Linear transformations preserving the strong q-log-convexity of polynomials

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Abstract

In this paper, we give a sufficient condition for the linear transformation preserving the strong q-log-convexity. As applications, we get some linear transformations (for instance, Morgan-Voyce transformation, binomial transformation, Narayana transformations of two kinds) preserving the strong q-log-convexity. In addition, our results not only extend some known results, but also imply the strong q-log-convexity of some sequences of polynomials.

Keywords: Log-concavity; Log-convexity; q-Log-convexity; Strong q-log-convexity

1 Introduction

Let $a_0, a_1, a_2, \ldots$ be a sequence of nonnegative real numbers. The sequence is called *log-concave* (resp. *log-convex*) if for all $k \geqslant 1$, $a_{k-1}a_{k+1} \leqslant a_k^2$ (resp. $a_{k-1}a_{k+1} \geqslant a_k^2$). The log-concave sequences arise often in combinatorics, algebra, geometry, analysis, probability and statistics and have been extensively investigated. We refer the reader to [21, 3, 25] for log-concavity and [15, 26] for log-convexity.

For a polynomial $f(q)$ with real coefficients, denote $f(q) \geqslant_q 0$ if it has only nonnegative coefficients. For a sequence of polynomials with nonnegative coefficients $\{f_n(q)\}_{n\geqslant 0}$, it is called *$q$-log-convex*, introduced by Liu and Wang [15], if

$$\mathcal{T}(f_n(q)) = f_{n+1}(q)f_{n-1}(q) - f_n(q)^2 \geqslant_q 0 \tag{1.1}$$

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for $n \geqslant 1$. If the opposite inequality in (1.1) holds, then it is called *q-log-concave*, first suggested by Stanley. It is called *strongly q-log-convex* if

$$f_{n+1}(q)f_{m-1}(q) - f_n(q)f_m(q) \geqslant_q 0$$

for any $n \geqslant m \geqslant 1$, see Chen *et al.* [8]. Clearly, their strong q-log-convexity of polynomial sequences implies the q-log-convexity. However, the converse does not hold, see Chen *et al.* [8]. It is easy to see that if the sequence $\{f_n(q)\}_{n\geqslant 0}$ is q-log-convex, then for each fixed nonnegative number q, the sequence $\{f_n(q)\}_{n\geqslant 0}$ is log-convex. The q-log-concavity and q-log-convexity of polynomials have been extensively studied, see [4, 5, 6, 7, 8, 10, 11, 13, 14, 15, 19, 20, 22, 26, 27] for instance.

It is a good way to obtain the log-concavity or log-convexity by some operators. For instance, Davenport and Pólya [9] demonstrated that the log-convexity is preserved under the binomial convolution. Wang and Yeh [25] also proved that the log-concavity is preserved under the binomial convolution. Brändén [2] studied some linear transformations preserving the Pólya frequency property of sequences. Liu and Wang [15] also studied the linear transformation preserving the log-convexity. However, there is no result about the linear transformation preserving the strong q-log-convexity. This is our motivation of this paper.

It has been found that many polynomials have the strong q-log-convexity. Let polynomials $\mathscr{A}_n(q) = \sum_{k=0}^{n} a(n,k)q^k$ for $n \geqslant 0$. Note that $\{q^k\}_{k\geqslant 0}$ is a strongly q-log-convex sequence. Thus it is natural to consider the strong q-log-convexity of

$$\mathscr{B}_n(q) = \sum_{k=0}^{n} a(n,k)f_k(q),$$

for $n \geqslant 0$ if $\{f_n(q)\}_{n\geqslant 0}$ is a strongly q-log-convex sequence.

The objective of this paper is to study the strong q-log-convexity of $\{\mathscr{B}_n(q)\}_{n\geqslant 0}$. In Section 2, we first give a sufficient condition implying the strong q-log-convexity of $\mathscr{B}_n(q)$, see Theorem 2.1. Then we apply it to some special linear transformations. As consequences, on the one hand, we extend some known results. On the other hand, we also get some new results on the strong q-log-convexity of some sequences of polynomials.

2 Strong q-log-convexity and linear transformations

Given a triangular array $\{a(n,k)\}_{0\leqslant k\leqslant n}$ of nonnegative real numbers and a strongly q-log-convex sequence $\{f_n(q)\}_{n\geqslant 0}$, define the polynomials

$$\mathscr{A}_n(q) = \sum_{k=0}^{n} a(n,k)q^k \quad \text{and} \quad \mathscr{B}_n(q) = \sum_{k=0}^{n} a(n,k)f_k(q),$$

for $n \geqslant 0$. For convenience, let $a(n,k) = 0$ unless $0 \leqslant k \leqslant n$. Suppose $m \geqslant n$. For $0 \leqslant t \leqslant m+n$, define

$$\begin{aligned} a_k(m,n,t) =& a(n-1,k)a(m+1,t-k) + a(m+1,k)a(n-1,t-k) \\ &- a(m,k)a(n,t-k) - a(n,k)a(m,t-k) \end{aligned}$$

if $0 \leqslant k < t/2$, and

$$a_k(m,n,t) = a(n-1,k)a(m+1,k) - a(n,k)a(m,k)$$

if t is even and $k = t/2$. Our main result of this paper is as follows.

Theorem 2.1. *Suppose that the triangle $\{a(n,k)\}$ of nonnegative real numbers satisfies the following two conditions:*

(C1) *The sequence of polynomials $\{\mathscr{A}_n(q)\}_{n\geqslant 0}$ is strongly q-log-convex.*

(C2) *There exists an index $r = r(m,n,t)$ such that $a_k(m,n,t) \geqslant 0$ for $k \leqslant r$ and $a_k(m,n,t) < 0$ for $k > r$.*

If the sequence $\{f_k(q)\}_{k\geqslant 0}$ is strongly q-log-convex, then the polynomials $\mathscr{B}_n(q)$ form a strongly q-log-convex sequence.

Proof. Let $m \geqslant n \geqslant 0$. By computation, we have

$$\mathscr{A}_{n-1}(q)\mathscr{A}_{m+1}(q) - \mathscr{A}_n(q)\mathscr{A}_m(q) = \sum_{t=0}^{m+n}\left[\sum_{k=0}^{\lfloor t/2\rfloor} a_k(m,n,t)\right]q^t,$$

and

$$\mathscr{B}_{n-1}(q)\mathscr{B}_{m+1}(q) - \mathscr{B}_n(q)\mathscr{B}_m(q) = \sum_{t=0}^{m+n}\left[\sum_{k=0}^{\lfloor t/2\rfloor} a_k(m,n,t)f_k(q)f_{t-k}(q)\right].$$

Let $A(m,n,t) = \sum_{k=0}^{\lfloor t/2\rfloor} a_k(m,n,t)$. Then the condition (C1) is equivalent to $A(m,n,t) \geqslant 0$ for $0 \leqslant t \leqslant m+n$. Since $\{f_k(q)\}_{k\geqslant 0}$ is strongly q-log-convex, we have $f_0(q)f_t(q) \geqslant_q f_1(q)f_{t-1}(q) \geqslant_q f_2(q)f_{t-2}(q) \geqslant_q \cdots$. By (C2) we find that

$$\sum_{k=0}^{\lfloor t/2\rfloor} a_k(m,n,t)f_k(q)f_{t-k}(q) \geqslant_q \sum_{k=0}^{\lfloor t/2\rfloor} a_k(m,n,t)f_r(q)f_{t-r}(q) = A(m,n,t)f_r(q)f_{t-r}(q).$$

Thus $\{\mathscr{B}_n(q)\}_{n\geqslant 0}$ is strongly q-log-convex. □

In what follows we will give some applications of Theorem 2.1.

Proposition 2.2. *If $\{f_k(q)\}_{k\geqslant 0}$ is strongly q-log-convex, then the polynomials $g_n(q) = \sum_{k=0}^{n}\binom{n+k}{n-k}f_k(q)$ form a strongly q-log-convex sequence.*

Proof. Let $a(n,k) = \binom{n+k}{n-k}$ for $0 \leqslant k \leqslant n$. Then by Theorem 2.1, it suffices to show that the triangle $\{a(n,k)\}$ satisfies the conditions (C1) and (C2).

Let $\mathscr{A}_n(q) = \sum_{k=0}^{n}\binom{n+k}{n-k}q^k$, which is the n-th Morgan-Voyce polynomial ([24]). By the recurrence relation of the binomial coefficients, we can obtain

$$\binom{n+1+k}{n+1-k} = 2\binom{n+k}{n-k} + \binom{n+k-1}{n-k+1} - \binom{n-1+k}{n-1-k}.$$

From this it follows that $\mathscr{A}_{n+1}(q) = (2+q)\mathscr{A}_n(q) - \mathscr{A}_{n-1}(q)$ with the initial conditions $\mathscr{A}_0(q) = 1, \mathscr{A}_1(q) = 1+q$ and $\mathscr{A}_2(q) = 1+3q+q^2$. Thus, by induction on k, it is easy to show $\mathscr{A}_{k+1}(q) \geqslant_q \mathscr{A}_k(q)$ for $k \geqslant 0$. This implies that

$$\mathscr{A}_{k+2}(q) - (1+q)\mathscr{A}_{k+1}(q) = \mathscr{A}_{k+1}(q) - \mathscr{A}_k(q) \geqslant_q 0.$$

Let $m = n+k \geqslant n$. Then we have

$$\begin{aligned}
& \mathscr{A}_{n-1}(q)\mathscr{A}_{m+1}(q) - \mathscr{A}_n(q)\mathscr{A}_m(q) \\
= \;& \mathscr{A}_{n-1}(q)\mathscr{A}_{n+k+1}(q) - \mathscr{A}_n(q)\mathscr{A}_{n+k}(q) \\
= \;& \mathscr{A}_{n-1}(q)[(2+q)\mathscr{A}_{n+k}(q) - \mathscr{A}_{n+k-1}(q)] - [(2+q)\mathscr{A}_{n-1}(q) - \mathscr{A}_{n-2}(q)]\mathscr{A}_{n+k}(q) \\
= \;& \mathscr{A}_{n-2}(q)\mathscr{A}_{n+k}(q) - \mathscr{A}_{n-1}(q)\mathscr{A}_{n-1+k}(q) \\
\vdots \;& \\
= \;& \mathscr{A}_0(q)\mathscr{A}_{2+k}(q) - \mathscr{A}_1(q)\mathscr{A}_{1+k}(q) \\
= \;& \mathscr{A}_{k+1}(q) - \mathscr{A}_k(q) \\
\geqslant_q \;& 0.
\end{aligned}$$

Thus the sequence $\{\mathscr{A}_n(q)\}_{n\geqslant 0}$ is strongly q-log-convex, and so the condition (C1) is satisfied.

In what follows we consider (C2) condition. Note that $a(n,k) = \binom{n+k}{n-k} = \binom{n+k}{2k}$. Let $m \geqslant n$, $0 \leqslant t \leqslant m+n$ and $0 \leqslant k \leqslant t/2$.

If t is even and $k = t/2$, then

$$a_k(m,n,t) = \binom{n-1+k}{2k}\binom{m+1+k}{2k} - \binom{n+k}{2k}\binom{m+k}{2k} < 0.$$

If $0 \leqslant k < t/2$, then

$$\begin{aligned}
a_k(m,n,t) &= \left\{\binom{n-1+k}{2k}\binom{m+1+t-k}{2(t-k)} - \binom{n+k}{2k}\binom{m+t-k}{2(t-k)}\right\} \\
&+ \left\{\binom{m+1+k}{2k}\binom{n-1+t-k}{2(t-k)} - \binom{m+k}{2k}\binom{n+t-k}{2(t-k)}\right\} \\
&= \underbrace{\frac{2[nt-(m+n+1)k]}{(n+k)[m+1-(t-k)]}\binom{n+k}{2k}\binom{m+t-k}{2(t-k)}}_{A} \\
&+ \underbrace{\frac{2[(m+n+1)k-(m+1)t]}{[n+(t-k)](m+1-k)}\binom{m+k}{2k}\binom{n+t-k}{2(t-k)}}_{B}.
\end{aligned}$$

It can be seen that if $0 \leqslant k < \frac{tn}{m+n+1}$, then $A > 0$ and $B < 0$; if $\frac{tn}{m+n+1} < k < t/2 < \frac{t(m+1)}{m+n+1}$, then $A < 0$ and $B < 0$. Thus $a_k(m,n,t) < 0$ if $\frac{tn}{m+n+1} < k < t/2$. And

$$\begin{aligned}
a_0(m,n,t) &= \binom{m+1+t}{2t} - \binom{m+t}{2t} + \binom{n-1+t}{2t} - \binom{n+t}{2t} \\
&= \frac{2t}{m+1-t}\binom{m+t}{2t} - \frac{2t}{n+t}\binom{n+t}{2t} > 0
\end{aligned}$$

Note that if $0 \leqslant k < \frac{tn}{m+n+1}$, then $A > 0$ and $B < 0$. In order to show that $A+B$ changes sign at most once (from nonnegative to nonpositive) for $k \in [0, \frac{tn}{m+n+1})$, we consider the monotonicity of $A/(-B)$. Let $\Delta = A/(-B)$. Then we have

$$\begin{aligned} \Delta &= \frac{[nt-(m+n+1)k][n+(t-k)](m+1-k)}{[(m+1)t-(m+n+1)k](n+k)[m+1-(t-k)]} \times \frac{\binom{n+k}{2k}\binom{m+t-k}{2(t-k)}}{\binom{m+k}{2k}\binom{n+t-k}{2(t-k)}} \\ &= \frac{[nt-(m+n+1)k][n+(t-k)](m+1-k)}{[(m+1)t-(m+n+1)k](n+k)[m+1-(t-k)]} \times \frac{\binom{m+(t-k)}{m-n}\binom{m-k}{m-n}}{\binom{m+k}{m-n}\binom{m-(t-k)}{m-n}}. \end{aligned}$$

Claim 2.3. *Δ is decreasing when k is increasing.*

Proof. If we assume that

$$Y_k = \frac{\binom{m+(t-k)}{m-n}\binom{m-k}{m-n}}{\binom{m+k}{m-n}\binom{m-(t-k)}{m-n}},$$

then it is not hard to prove that

$$\frac{Y_{k+1}}{Y_k} = \frac{(n+t-k)(n-k)(n+1-t+k)(n+1+k)}{(m+t-k)(m-k)(m+1-t+k)(m+1+k)} \leqslant 1,$$

which implies that Y_k is decreasing in k. On the other hand, it is easy to see that both $\frac{m+1-k}{m+1-(t-k)}$ and $\frac{n+(t-k)}{n+k}$ are decreasing in k. In addition,

$$\frac{nt-(m+n+1)k}{(m+1)t-(m+n+1)k} = 1 - \frac{(m-n+1)t}{(m+1)t-(m+n+1)k}$$

is also decreasing in k. Thus Δ is decreasing when k increasing. □

By Claim 2.3, it follows that $\frac{A}{-B} - 1$ changes sign at most once (from nonnegative to nonpositive) for $k \in [0, t/2]$. So does $A+B$. It follows that the triangle $a(n,k) = \binom{n+k}{n-k}$ satisfies the condition (C2). The proof is complete. □

In [1], Bonin, Shapiro and Simion introduced a q-analog of the large Schröder number r_n, called the q-Schröder number $r_n(q)$. It is defined as:

$$r_n(q) = \sum_P q^{\mathrm{diag}(P)},$$

where P takes over all Schröder paths from $(0,0)$ to (n,n) and $\mathrm{diag}(P)$ denotes the number of diagonal steps in the path P. Obviously, the large Schröder numbers $r_n = r_n(1)$. In addition

$$r_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k} C_k q^{n-k},$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$. In [26], Zhu proved the strong q-log-convexity of q-Schröder numbers, which also immediately follows from Proposition 2.2.

Corollary 2.4. [26] *The q-Schröder numbers $r_n(q)$ form a strongly q-log-convex sequence.*

The q-central Delannoy numbers

$$D_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k} \binom{2k}{k} q^{n-k},$$

see Sagan [18]. Liu and Wang [15] proved that numbers $D_n(q)$ form a q-log-convex sequence. Zhu [26] demonstrated the strong q-log-convexity of q-central Delannoy numbers, which can also been obtained from Proposition 2.2.

Corollary 2.5. [26] *The q-central Delannoy numbers $D_n(q)$ form a strongly q-log-convex sequence.*

The Bessel polynomials are defined by

$$B_n(q) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} \left(\frac{q}{2}\right)^k,$$

and they have been extensively studied. Chen, Wang and Yang [8] obtained the strong q-log-convexity of $B_n(q)$. Note that

$$B_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k} \binom{2k}{k} k! \left(\frac{q}{2}\right)^k$$

and it is not hard to prove that the sequence $\{\binom{2k}{k} k! \left(\frac{q}{2}\right)^k\}_{k \geqslant 0}$ is strongly q-log-convex. Thus we have the following corollary by Proposition 2.2.

Corollary 2.6. [8] *The Bessel polynomials $B_n(q)$ form a strongly q-log-convex sequence.*

Proposition 2.7. *If $\{f_k(q)\}_{k \geqslant 0}$ is strongly q-log-convex, then the polynomials $b_n(q) = \sum_{k=0}^{n} \binom{n}{k} f_k(q)$ for $n \geqslant 0$ form a strongly q-log-convex sequence.*

Proof. Let $a(n,k) = \binom{n}{k}$ for $0 \leqslant k \leqslant n$. Then by Theorem 2.1, it suffices to show that the triangle $\{a(n,k)\}$ satisfies the conditions (C1) and (C2).

Let $g_n(q) = \sum_{k=0}^{n} \binom{n}{k} q^k = (1+q)^n$ for $n \geqslant 0$. It follows that

$$g_{n-1}(q) g_{m+1}(q) - g_n(q) g_m(q) = 0$$

for any $m \geqslant n$. Thus the sequence $\{g_n(q)\}_{n \geqslant 0}$ is strongly q-log-convex, and so the condition (C1) is satisfied.

We proceed to demonstrating the condition (C2) as follows. Note that $a(n,k) = \binom{n}{k}$. Let $m \geqslant n$, $0 \leqslant t \leqslant m+n$ and $0 \leqslant k \leqslant t/2$. If t is even and $k = t/2$, then

$$a_k(m,n,t) = \binom{n-1}{k} \binom{m+1}{k} - \binom{n}{k} \binom{m}{k} < 0.$$

If $0 \leqslant k < t/2$, then

$$\begin{aligned} a_k(m,n,t) &= \left\{ \binom{n-1}{k}\binom{m+1}{t-k} - \binom{n}{k}\binom{m}{t-k} \right\} \\ &+ \left\{ \binom{m+1}{k}\binom{n-1}{t-k} - \binom{m}{k}\binom{n}{t-k} \right\} \\ &= \underbrace{\frac{nt-(m+n+1)k}{n[m+1-(t-k)]}\binom{n}{k}\binom{m}{t-k}}_{A_1} \\ &+ \underbrace{\frac{(m+n+1)k-(m+1)t}{n(m+1-k)}\binom{m}{k}\binom{n}{t-k}}_{B_1}. \end{aligned}$$

We find that if $0 \leqslant k < \frac{tn}{m+n+1}$, then $A_1 > 0$ and $B_1 < 0$; if $\frac{tn}{m+n+1} \leqslant k < t/2 < \frac{t(m+1)}{m+n+1}$, then $A_1 \leqslant 0$ and $B_1 < 0$. Thus $a_k(m,n,t) < 0$ if $\frac{tn}{m+n+1} \leqslant k < t/2$. And

$$\begin{aligned} a_0(m,n,t) &= \binom{m+1}{t} - \binom{m}{t} + \binom{n-1}{t} - \binom{n}{t} \\ &= \frac{t}{m+1-t}\binom{m}{t} - \frac{t}{n}\binom{n}{t} > 0. \end{aligned}$$

In order to show that A_1+B_1 changes sign at most once (from nonnegative to nonpositive) for $k \in [0, \frac{tn}{m+n+1})$, we consider the monotonicity of $A_1/(-B_1)$. Let $\Delta_1 = A_1/(-B_1)$. Then we have

$$\Delta_1 = \frac{[nt-(m+n+1)k](m+1-k)}{[(m+1)t-(m+n+1)k][m+1-(t-k)]} \times \frac{\binom{n}{k}\binom{m}{t-k}}{\binom{m}{k}\binom{n}{t-k}}.$$

In the following we will prove that Δ_1 is decreasing in k.

If we assume that

$$y_k = \frac{\binom{n}{k}\binom{m}{t-k}}{\binom{m}{k}\binom{n}{t-k}},$$

then it is not hard to prove that

$$\frac{y_{k+1}}{y_k} = \frac{(n-t+k+1)(n-k)}{(m-t+k+1)(m-k)} \leqslant 1.$$

It follows that y_k is decreasing in k. On the other hand, we have known that $\frac{m+1-k}{m+1-(t-k)}$ and $\frac{nt-(m+n+1)k}{(m+1)t-(m+n+1)k}$ are decreasing in k. Thus Δ_1 is decreasing in k. It follows that $a_k(m,n,t)$ changes sign at most once (from nonnegative to nonpositive). As a consequence, we know that $a(n,k) = \binom{n}{k}$ satisfies the condition $(C2)$ of Theorem 2.1. This completes the proof. $\square$

Let the Bell polynomial $B(n,q) = \sum_{k\geqslant 0}^{n} S(n,k)q^k$, where $S(n,k)$ is the Stirling number of the second kind. It was proved that the polynomials $B(n,q)$ form a strongly q-log-convex sequence, see Chen *et al.* [8] and Zhu [26, 27]. Note that $B(n+1,q) = \sum_{k=0}^{n} \binom{n}{k} B(n,q)$. Thus, by induction and Proposition 2.7, we can give a new proof for the strong q-log-convexity of $B(n,q)$.

Let the polynomial

$$W_n(q) = \sum_{k\geqslant 0} \binom{n}{k}^2 q^k,$$

which is called the Narayana polynomials of type B, see [6]. Liu and Wang [15] conjectured that $\{W_n(q)\}_{n\geqslant 0}$ is q-log-convex, which was proved by Chen *et al.* [6] using the theory of symmetric functions. In addition, Zhu [26] proved the strong q-log-convexity of $W_n(q)$. Now we can extend it to the following by Theorem 2.1.

Proposition 2.8. *If $\{f_k(q)\}_{k\geqslant 0}$ is strongly q-log-convex, then the polynomials $s_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 f_k(q)$ form a strongly q-log-convex sequence.*

Proof. Note that the strong q-log-convexity of $W_n(q)$ has been proved, see Zhu [26]. So the condition $(C1)$ of Theorem 2.1 is satisfied.

Note that $a(n,k) = \binom{n}{k}^2$, $m \geqslant n$, $0 \leqslant t \leqslant m+n$ and $0 \leqslant k \leqslant t/2$.
If t is even and $k = t/2$, then

$$a_k(m,n,t) = \binom{n-1}{k}^2 \binom{m+1}{k}^2 - \binom{n}{k}^2 \binom{m}{k}^2 < 0.$$

If $0 \leqslant k < t/2$, then

$$\begin{aligned} a_k(m,n,t) =& \left\{ \binom{n-1}{k}^2 \binom{m+1}{t-k}^2 - \binom{n}{k}^2 \binom{m}{t-k}^2 \right\} \\ &+ \left\{ \binom{m+1}{k}^2 \binom{n-1}{t-k}^2 - \binom{m}{k}^2 \binom{n}{t-k}^2 \right\} \\ =& \underbrace{\frac{nt-(m+n+1)k}{n[m+1-(t-k)]} \frac{n(2m+2-t)-(m-n+1)k}{n[m+1-(t-k)]} \binom{n}{k}^2 \binom{m}{t-k}^2}_{A_2} \\ &+ \underbrace{\frac{(m+n+1)k-(m+1)t}{n(m+1-k)} \frac{(m+1)(2n-t)+(m-n+1)k}{n(m+1-k)} \binom{m}{k}^2 \binom{n}{t-k}^2}_{B_2}. \end{aligned}$$

We find that if $0 \leqslant k < \frac{tn}{m+n+1}$, then $A_2 > 0$ and $B_2 < 0$; if $\frac{tn}{m+n+1} \leqslant k < t/2 < \frac{t(m+1)}{m+n+1}$, then $A_2 \leqslant 0$ and $B_2 < 0$. Thus $a_k(m,n,t) < 0$ if $\frac{tn}{m+n+1} \leqslant k < t/2$. And

$$\begin{aligned} a_0(m,n,t) &= \binom{m+1}{t}^2 - \binom{m}{t}^2 + \binom{n-1}{t}^2 - \binom{n}{t}^2 \\ &= \frac{t}{m+1-t}\binom{m}{t}\left\{\binom{m+1}{t} + \binom{m}{t}\right\} - \frac{t}{n}\binom{n}{t}\left\{\binom{n-1}{t} + \binom{n}{t}\right\} > 0. \end{aligned}$$

Note that if $0 \leqslant k < \frac{tn}{m+n+1}$, then $A_2 > 0$ and $B_2 < 0$. Let $\Delta_2 = A_2/(-B_2)$. Then we have

$$\Delta_2 = y_k \times \Delta_1 \times \frac{n(2m+2-t)-(m-n+1)k}{(m+1)(2n-t)+(m-n+1)k} \times \frac{m+1-k}{m+1-(t-k)}$$

Noticing that when k is increasing, we have known that y_k, Δ_1 and $\frac{m+1-k}{m+1-(t-k)}$ are decreasing, respectively. Moreover, it is easy to see that $\frac{n(2m+2-t)-(m-n+1)k}{(m+1)(2n-t)+(m-n+1)k}$ is decreasing when k increasing. So is Δ_2. As a result, we obtain that $a(n,k) = \binom{n}{k}^2$ satisfies the condition $(C2)$ of Theorem 2.1. This completes the proof. □

In [12], Drivera, *et al.* proved that a conjecture of C. Greene and H. Wilf that all zeros of the hypergeometric polynomial

$$P_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} q^k$$

are real. It is known that $P_n(1)$ is related to the Franel numbers, and Sun [23] studied the congruence properties of $P_n(q)$. In [11], it was proved that polynomials $P_n(q)$ form a q-log-convex sequence. By Proposition 2.8, we have the following stronger result.

Corollary 2.9. *The hypergeometric polynomials $P_n(q)$ form a strongly q-log-convex sequence.*

Note that the rook polynomial of a square of side n, denoted by $S_n(q)$, is given by

$$S_n(q) = \sum_{k=0}^{n} \binom{n}{k}^2 k! q^k,$$

see [17, Chapter 3. Problems 18] for instance, which also has only real zeros in term of the rook theory. The following result is immediate from Proposition 2.8.

Corollary 2.10. *The rook polynomials $S_n(q)$ form a strongly q-log-convex sequence.*

The Narayana number $N(n,k)$ is defined as the number of Dyck paths of length $2n$ with exactly k peaks (a peak of a path is a place at which the step $(1,1)$ is directly followed by the step $(1,-1)$). The Narayana numbers have an explicit expression $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. Liu and Wang [15] conjectured that the Narayana polynomials $N_n(q) = \sum_{k=0}^{n} N(n,k)q^k$ are q-log-convex. Using the technique of the symmetric functions, Chen, et al. [6] proved the strong q-log-convexity of $N_n(q)$. Recently, Zhu [26] gave a simple proof based on certain recurrence relation. Now, we can extend it to the next result, whose proof is omitted for brevity.

Proposition 2.11. *If $\{f_k(q)\}_{k\geqslant 0}$ is strongly q-log-convex, then the polynomials $\mathcal{N}_n(q) = \sum_{k=0}^{n} N(n,k) f_k(q)$ form a strongly q-log-convex sequence.*

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