}_{p+\epsilon}$, where b is an involution and, if r divides $p+\epsilon$, we may choose b such that b centralizes H. Then, for $1 \leqslant i < \frac{p+\epsilon}{2}$, the coset graph $\mathsf{Cos}(T,H,Ha^ibH)$ is a connected T-arc-transitive graph of valency r.

(2) Let $G = T$ or $\mathrm{PGL}(2,p)$ and $\mathbb{Z}_2^2 \cong K < H \cong \mathrm{D}_{4r}$ for an odd prime r with $|G : H|$ square-free. Suppose that G contains a subgroup isomorphic to S_4. Then $\mathbf{N}_G(K) = K{:}\langle y, z\rangle \cong \mathrm{S}_4$, where z is an involution with $y^z = y^{-1}$. Then $\mathsf{Cos}(G, T_\alpha, T_\alpha yzT_\alpha)$ is a G-arc-transitive graph of valency r.

Example 21. Let $\mathrm{A}_4 \cong H < T = \mathrm{PSL}(2,p) < G = \mathrm{PGL}(2,p)$ with $|T : H|$ square-free and $\mathbb{Z}_3 \cong K < H$. Let $\epsilon = \pm 1$ with 3 dividing $p+\epsilon$. Then $\mathbf{N}_T(K) \cong \mathrm{D}_{p+\epsilon}$ and $\mathbf{N}_G(K) \cong \mathrm{D}_{2(p+\epsilon)}$. Moreover,

(1) $\mathsf{Cos}(T,H,HxH)$ is a connected $(T,2)$-arc-transitive graph of valency 4, where $x \in \mathbf{N}_T(K) \setminus \mathbf{N}_T(H)$ is an involution;

(2) $\mathsf{Cos}(G,H,HxH)$ is a connected $(G,2)$-arc-transitive graph of valency 4, where $x \in \mathbf{N}_G(K) \setminus (T \cup \mathbf{N}_G(H))$ is an involution.

Example 22. Let $\mathrm{S}_4 \cong H < T = \mathrm{PSL}(2,p)$ with $|T : H|$ square-free.

(1) Let $\mathrm{D}_8 \cong K < H$, and $X = T$ or $\mathrm{PGL}(2,p)$ such that $|X : H|$ is square-free and X has a Sylow 2-subgroup isomorphic to D_{16}. Then $\mathrm{D}_{16} \cong \mathbf{N}_X(K) = K{:}\langle z\rangle$ for an involution $z \in X \setminus H$, and $\mathsf{Cos}(X,H,HzH)$ is a connected $(X,2)$-arc-transitive graph of valency 3.

(2) Let $\mathrm{S}_3 \cong K < H$ and $G = \mathrm{PGL}(2,p)$, and $\epsilon = \pm 1$ with 3 dividing $p+\epsilon$. Then $\mathbf{N}_G(K) = \langle o\rangle \times K$ for an involution o. Set $X = \langle o, H\rangle$. Then $X = T$ or $\mathrm{PGL}(2,p)$ depending on whether or not 12 divides $p+\epsilon$. Thus $\mathsf{Cos}(X,H,HoH)$ is a connected $(X,2)$-arc-transitive graph of valency 4.

Example 23. Let $\mathrm{A}_5 \cong H < T = \mathrm{PSL}(2,p) < G = \mathrm{PGL}(2,p)$ and $K < H$ with $K \cong \mathrm{A}_4$, D_{10} or S_3. Then $\mathbf{N}_G(K) = K{:}\langle z\rangle \cong \mathrm{S}_4$, D_{20} or D_{12}, respectively, where $z \in G \setminus H$ is an involution. Set $X = \langle z, H\rangle$. Then $X = T$ or $\mathrm{PGL}(2,p)$, and $\mathsf{Cos}(X,H,HzH)$ is either a connected $(X,2)$-arc-transitive graph of valency 5 or 6, or a connected X-locally primitive graph of valency 10.

6.2 A characterization

Let $\Gamma = (V, E)$ be a connected G-edge-transitive graph of square-free order and valency $k \geqslant 3$, where $G \leqslant \mathsf{Aut}\Gamma$. Assume that $T := \mathsf{soc}(G) = \mathrm{PSL}(2, p)$ for a prime $p \geqslant 5$, and that G acts transitively on V.

Let $\alpha \in V$. Then $|T : T_\alpha|$ is square-free; in particular, T_α has even order. Since $|G : T| \leqslant 2$, either T is transitive on V, or T has two orbits on V of the same length $\frac{|V|}{2}$. Thus $|V| = |T : T_\alpha|$ or $2|T : T_\alpha|$.

Note that the subgroups of T are known, refer to [11, II.8.27]. We next analyze one by one the possible candidates for T_α.

Lemma 24. *Assume that T_α is cyclic. Then $T_\alpha \cong \mathbb{Z}_m$ for an even divisor m of $\frac{p\pm1}{2}$, T is transitive on V, Γ is not G-locally-primitive, and one of the following holds:*

(i) *Γ is T-edge-transitive, and $k = m$ or $2m$;*

(ii) *$G = \mathrm{PGL}(2, p)$, $G_\alpha \cong \mathbb{Z}_{2m}$ or D_{2m}, and $k = 2m$ or $4m$.*

Proof. Note T_α is a cyclic group of even order. By Lemma 7, T_α is faithful and semiregular on $\Gamma(\alpha)$. It is easy to check that no primitive group contains a normal semiregular cyclic subgroup of even order. Thus Γ is not G-locally-primitive. By [11, II.8.5], T_α is contained in a subgroup conjugate to $\mathbb{Z}_{\frac{p\pm1}{2}}$ in T. Thus $T_\alpha \cong \mathbb{Z}_m$ for an even divisor m of $\frac{p\pm1}{2}$. Then $p(p \mp 1)$ is a divisor of $|T : T_\alpha|$, and so $|T : T_\alpha|$ is even. It follows that T is transitive on V. Note that $|G_\alpha| = m$ or $2m$. It follows that Γ has valency m, $2m$ or $4m$. Then (i) or (ii) is associated with the case that T is transitive or intransitive on E, respectively. □

Lemma 25. *Assume that $|T_\alpha|$ is divisible by p. Then $T_\alpha \cong \mathbb{Z}_p{:}\mathbb{Z}_l$, T is transitive on V and Γ has valency divisible by p, where l is an even divisor of $\frac{p-1}{2}$ with $\frac{p-1}{2l}$ odd. If Γ is G-locally primitive, then Γ is isomorphic to the graph in Example* 19.

Proof. By [11, II.8.27], recalling that T_α has even order, $T_\alpha \cong \mathbb{Z}_p{:}\mathbb{Z}_l$ for an even divisor l of $\frac{p-1}{2}$. Since $|T : T_\alpha| = \frac{p^2-1}{2l} = (p+1)\frac{p-1}{2l}$ is even and square-free, $\frac{p-1}{2l}$ is odd and T is transitive on V. By Lemma 7, noting that T_α is a Frobenius group, T_α acts faithfully on $\Gamma(\alpha)$. In particular, each T_α-orbit on $\Gamma(\alpha)$ has size divisible by p.

Assume that Γ is G-locally primitive. Then T_α is transitive on $\Gamma(\alpha)$ as $T_\alpha \lhd G_\alpha$. It implies that Γ has valency p and Γ is T-arc-transitive. Then $\Gamma \cong \mathsf{Cos}(T, T_\alpha, T_\alpha x T_\alpha)$ for some $x \in \mathbf{N}_T(T_{\alpha\beta})$ with $x^2 \in T_{\alpha\beta}$ and $\langle x, T_\alpha\rangle = T$, where $\beta \in \Gamma(\alpha)$. Note that $\mathbf{N}_T(T_{\alpha\beta}) \cong \mathrm{D}_{p-1}$. We write $\mathbf{N}_T(T_{\alpha\beta}) = \langle a\rangle{:}\langle b\rangle$. Let M be a maximal subgroup of T with $T_\alpha \leqslant M \cong \mathbb{Z}_p{:}\mathbb{Z}_{\frac{p-1}{2}}$. Then $\mathbb{Z}_{\frac{p-1}{2}} \cong \mathbf{N}_M(T_{\alpha\beta}) \leqslant \mathbf{N}_T(T_{\alpha\beta})$. Thus $a \in M$. Write $\frac{p-1}{2} = ij$, where i is odd and j is a power of 2. Then $\langle a\rangle = \langle a^i\rangle \times \langle a^j\rangle$. Since $T_{\alpha\beta} \cong \mathbb{Z}_l$ and $\frac{p-1}{2l}$ is odd, we have $a^i \in T_{\alpha\beta} \leqslant T_\alpha$. Since l is even, $j \neq 1$. It follows from $\langle x, T_\alpha\rangle = T$ that $x = a^{si}a^{tj}b$ for some s and t. Then $T_\alpha x T_\alpha = T_\alpha a^{tj} b T_\alpha = (T_\alpha b T_\alpha)^{a^{-\frac{tj}{2}}}$. Noting that $a^{-\frac{tj}{2}}$ normalizes T_α, we have $\Gamma \cong \mathsf{Cos}(T, T_\alpha, T_\alpha x T_\alpha) \cong \mathsf{Cos}(T, T_\alpha, T_\alpha b T_\alpha)$ as constructed in Example 19. □

Lemma 26. *Assume that* $T_\alpha \cong \mathrm{D}_{2m}$ *with* $m > 1$ *coprime to* p*. Then* m *is a divisor of* $\frac{p\pm1}{2}$*, and* Γ *has valency divisible by* $\frac{m}{2}$ *or* m*. If* Γ *is* G*-locally-primitive, then* Γ *has odd prime valency* r*,* $T_\alpha \cong \mathrm{D}_{2r}$ *or* D_{4r}*, and* Γ *is isomorphic to one of the graphs given in Example* 20*.*

Proof. The first part follows from that $|T_\alpha|$ is a divisor of $|T| = \frac{p(p^2-1)}{2}$.

Let $\{\alpha,\beta\}$ be an edge of Γ. Suppose that $T_{\alpha\beta}$ contains a cyclic subgroup C of order no less than 3. Then C is the unique subgroup of order $|C|$ in both T_α and T_β. For an arbitrary edge $\{\gamma,\delta\}$, since Γ is G-edge-transitive, $\{\gamma,\delta\} = \{\alpha,\beta\}^x$ for $x \in G$, so $T_{\gamma\delta} = T_{\alpha^x\beta^x} = T \cap G_{\alpha^x\beta^x} = T \cap (G_{\alpha\beta})^x = (T_{\alpha\beta})^x$. Then C^x is the unique subgroup of order $|C|$ in both T_γ and T_δ. So $C \leqslant T_\gamma$ for $\gamma \in \Gamma(\alpha) \cup \Gamma(\beta)$. Since Γ is connected, C fixes each vertex of Γ, and so $C = 1$ as $C \leqslant \mathsf{Aut}\Gamma$, a contradiction. Thus $|T_{\alpha\beta}|$ is a divisor of 4, and hence Γ has valency divisible by $\frac{m}{2}$ or m.

Assume that Γ is G-locally primitive. Then $T_\alpha^{\Gamma(\alpha)}$ contains a transitive normal cyclic subgroup. Thus $|\Gamma(\alpha)| = r$ is an odd prime, and $T_\alpha^{\Gamma(\alpha)} \cong T_\alpha/T_\alpha^{[1]} \cong (T_\alpha G_\alpha^{[1]})/G_\alpha^{[1]} \cong \mathrm{D}_{2r}$. Note that $T_\alpha^{[1]}$ is a normal cyclic subgroup of T_α. By the argument in above paragraph, $|T_\alpha^{[1]}| \leqslant 2$. It follows that $T_\alpha \cong \mathrm{D}_{2r}$ or D_{4r}.

Let $T_\alpha \cong \mathrm{D}_{2r}$. Then $|T : T_\alpha|$ is even, so T is transitive on V, and hence Γ is T-arc-transitive. Then $\Gamma \cong \mathsf{Cos}(T, T_\alpha, T_\alpha x T_\alpha)$ for some $x \in \mathbf{N}_T(T_{\alpha\beta})$ with $x^2 \in T_{\alpha\beta}$ and $\langle x, T_\alpha\rangle = T$. Let $\epsilon = \pm1$ such that 4 divides $p+\epsilon$. Then $\mathbf{N}_T(T_{\alpha\beta}) = T_{\alpha\beta}\times\langle a\rangle{:}\langle b\rangle \cong \mathbb{Z}_2\times\mathrm{D}_{\frac{p+\epsilon}{2}} \cong \mathrm{D}_{p+\epsilon}$. It implies that x is an involution. If r does not divides $p+\epsilon$, then $x = a^i b$ for some $1 \leqslant i \leqslant \frac{p+\epsilon}{2}$. Assume that r is a divisor of $p+\epsilon$. Then T_α is contained in a maximal subgroup $M \cong \mathrm{D}_{p+\epsilon}$ of T, and $\mathbf{N}_M(T_{\alpha\beta}) \cong \mathbb{Z}_2^2$ contains the center of M. Without loss of generality, we choose b in the center of M, and so $x = a^i b$ for $1 \leqslant i < \frac{p+\epsilon}{2}$. Thus Γ is isomorphic to a graph given in Example 20 (1).

Now let $T_\alpha \cong \mathrm{D}_{4r}$. Then $T_{\alpha\beta} \cong \mathbb{Z}_2^2$. If T is not transitive on $V\Gamma$, then $G = \mathrm{PGL}(2,p)$, Γ is a bipartite graph, and $T_\alpha = G_\alpha$. Thus we set $X = \mathrm{PSL}(2,p)$ or $\mathrm{PGL}(2,p)$ depending respectively on whether or not T is is transitive on $V\Gamma$. Then $\Gamma \cong \mathsf{Cos}(X, T_\alpha, T_\alpha x T_\alpha)$ for some $x \in \mathbf{N}_X(T_{\alpha\beta}) \setminus T_{\alpha\beta}$ with $x^2 \in T_{\alpha\beta}$; in particular, $\mathbf{N}_X(T_{\alpha\beta})/T_{\alpha\beta}$ is of even order It implies that $\mathbf{N}_T(T_{\alpha\beta}) \cong \mathrm{S}_4$. Let M be the maximal subgroup of X with $T_\alpha \leqslant M$. Then 8 divides $|M|$, and $\mathbf{N}_M(T_{\alpha\beta}) \cong \mathrm{D}_8$. Take $\mathrm{D}_{8r} \cong M_1 \geqslant T_\alpha$. Then $\mathbf{N}_M(T_{\alpha\beta}) = \mathbf{N}_{M_1}(T_{\alpha\beta})$. We write $\mathbf{N}_X(T_{\alpha\beta}) = T_{\alpha\beta}{:}(\langle y\rangle{:}\langle z\rangle)$, where $z \in \mathbf{N}_M(T_{\alpha\beta})$ and $\langle y\rangle{:}\langle z\rangle \cong \mathrm{S}_3$. Noting that $x \notin \mathbf{N}_M(T_{\alpha\beta})$ and x is of even order, we have $x = x_1 y^i z$ for some $x_1 \in T_{\alpha\beta}$ and $i = 1$ or 2. Noting that z normalizes T_α and $y^z = y^{-1}$, we have $\mathsf{Cos}(X, T_\alpha, T_\alpha x T_\alpha) = \mathsf{Cos}(X, T_\alpha, T_\alpha y^i z T_\alpha) \cong \mathsf{Cos}(X, T_\alpha, T_\alpha yz T_\alpha)$. Thus Γ is isomorphic to the graph given in Example 20 (2). □

Theorem 27. *Let* $\Gamma = (V, E)$ *be a connected* G*-edge-transitive graph of square-free order and valency* $k \geqslant 3$*, where* $G \leqslant \mathsf{Aut}\Gamma$*. Assume that* $\mathsf{soc}(G) = \mathrm{PSL}(2,p)$ *for a prime* $p \geqslant 5$*, and that* G *is transitive on* V*. Then, for* $\alpha \in V$*, the pair* $(\mathsf{soc}(G)_\alpha, k)$ *lies in Table* 1*. Further, if* Γ *is* G*-locally primitive, then* $(\mathsf{soc}(G)_\alpha, k)$ *lies in Table* 2*.*

$\mathsf{soc}(G)_\alpha$	k	remark
$\mathbb{Z}_m$	$m, 2m, 4m$	m is an even divisor of $\frac{p\pm1}{2}$
$\mathbb{Z}_p{:}\mathbb{Z}_l$	$pm, 2pm, 4pm$	$\frac{(p-1)}{2l}$ is odd, $m \mid l$
D_{2m}	$\frac{m}{2}, m, 2m, 4m$	m divides $\frac{p\pm1}{2}$
A_4	$l, 2l$	$l \in \{4, 6, 12\}, 32 \nmid p^2-1, T^E$ is transitive
	$2l, 4l$	$p \equiv \pm3(\mathsf{mod}\ 8), G = \mathrm{PGL}(2,p)$
S_4	$l, 2l$	$l \geqslant 3, l \mid 24, p \equiv \pm1(\mathsf{mod}\ 8), G_\alpha = T_\alpha$
A_5	$l, 2l$	$l \geqslant 5, l \mid 60, p \equiv \pm1(\mathsf{mod}\ 10), G_\alpha = T_\alpha$

Table 1:

$\mathsf{soc}(G)_\alpha$	k	Γ	remark
$\mathbb{Z}_p{:}\mathbb{Z}_l$	p	Example 19	$(p-1)/2l$ is odd
D_{4r}	r	Example 20 (1)	prime $r \neq p,\ 32 \nmid (p^2-1)$
D_{2r}	r	Example 20 (2)	prime $r \neq p,\ 16 \nmid (p^2-1)$
A_4	4	Example 21	$32 \nmid (p^2-1)$
S_4	3, 4	Example 22	$p \equiv \pm1(\mathsf{mod}\ 8)$
A_5	5, 6, 10	Example 23	$p \equiv \pm1(\mathsf{mod}\ 10)$

Table 2:

Proof. Let $\Gamma = (V, E)$ be a connected G-edge-transitive graph of square-free order and valency $k \geqslant 3$, where $G \leqslant \mathsf{Aut}\Gamma$. Assume that $T := \mathsf{soc}(G) = \mathrm{PSL}(2,p)$ for a prime $p \geqslant 5$, and that G acts transitively on V. Let $\{\alpha, \beta\} \in E$.

Noting that $|G : G_\alpha| = |T : T_\alpha|$ or $2|T : T_\alpha|$, we have $|G_\alpha : T_\alpha| = 1$ or 2. Then T_α has at most two orbits on each G_α-orbits on $\Gamma(\alpha)$. By Lemma 9, we have $k = |\Gamma(\alpha)| = l$, $2l$ or $4l$, where $l = |T_\alpha : T_{\alpha\beta}|$. By Lemmas 24, 25 and 26, we need only consider the remaining case: $T_\alpha \cong \mathrm{A}_4$, S_4 or A_5.

Let $T_\alpha \cong \mathrm{S}_4$ or A_5. Checking the maximal subgroups of $\mathrm{PGL}(2,p)$ (see [3], for example), we know that $\mathrm{PGL}(2,p)$ has no subgroups of order $2|T_\alpha|$. It follows that $G_\alpha = T_\alpha$. Then $k = l$ or $2l$ depending whether or not $T_\alpha^{\Gamma(\alpha)}$ is transitive. If $T_\alpha \cong \mathrm{S}_4$, then $T_\alpha^{\Gamma(\alpha)} \cong \mathrm{S}_3$ or S_4, which implies that $l \geqslant 3$ and l divides 24. If $T_\alpha \cong \mathrm{A}_5$, Then $l \geqslant 5$ is a divisor of 60.

Let $T_\alpha \cong \mathrm{A}_4$. Assume that T is transitive on E. Then $k = l$ or $2l$, where $l = |T_\alpha : T_{\alpha\beta}|$ for $\alpha \in \Gamma(\alpha)$. By Lemma 7, $l \neq 3$. Thus $l \in \{4, 6, 12\}$. Assume that T is intransitive on E. Then $G = \mathrm{PGL}(2,p)$ and $G_\alpha \cong \mathrm{S}_4$, and hence $p \equiv \pm3(\mathsf{mod}\ 8)$ by checking the maximal subgroups of G. By Lemma 7, we conclude that $T_\alpha^{\Gamma(\alpha)} \cong \mathrm{A}_4$ and $G_\alpha^{\Gamma(\alpha)} \cong \mathrm{S}_4$. It follows that $k = 2l$ or $4l$ for $l \in \{4, 6, 12\}$.

Further, if Γ is G-locally primitive, then $k = 4$ for $T_\alpha \cong \mathrm{A}_4$, $k = 3$ or 4 for $T_\alpha \cong \mathrm{S}_4$, and $k = 5, 6$ or 10 for $T_\alpha \cong \mathrm{A}_5$. Next we determine the G-locally primitive graphs.

Let $T_\alpha \cong \mathrm{A}_4$. Then $T_{\alpha\beta} \cong \mathbb{Z}_3$, and Γ is $(G,2)$-arc-transitive and of valency 4. Let $X = T$ or $\mathrm{PGL}(2,p)$ depending T is transitive or intransitive on V. Then $\mathbf{N}_X(T_{\alpha\beta}) \cong \mathrm{D}_{t(p+\epsilon)}$, where $t = |X : T|$ and $\epsilon = \pm1$ such that 3 divides $p+\epsilon$. Let $x \in \mathbf{N}_X(T_{\alpha\beta})$ with $x^2 \in T_{\alpha\beta}$ and $\langle x, T_\alpha \rangle = X$. Then x is either an involution or of order 6, and xy is an involution

M	T_α	k	T-orbits	remark
$\mathbb{Z}_m$	$\mathbb{Z}_p{:}\mathbb{Z}_l$	p	1	m and $(p-1)/2ml$ are odd
1	D_{4r}	r	$1,2$	prime $r \neq p,\ 32 \nmid (p^2-1)$
1	D_{2r}	r	$1,2$	prime $r \neq p,\ 16 \nmid (p^2-1)$
1	A_4	4	$1,2$	$32 \nmid (p^2-1)$
$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	4	$1,2$	$32 \nmid (p^2-1)$
$\mathbb{Z}_6, \mathrm{S}_3$	$\mathbb{Z}_2^2$	4	2	$16 \nmid (p^2-1)$
1	S_4	$3,\ 4$	$1,2$	$p \equiv \pm 1 (\mathsf{mod}\ 8)$
$\mathbb{Z}_2$	A_4	4	1	$32 \nmid (p^2-1)$
S_3	$\mathbb{Z}_2^2$	4	1	$32 \nmid (p^2-1)$
$\mathbb{Z}_2$	S_4	4	2	$32 \nmid (p^2-1)$
1	A_5	$5,\ 6,\ 10$	1	$p \equiv \pm 1 (\mathsf{mod}\ 10)$
$\mathbb{Z}_2$	A_5	$6,\ 10$	2	$p \equiv \pm 1 (\mathsf{mod}\ 10), 16 \nmid (p^2-1)$

Table 3:

for some $y \in T_{\alpha\beta}$. Note that $T_\alpha x T_\alpha = T_\alpha xy T_\alpha$. Thus, writing Γ as a coset graph, Γ is isomorphic to one of the graphs in Example 21.

Let $T_\alpha \cong \mathrm{S}_4$. Then $G_\alpha = T_\alpha$. If Γ has valency 3, then Γ is isomorphic the graph given in Example 22 (1). If Γ has valency 4, then $G_{\alpha\beta} \cong \mathrm{S}_3$ and $\mathbf{N}_G(G_{\alpha\beta}) = \mathbb{Z}_2 \times \mathrm{S}_3$, it follows that Γ is isomorphic the graph given in Example 22 (2).

Finally, if $T_\alpha = \mathrm{A}_5$ then $G_\alpha = T_\alpha$ and $G_{\alpha\beta} \cong \mathrm{A}_4$, D_{10} or S_3, and thus Γ is isomorphic one of the graphs given in Example 23. □

7 Locally primitive arc-transitive graphs

In this section we give a proof of Theorem 4. We first prove a technical lemma.

Lemma 28. *Let G be a transitive permutation group on V of square-free degree and let M be a normal subgroup of G. Assume that M is semiregular on V and G/M acts faithfully on the M-orbits. Then there is $X \leqslant G$ such that $G = M{:}X$.*

Proof. The result is trivial if $M = 1$. Thus we assume that $M \neq 1$. Note that M has square-free order. Let p be the largest prime divisor of $|M|$ and P be the Sylow p-subgroup of M. Then P is cyclic and is normal in G. Let $\alpha \in V$ and B be the P-orbit with $\alpha \in B$. Let V_P be the set of P-orbits. Then $|B| = p$ is coprime to $|V_P|$. Then $G_B = P{:}G_\alpha$ contains a Sylow p-subgroup $P \times Q$ of G, where Q is a Sylow p-subgroup of G_α. It follows from [2, 10.4] that the extension $G = P.(G/P)$ splits over P. Thus $G = P{:}H$ for some $H < G$ with $H \cap P = 1$. If $M = P$, then the result follows. We assume $M \neq P$ in the following.

Let K be the kernel of G acting on V_P. Noting that each M-orbit on V consists of several P-orbits, we know that K fixes each M-orbits set-wise. It follows from the assumptions that $K \leqslant M$. Then, considering the action of M on its an orbit, we conclude that $K = P$. Thus H is faithful and transitive on V_P. Further, $M = M \cap PH = P(M \cap H)$

implies that $M \cap H$ is semiregular on V_P. It is easily shown that $H/(M \cap H)$ acts faithfully on the $(M \cap H)$-orbits on V_P. Noting that $|V_P| < |V|$, we may assume by induction that $H = (M \cap H)X$ with $X \cap (M \cap H) = 1$. Then $G = P((M \cap H)X) = MX$, and $M \cap X \leqslant M \cap H$ yielding $M \cap X \leqslant M \cap H \cap X = 1$, hence our result follows. □

Let $\Gamma = (V, E)$ be a connected G-locally primitive graph. Suppose that G has a normal subgroup N which has at least three orbits on V. Then either the quotient graph Γ_N is a star, or Γ is a normal cover of Γ_N, refer to [10, Theorem 1.1]. Then following lemma is easily shown.

Lemma 29. *Let $\Gamma = (V, E)$ be a connected G-locally primitive and G-symmetric graph. Let N be a normal subgroup of G. If N is not semiregular on V, then N is transitive on E and has at most two orbits on V.*

Theorem 30. *Let $\Gamma = (V, E)$ be a connected G-locally primitive graph of square-free order and valency $k > 2$. Let $M \lhd G$ be maximal subject to that M has at least three orbits on V. Assume further that Γ_M is not a star. Then one of the following holds.*

(1) *$M = 1$, Γ and G are described as in (1) or (5) of Lemma 13;*

(2) *Γ is a bipartite graph, $G \cong \mathrm{D}_{2n}{:}\mathbb{Z}_k$, $\mathbb{Z}_n{:}\mathbb{Z}_k$ or $\mathbb{Z}_{\frac{n}{k}}{:}\mathbb{Z}_k^2$, and k is the smallest prime divisor of nk;*

(3) *$G = M{:}X$, $M\mathsf{soc}(X) = M \times \mathsf{soc}(X)$ and $\mathsf{soc}(X)$ is a simple group descried in (3)-(6) and (8) of Theorem 1.*

Proof. Since Γ_M is not a star, Γ is a normal cover of Γ_M, hence M is semiregular on V; in particular, $|M|$ is coprime to $|V_M|$. By the choice of M, we know that G/M is faithful on either V_M or one of two G/M-orbits on V_M. Then, by Lemma 28, we have $G = M{:}X$. Note that Γ_M is G/M-locally primitive, and the pair G/M and Γ_M satisfies the assumptions in Theorem 1. Let $Y = \mathsf{soc}(X)$. Then, by Lemma 13, $\Gamma_M \cong \mathsf{K}_{k,k}$ and $Y \cong T^2$ for a simple group T, or Y is a minimal normal subgroup of X.

Since $|M|$ is square-free, M has soluble automorphism group $\mathsf{Aut}(M)$. Noting that $G/\mathbf{C}_G(M) = \mathbf{N}_G(M)/\mathbf{C}_G(M) \lesssim \mathsf{Aut}(M)$, it follows that $G/\mathbf{C}_G(M)$ is soluble. If Y is a nonabelian simple group then $Y \leqslant \mathbf{C}_G(M)$, and hence $MY = M \times Y$, and so part (3) of this theorem occurs. We next complete the proof in two cases.

Case 1. $\Gamma_M \cong \mathsf{K}_{k,k}$ and $Y \cong T^2$ for a simple group T. In this case, by Lemma 13, X is transitive on V_M, and so Γ_M is X-arc-transitive. Then Γ is G-arc-transitive. Moreover, Y has exactly two orbits on V_M of size k. Thus MY has exactly two orbits U and W on V of length $k|M|$. Let U_M and W_M be the sets of M-orbits on U and W, respectively. Then U_M and W_M are Y-orbits on V_M.

Assume first that T is a nonabelian simple group. Then part (5) of Lemma 13 holds for the pair (X, Γ_M). In particular, Y is the unique minimal normal subgroup of X. Let Δ be an M-orbit on V. Suppose that $T \cong \mathrm{A}_7$. Then $k = 105$ and $T_\Delta \cong \mathrm{A}_6 \times \mathrm{PSL}(3, 2)$. It is easily shown that Γ_M is not X-locally primitive, which is not the case. Thus Y is unfaithful on both U_M and W_M. Let K be the kernel of Y acting on U_M. Then $K \cong T$

and, $Y = K \times K^x$ for $x \in X \setminus Y$. It is easily shown that $K \cong T$ is transitive on W_M. Recalling that $G/\mathbf{C}_G(M)$ is soluble, it follows that $K \leqslant \mathbf{C}_{MK}(M)$ and so $MK = M \times K$. Considering the action of MK on Δ, we conclude that K acts trivially on Δ. Then K acts trivially on U. Since K is transitive on W_M, we conclude that $\Gamma \cong \mathsf{K}_{k,k}$. It follows that $M = 1$, and so (1) of this theorem occurs.

Now let $T \cong \mathbb{Z}_p$ for an odd prime p. Then $k = p$ is coprime to $|M|$, and so $|V| = 2k|M|$. Noting that Γ_M has odd valency k, it implies that Γ_M has even order, and so $|M|$ is odd. Moreover, by Lemma 13, $X \cong G/M \cong (\mathbb{Z}_k^2{:}\mathbb{Z}_l).\mathbb{Z}_2$ is nonabelian, where l is a divisor of $k-1$. Since $|M|$ is square-free, M is soluble, and so G is soluble. Let F be the Fitting subgroup of G. Then $\mathbf{C}_G(F) \leqslant F \neq 1$. Suppose that F has at least three orbits on V. Since Γ is G-locally primitive and G-vertex-transitive, Γ is a normal cover of Γ_F; in particular, F has square-free order. Then G/F is isomorphic to a subgroup of $\mathsf{Aut}\Gamma_F$ acting transitively on the arcs of Γ_F, and so G/F is not abelian. On other hand, since $|F|$ is square-free, F is cyclic, and hence $\mathbf{C}_G(F) = F$ and $\mathsf{Aut}(F)$ is abelian. Since $G/F = \mathbf{N}_G(F)/\mathbf{C}_G(F) \lesssim \mathsf{Aut}(F)$, we know that G/F is abelian, a contradiction. Thus F has one or two orbits on V. Suppose that $|F|$ is even. Let Q be the Sylow 2-subgroup of F. Then $Q \lhd G$. Consider the quotient Γ_Q. Since $|V|$ is square-free and Γ is G-vertex-transitive, we get a graph of odd order $k|M|$ and odd valency k, which is impossible. Then F has odd order, and hence F has exactly two orbits on V.

Assume $|F|$ is divisible by k^2. Let P be the Sylow k-subgroup of F. Then $\mathbb{Z}_k^2 \cong Y = \mathsf{soc}(X) = P \lhd G$. By Lemma 29, we conclude that $\Gamma \cong \mathsf{K}_{k,k}$. This implies that $M = 1$, and Γ and G are described as in (1) of Lemma 13. Then (1) of this theorem occurs.

Assume that $|F|$ is not divisible by k^2. Then $M \neq 1$; otherwise $\mathbb{Z}_k^2 \cong Y \leqslant F$, a contradiction. Since F has exactly two orbits on V, we know that $|F|$ is divisible by $k|M|$. Let P be the Sylow k-subgroup of F. Then $\mathbb{Z}_k \cong P \lhd G$. Let q be the smallest prime divisor of $|M|$, and the let N be the q'-Hall subgroup of M. Then NP is a normal subgroup of G. It is easy to see that NP is intransitive on both U and W. Then the quotient graph Γ_{NP} is bipartite and of order $2q$ and valency k, and so $q > k$. Thus, since l is a divisor of $k-1$, each possible prime divisor of l is less than q. Note that FM is a subgroup of G. Then $|G| = 2lk^2|M|$ is divisible by $|FM| = \frac{|F||M|}{|F\cap M|}$. Recalling that $|F|$ is divisible by $k|M|$, it follows that $M \leqslant F$. Let r be an arbitrary prime divisor of $|F|$, and let R be the Sylow r-subgroup of F. Then $R \lhd G$ and r is odd. Since G is transitive on V, all R-orbits on V have the same length. It implies that r is a divisor of $|V|$, and so r is a divisor of $k|M|$. The above argument yields that $|F| = k|M|$, and so $|F|$ is square-free. Then F is cyclic and semiregular on V, $\mathbf{C}_G(F) = F$ and $\mathsf{Aut}(F)$ is abelian. Since $G_\alpha \cong G_\alpha F/F \leqslant G/F = \mathbf{N}_G(F)/\mathbf{C}_G(F) \lesssim \mathsf{Aut}(F)$, we know that both G_α and G/F are abelian. By Lemma 8, $G_\alpha \cong \mathbb{Z}_k$. Since $|G : (FG_\alpha)| = 2$, we have $G = F.\mathbb{Z}_{2k}$. Thus G has a normal regular subgroup $F{:}\mathbb{Z}_2$. Then Γ is isomorphic a Cayley graph $\mathsf{Cay}(F{:}\mathbb{Z}_2, S)$, where $S = \{s^{\tau^i} \mid 0 \leqslant i \leqslant k-1\}$ for an involution $s \in F{:}\mathbb{Z}_2$ and $\tau \in \mathsf{Aut}(F{:}\mathbb{Z}_2)$ of order k such that $\langle S\rangle = F{:}\mathbb{Z}_2$. Noting that $|F{:}\mathbb{Z}_2|$ is square-free, it follows that $F{:}\mathbb{Z}_2$ is a dihedral group, say D_{2n}. Then part (2) of this theorem occurs.

Case 2. $\mathsf{soc}(G/M) \cong \mathsf{soc}(X) = Y \cong \mathbb{Z}_p$. Since Γ_M is X-locally primitive, by Lemma 13, either $X \cong \mathbb{Z}_p{:}\mathbb{Z}_k$, or $X \cong \mathbb{Z}_p{:}\mathbb{Z}_{2k}$ and X is transitive on V_M. Moreover,

$|V_M| = 2p$, $(p, |M|) = 1$, $p > k$ and k is an odd prime. Let $L = MY$. Then L is a semiregular normal subgroup of G, and L has exactly two orbits U and W on V.

Let $X \cong \mathbb{Z}_p{:}\mathbb{Z}_k$. Then $|G| = kp|M| = k|L|$. Assume that $|L|$ has a prime divisor q such that either a Sylow q-subgroup of L is not normal in L or q is the smallest prime divisor of $|L|$. It is easily shown that L has a unique q'-Hall subgroup N; in particular, N is normal in L. Then N is normal in G, and N has q-orbits on each of U and W. Thus the quotient graph Γ_N is bipartite and of order $2q$ and valency k. In particular, $k \leqslant q$. Further, $G/N = \mathbb{Z}_q{:}\mathbb{Z}_k$ is not abelain unless $q = k$. Since $|N|$ is square-free, the outer automorphism group $\mathsf{Out}(N)$ of N is abelian, refer to [12]. Note that $G/(N\mathbf{C}_G(N))$ is isomorphic a quotient of a subgroup of $\mathsf{Out}(N)$. Then $G/(N\mathbf{C}_G(N))$ is abelian. Thus either $q = k$, or $N\mathbf{C}_G(N)$ has order divisible by q. Suppose that $q > k$. Then q is not a divisor of $|N|$ as $N \leqslant L$ and $|L|$ is square-free. Note that $N\mathbf{C}_G(N)/N \cong \mathbf{C}_G(N)/(N \cap \mathbf{C}_G(N))$. It follows that $|\mathbf{C}_G(N)|$ is divisible by q. Let Q be a Sylow q-subgroup of $\mathbf{C}_G(N)$. Then Q is also a Sylow q-subgroup of G, and hence $Q \leqslant L$. Moreover, $NQ/N \lhd G/N$, and so $NQ \lhd G$. Since $NQ = N \times Q$, we know that $Q \lhd G$, which contradicts the choice of q. Therefore, $q = k$. This says that k is the smallest prime divisor of $|G|$, and either $L \cong \mathbb{Z}_n$ or $L \cong \mathbb{Z}_{\frac{n}{k}}{:}\mathbb{Z}_k$, where $n = |L|$. Thus $G = \mathbb{Z}_n{:}\mathbb{Z}_k$ or $\mathbb{Z}_{\frac{n}{k}}{:}\mathbb{Z}_k^2$, and k is the smallest prime divisor of nk.

Now let $X \cong \mathbb{Z}_p{:}\mathbb{Z}_{2k}$. Then G has a normal regular subgroup $R = L{:}\mathbb{Z}_2$, and Γ is isomorphic a Cayley graph $\mathsf{Cay}(R, S)$, where $S = \{s^{\tau^i} \mid 0 \leqslant i \leqslant k-1\}$ for an involution $s \in R$ and an automorphism $\tau \in \mathsf{Aut}(R)$ of order k such that $\langle S \rangle = R$. Noting that $|R|$ is square-free, it follows that R is a dihedral group, say D_{2n}. Then $G = \mathrm{D}_{2n}{:}\mathbb{Z}_k$. Let q be the smallest prime divisor of n. Then G has a normal subgroup N with $|G : N| = 2qk$. It is easily shown that the quotient graph Γ_N is bipartite and of valency k and order $2q$. Then $k \leqslant q$, and so k is the smallest prime divisor of nk. Thus part (2) follows. □

Now we are ready to give a proof of Theorem 4.

Proof of Theorem 4. Let $\Gamma = (V, E)$ be a G-locally primitive arc-transitive graph, and let $M \lhd G$ be maximal subject to that M has at least three orbits on V. Then Γ is a normal cover of $\Sigma := \Gamma_M$. Note that Γ and Σ has even valency if $|M|$ is even.

If G is soluble then, by Theorem 30, one of part (1) of Theorem 4 occurs. Thus we assume that G is insoluble. Then $G = M{:}X$, where $T := \mathsf{soc}(X)$ is a simple group descried in (3)-(6) and (8) of Theorem 1. By Lemma 29, we conclude that either Γ is T-arc-transitive, or Γ is T-edge-transitive and T has exactly two orbits on V. We next consider the case where $T = \mathrm{PSL}(2, p)$ for a prime $p \geqslant 5$.

Let Δ be an M-orbit on V. Then either T_Δ is transitive on Δ; or T_Δ has exactly two orbits on Δ and, in this case, T is intransitive on V and $M \times T$ is transitive on V. We take a normal subgroup N of G such that $N = M$ if the first case occurs, or N is the $2'$-Hall subgroup of M if the second case occurs. Let Δ_1 be an N-orbit contained in Δ. Then $T_\Delta = T_{\Delta_1}$ is transitive on Δ_1 and N is regular on Δ_1. Considering the action of $N \times T_\Delta$, we conclude that $N \cong T_\Delta/K$, where K is the kernel of T_Δ on Δ_1. Note that T_Δ is known by Theorem 27, and that $|V| = |T : T_\alpha|$ or $2|T : T_\alpha|$ is square-free. Then we get Table 3 by checking possible normal subgroups of T_Δ with square-free index. □

References

[1] B. Alspach and M. Y. Xu. $\frac{1}{2}$-transitive graphs of order $3p$. *J. Algebraic Combin.*, 3:347–355, 1994.

[2] M. Aschbacher. *Finite group theory.* Cambridge University Press, Cambridge, 1993.

[3] P. J. Cameron, G. R. Omidi, and B. Tayfeh-Rezaie. 3-Design from $\mathrm{PGL}(2, q)$. *The Electronic J. Combin.*, 13:#R50, 2006.

[4] C. Y. Chao. On the classification of symmetric graphs with a prime number of vertices. *Trans. Amer. Math. Soc.*, 158:247–256, 1971.

[5] Y. Cheng and J. Oxley. On weakly symmetric graphs of order twice a prime. *J. Combin. Theory Ser. B*, 42:196–211, 1987.

[6] M. Conder and P. Dobcsányi. Trivalent symmetric graphs on up to 768 vertices. *J. Combin. Math. Combin. Comput.*, 40:41–63, 2002.

[7] M. D. Conder, C. H. Li, and C. E. Praeger. On the Weiss conjucture for finite locally primitive graphs. *Pro. Edinburgh Math. Soc.*, 43:129–138, 2000.

[8] J. H. Conway, R. T. Curtis, S. P. Noton, R. A. Parker, and R. A. Wilson. *Atlas of Finite Groups.* Clarendon Press, Oxford, 1985.

[9] Y. Q. Feng and Y. T. Li. One-regular graphs of square-free order of prime valency. *European J. Combin.*, 32:165–175, 2011.

[10] M. Giudici, C. H. Li, and C. E. Praeger. Analysing finite locally s-arc transitive graphs. *Trans. Amer. Math. Soc.*, 356:291–317, 2004.

[11] B. Huppert. *Endliche Gruppen I.* Springer-Verlag, 1967.

[12] C. H. Li, Z. Liu, and Z. P. Lu. Tetravalent edge-transitive Cayley graphs of square free order. *Discrete Math.*, 312:1952–1967, 2012.

[13] C. H. Li, Z. P. Lu, and G. X. Wang. Vertex-transitive cubic graphs of square-free order. *J. Graph Theory*, 75:1–19, 2014.

[14] C. H. Li, Z. P. Lu, and G. X. Wang. The vertex-transitive and edge-transitive tetravalent graphs of square-free order. *J. Algebraic Combin.*, 42:25–50, 2015.

[15] C. H. Li and Ákos Seress. The primitive permutation groups of square free degree. *Bull. London Math. Soc.*, 35:635–644, 2003.

[16] G. X. Liu and Z. P. Lu. On edge-transitive cubic graphs of square-free order. *European J. Combin.*, 45:41–46, 2015.

[17] C. E. Praeger, R. J. Wang, and M. Y. Xu. Symmetric graphs of order a product of two distinct primes. *J. Combin. Theory Ser. B*, 58:299–318, 1993.

[18] C. E. Praeger and M. Y. Xu. Vertex-primitive graphs of order a product of two distinct primes. *J. Combin. Theory Ser. B*, 59:245–266, 1993.

[19] M. W. Short. *The Primitive Soluble Permutation Groups of Degree less than* 256. Springer-Verlag, 1992.

[20] M. Suzuki. On a class of doubly transitive groups. *Ann. Math.*(2), 75:105–145, 1962.

[21] R. J. Wang. Half-transitive graphs of order a product of two distinct primes. *Comm. Alg.*, 22:915–927, 1994.

[22] R. J. Wang and M. Y. Xu. A classification of symmetric graphs of order $3p$. *J. Combin. Theory Ser. B*, 58:197–216, 1993.

[23] J. X. Zhou and Y. Q. Feng. Tetravalent one-regular graphs of order $2pq$. *J. Algebraic Combin.*, 29:457–471, 2009.