Weight of 3-paths in sparse plane graphs

V. A. Aksenov*
Novosibirsk State University
Novosibirsk, Russia
akc@belka.sm.nsc.ru

O. V. Borodin†
Sobolev Institute of Mathematics
Novosibirsk, Russia
brdnoleg@math.nsc.ru

A. O. Ivanova‡
Ammosov North-Eastern Federal University
Yakutsk, Russia
shmgnanna@mail.ru

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Abstract

We prove precise upper bounds for the minimum weight of a path on three vertices in several natural classes of plane graphs with minimum degree 2 and girth g from 5 to 7. In particular, we disprove a conjecture by S. Jendrol' and M. Maceková concerning the case $g = 5$ and prove the tightness of their upper bound for $g = 5$ when no vertex is adjacent to more than one vertex of degree 2. For $g \geqslant 8$, the upper bound recently found by Jendrol' and Maceková is tight.

Keywords: Plane graph, Girth, 3-Path, Weight

1 Introduction

A normal plane map (NPM) is a plane pseudograph in which loops and multiple edges are allowed, but the degree of each vertex and face is at least three.

The degree of a vertex v or a face f, that is, the number of edges incident with v or f (loops and cut-edges are counted twice), is denoted by $d(v)$ or $d(f)$, respectively. A *k-vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-vertex v satisfies $d(v) \geqslant k$, etc.

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Let $\delta(G)$ be the minimum vertex degree, $w_k(G)$ be the minimum degree-sum of a path on k vertices (hereafter called a *k-path*) in a plane graph, and $g(G)$ be its girth, that is the length of a shortest cycle. We will often drop the argument when the graph is clear from context.

An edge uv is an *(i,j)-edge* if $d(u) \leqslant i$ and $d(v) \leqslant j$. More generally, a path $v_i \dots v_k$ is a *path of type* $(i_1, \dots, i_k)$ if $d(v_j) \leqslant i_j$ whenever $1 \leqslant j \leqslant k$.

Already in 1904, Wernicke [20] proved that every NPM M_5 with $\delta(M_5) = 5$ satisfies $w_2 \leqslant 11$, and Franklin [12] strengthened this to the existence of at least two 6^--neighbors for a 5-vertex, which implies that M_5 satisfies $w_3(M_5) \leqslant 17$. Franklin's bound 17 is precise, as shown by putting a vertex inside each face of the dodecahedron and joining it with the five boundary vertices.

It follows from Lebesgue's results in [18] that each NPM has an edge of weight at most 14 incident with a 3-vertex, or an edge of weight at most 11, where 11 is sharp. For 3-connected plane graphs, Kotzig [17] proved a precise result: $w_2 \leqslant 13$.

In 1972, Erdős (see [13]) conjectured that Kotzig's bound $w_2 \leqslant 13$ holds for all planar graphs with $\delta \geqslant 3$. Barnette (see [13]) announced to have proved this conjecture, but the proof has never appeared in print. The first published proof of Erdős' conjecture is due to Borodin [2]. More generally, Borodin [4, 5, 6] proved that every NPM contains a $(3,10)$-, or $(4,7)$-, or $(5,6)$-edge (as easy corollaries of some stronger structural facts having applications to coloring of plane graphs, see [10]).

Note that $\delta(K_{2,t}) = 2$ and $w_2(K_{2,t}) = t + 2$, so w_2 is unbounded if $\delta \leqslant 2$. Anyway, its finiteness may be enforced by certain additional constraints based, for example, on degree properties of particular subgraphs. For example, an induced cycle $v_1 \dots v_{2k}$ in a graph is *2-alternating* (Borodin [3]) if $d(v_1) = d(v_3) = \dots = d(v_{2k-1}) = 2$. This notion, along with its more sophisticated analogues (t-alternating subgraph, 3-alternator (Borodin, Kostochka, and Woodall [8]), cycle consisting of 3-paths (Borodin–Ivanova [9]), etc.), turns out to be useful for the study of graph coloring, since it sometimes provides crucial reducible configurations in coloring and partition problems (more often, on sparse plane graphs, see Borodin [10]). Its first application was to show that the total chromatic number of planar graphs with maximum degree Δ at least 14 equals $\Delta + 1$ (Borodin [3]).

In particular, forbidding 2-alternating 4-cycles implies $w_2 \leqslant 17$ (Borodin [2]), while forbidding all 2-alternating cycles implies $w_2 \leqslant 15$ (Borodin [3]), where both bounds are tight.

Nowadays, the maximum weight of edges is known for all most interesting classes of plane graphs with given girth (further examples and references can be found in Borodin [4, 5, 6, 10]).

We now switch to the maximum weight w_3 of 3-paths. In 1993, Ando, Iwasaki, and Kaneko [1] proved that every 3-polytope satisfies $w_3 \leqslant 21$, which is sharp due to the Jendrol' construction [15]. Jendrol' [14] proves that each 3-polytope has a 3-path uvw such that $\max\{d(u), d(v), d(w)\} \leqslant 15$ (the bound is precise). Jendrol' [15] further shows that such a path must belong to one of ten types, in which $d(u) + d(v) + d(w)$ varies from 23 to 16.

Note that the graphs of 3-polytopes are precisely the 3-connected planar graphs due to Steinitz's famous theorem [19]. The requirement of 3-connectedness is essential for the finiteness of w_3, as shown by the construction $K^*_{2,2t}$ obtained from the double $2t$-pyramid by deleting a t-matching from the $2t$-cycle formed by 4-vertices (Borodin [7]).

Moreover, Borodin [7] showed that only the presence in a NPM of $K^*_{2,4}$ is responsible for the unboundedness of w_3. The following refinement of the bound $w_3 \leqslant 21$ by Ando, Iwasaki, and Kaneko [1] holds:

Theorem 1 (Borodin [7]). *Every normal plane map without $K^*_{2,4}$ has*
(*i*) *either $w_3 \leqslant 18$ or a vertex of degree $\leqslant 15$ adjacent to two 3-vertices, and*
(*ii*) *either $w_3 \leqslant 17$ or $w_2 \leqslant 7$.*

As mentioned above, the bounds $w_3 \leqslant 21$ and $w_3 \leqslant 17$ are tight. For a long time, it was not known whether the bound $w_3 \leqslant 18$ in Theorem 1 is sharp or not; its sharpness was recently confirmed in Borodin et al. [11]. In particular, Ando, Iwasaki, and Kaneko's [1] precise bound $w_3 \leqslant 21$ is valid for all NPMs with $w_2 > 6$ (Borodin [7]). Also, Theorem 1 immediately implies that Franklin's precise bound $w_3 \leqslant 17$ is valid for all normal plane maps with $\delta \geqslant 4$.

Recently, Borodin et al. [11] precisely described 3-paths in all normal plane maps without $K^*_{2,4}$ (in particular, in planar graphs with $\delta \geqslant 3$ and in 3-polytopes) by showing that they belong to eight specific types having weight from 17 to 21, where all parameters are best possible.

Note that the star graph $K_{1,n}$ satisfies $\delta = 1$ and $w_3 = n+2$. The behavior of 3-paths with low degree-sum in sparse planar graphs with $\delta = 2$ was recently studied by Jendrol' and Maceková [16]. As observed in [16], if we join vertices a and b by independent paths ax_iy_ib with $1 \leqslant i \leqslant n$, then $w_3 = n + 4$.

Theorem 2 ([16]). *Every planar graph G with $\delta = 2$ and girth $g(G) \geqslant g \geqslant 5$ has a 3-path of one of the following types:*
(*i*) $(2, \infty, 2)$, $(2, 2, 6)$, $(2, 3, 5)$, $(2, 4, 4)$ *or* $(3, 3, 3)$ *if* $g = 5$,
(*ii*) $(2, 2, \infty)$, $(2, 3, 5)$, $(2, 4, 3)$ *or* $(2, 5, 2)$ *if* $g = 6$,
(*iii*) $(2, 2, 6)$, $(2, 3, 3)$ *or* $(2, 4, 2)$ *if* $g = 7$,
(*iv*) $(2, 2, 5)$ *or* $(2, 3, 3)$ *if* $8 \leqslant g \leqslant 9$,
(*v*) $(2, 2, 3)$ *or* $(2, 3, 2)$ *if* $10 \leqslant g \leqslant 15$, *and*
(*vi*) $(2, 2, 2)$ *if* $g \geqslant 16$.

In particular, Theorem 2 yields the following bounds for w_3.

Corollary 3. *Every planar graph G with $\delta = 2$ and girth $g(G) \geqslant g \geqslant 5$ has:*
(*i*) *either a $(2, \infty, 2)$-path or $w_3 \leqslant 10$ if $g = 5$,*
(*ii*) *either a $(2, 2, \infty)$-path or $w_3 \leqslant 10$ if $g = 6$,*
(*iii*) *$w_3 \leqslant 10$ if $g = 7$,*
(*iv*) *$w_3 \leqslant 9$ if $8 \leqslant g \leqslant 9$,*
(*v*) *$w_3 \leqslant 7$ if $10 \leqslant g \leqslant 15$, and*
(*vi*) *$w_3 = 6$ if $g \geqslant 16$,*
where the bounds for $g \geqslant 8$ are sharp.

Also, they conjectured that the bound in Corollary 3(i) can be lowered to 9.

Conjecture 4 ([16])**.** Every planar graph with $\delta = 2$ and girth $g = 5$ has either a $(2, \infty, 2)$-path or $w_3 \leqslant 9$.

The purpose of this paper is to establish precise upper bounds on w_3 whenever $5 \leqslant g \leqslant 7$ under the assumptions of Corollary 3, and also in a broader class of planar graphs with $g = 6$. In particular, we disprove Conjecture 4.

Our new results are in Theorems 5–7 below.

Theorem 5. *There is a plane graph with* $\delta = 2$*,* $g = 5$*, and* $w_3 = 10$*, having neither a* $(2, \infty, 2)$*-path nor* $(2, 2, \infty)$*-path.*

In particular, Theorem 5 shows the tightness of the upper bound in Corollary 3(i).

Theorem 6. *Every plane graph with* $\delta = 2$ *and* $g = 6$ *has either a* $(2, 2, \infty, 2)$*-path or* $w_3 \leqslant 9$*, which bound is tight.*

We see that Theorem 6 extends Corollary 3(ii) and improves the upper bound in it.

Theorem 7. *Every plane graph with* $\delta = 2$ *and* $g \geqslant 7$ *has* $w_3 \leqslant 9$*, which bound is tight whenever* $7 \leqslant g \leqslant 9$*.*

So, Theorem 7 improves the upper bound in Corollary 3(iii).

2 Proof of Theorem 5

Proof of Theorem 5. In Fig. 1, we see a half of a plane graph with the desired properties: $\delta = 2$, $g = 5$, $w_3 = 10$, and no $(2, \infty, 2)$-path.

More specifically, the bounding cycle of the graph to be obtained may be encoded as $5, 3, 5, 3, \ldots$ according to the degrees of its vertices. Moreover, its internal half may be encoded as $5_2, 3_1, 5_1, 3_0 \ldots$, where the subscripts show the number of ingoing edges. For the exterior half, we have a similar encoding $5_1, 3_0, 5_2, 3_1 \ldots$, so the two halves can be glued in this order. ☐

3 Proof of Theorems 6 and 7

Proof of Theorems 6 and 7. Forbidding $(2, 2, \infty, 2)$-paths in Theorem 6 is justified by the already mentioned graph with $w_3 = \infty$ and $g = 6$ in which vertices a and b are joined by independent paths ax_iy_ib with $1 \leqslant i \leqslant n$. We note that forbidding $(2, 2, \infty, 2)$-paths still allows both $(2, 2, \infty)$-paths and $(2, \infty, 2)$-paths. The sharpness of the bound on w_3 follows by putting a 2-vertex on every edge of the icosahedron, which results in $w_3 = 2 + 5 + 2$ under the absence of $(2, 2, \infty, 2)$-paths.

To confirm the tightness of Theorem 7, we put two 2-vertices on every edge of the icosahedron. If desired, we can then fix one of 9-faces and contract any two 2-vertices in its boundary to obtain $g = 7$.

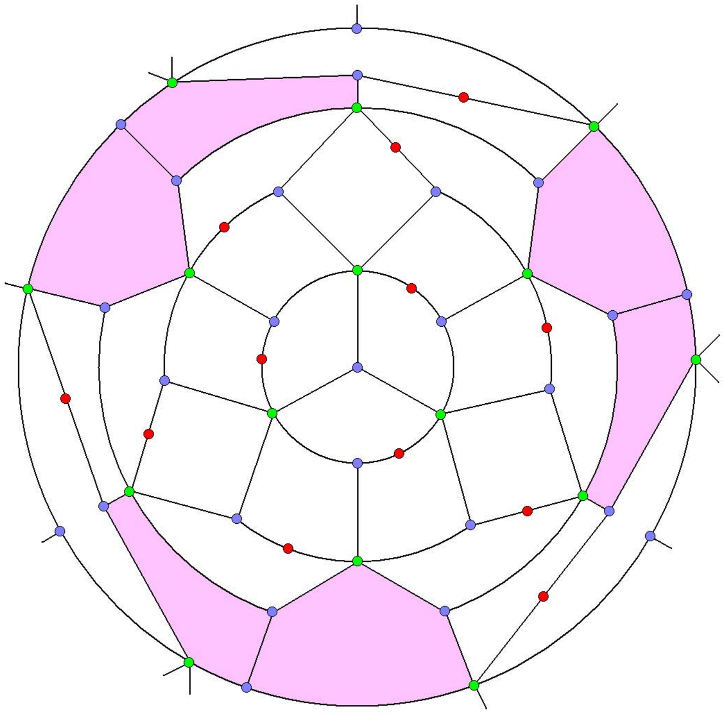

Figure 1: A counterexample to Conjecture 4.

3.1 Discharging and its consequences

Let M be a counterexample to the upper bounds on w_3 in Theorems 6 or 7. Without loss of generality, we can assume that M is connected. Let V, E, and F be the sets of vertices, edges and faces of M, respectively. Euler's formula $|V| - |E| + |F| = 2$ for M may be rewritten as

$$\sum_{v \in V}(d(v) - 6) + \sum_{f \in F}(2d(f) - 6) = -12. \tag{1}$$

Every vertex v contributes the *charge* $\mu(v) = d(v) - 6$ to (1), so only the charges of 5^--vertices are negative. Every face f contributes the non-negative *charge* $\mu(f) = 2d(f) - 6$ to (1). Using the properties of M as a counterexample, we define a local redistribution of μ's, preserving their sum, such that the *new charge* $\mu'(x)$ is non-negative for all $x \in V \cup F$. This will contradict the fact that the sum of the new charges is, by (1), equal to -12.

Throughout the paper, we denote the vertices adjacent to a vertex or incident with face x in a cyclic order by $v_1, \dots, v_{d(x)}$. Let $\partial(f)$ be the boundary of a face f, $n_i(f)$ be the number of i-vertices in $\partial(f)$, and $\rho(f) = \mu(f) - 2n_2(f) - n_3(f) - \frac{1}{2}n_4(f)$.

Now we apply the following rule of discharging.

R. *Every face f gives to each incident vertex v:*
(a) 2 *if* $d(v) = 2$,

(*b*) 1 *if* $d(v) = 3$,
(*c*) $\frac{1}{2}$ *if* $d(v) = 4$, *and*
(*d*) $\max\{\frac{\rho(f)}{n_5(f)}, \frac{1}{2}\}$ *if* $d(v) = 5$.

Thus, the 6^+-vertices do not receive any charge from faces.

If the degrees of vertices incident with a face f constitute a multi-set $\{d_1, \ldots, d_{d(f)}\}$ with $d_1 \leqslant \ldots d_{d(f)}$, then f is a $(d_1, \ldots, d_{d(f)})$*-face*. For example, a $(2, 2, 3, 3, 4, \ldots)$-face is incident with precisely two 2-vertices and two 3-vertices.

Lemma 8. *Every* 6^+*-face* f *(not necessarily in* M*) such that each* 3*-path in* $\partial(f)$ *has weight at least* 10 *gives the following charge to each incident* 5*-vertex according to the rule* **R***:*
(*i*) 0 *if* f *is a* $(2, 2, 3, 3, \ldots)$*-face with* $d(f) = 6$*,*
(*ii*) *at least* $\frac{1}{4}$ *if* f *is a* $(2, 2, 3, 4^+, \ldots)$*-face with* $d(f) = 6$*, and*
(*iii*) $\frac{1}{2}$ *otherwise.*

Proof. CASE 1. $d(f) = 6$. Due to the absence of $(2, 2, \infty, 2)$-paths in M, we see that f is incident with at most three 2-vertices. Moreover, the only possibility for three 2-vertices is $d(v_1) = d(v_3) = d(v_5) = 2$, in which case the other three incident vertices have degrees at least 6.

Note that f is incident with at most four 3^--vertices, for otherwise we would have three consecutive 3^--vertices in $\partial(f)$, which is a contradiction.

First suppose f is incident with precisely two 2-vertices. If there are also (precisely) two incident 3-vertices (see Fig. 2), then $\mu(f) = 6$ is shared in full among the four vertices of smallest degree by **R**, and 0 is given to each incident 5-vertex if any, which proves Lemma 8(i).

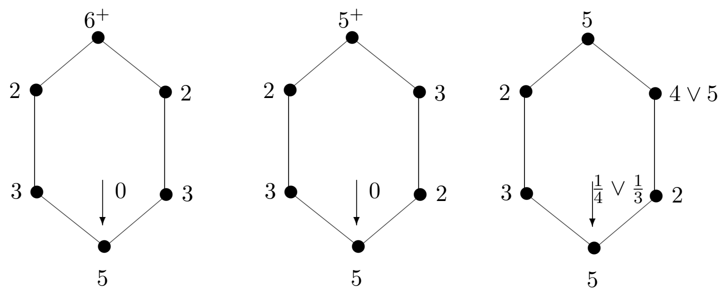

Figure 2: The only 6^+-faces giving less than $\frac{1}{2}$ to their 5-vertices.

Now suppose f is incident with two 2-vertices and one 3-vertex. Thus five units of charge are given to these three vertices. If f is incident with at least one 6^+-vertex, then the two remaining 4^+-vertices receive $\frac{1}{2}$ each, as desired. So suppose f is also incident with three vertices of degree from 4 to 5. The only possible such face contains a $(5, 2, 3, 5)$-path, is shown in Fig. 2, and therefore gives $\frac{1}{4}$ or $\frac{1}{3}$ to each incident 5-vertex, which proves Lemma 8(ii).

To complete Case 1, we observe that any other 6-face is either a $(2, 2, 4^+ \dots)$-face or $(2^+, 3^+, 3^+ \dots)$-face. In both cases, f gives at most $\frac{9}{2}$ to its three vertices of smallest degrees, which means that each incident 5-vertex receives $\frac{1}{2}$ from f, as stated in Lemma 8(iii).

Remark 9. From now on, we run induction on $d(f)$, where $d(f) \geqslant 7$. In particular, we are in the situation of Lemma 8(iii). Furthermore, our proof below does not use the absence of $(2, 2, \infty, 2)$-paths in $\partial(f)$.

CASE 2. $d(f) = 7$ (Induction Base). Recall that $\mu(f) = 8$. Note that f is incident with at most four 3^--vertices, for otherwise we would have three consecutive 3^--vertices in $\partial(f)$. On the other hand, if there are at most three 3^--vertices in $\partial(f)$, then f gives them at most 6 units of charge and thus can afford giving $\frac{1}{2}$ to each of the four remaining vertices, as desired.

Therefore, we are done unless f is incident with precisely four 3^--vertices. Furthermore, if there are at least two 3-vertices among them, then f has at least 2 to give to the three 4^+-vertices, which is enough to satisfy Lemma 8(iii). Thus we have two subcases to consider.

Subcase 2.1. f is incident with four 2-vertices and three 4^+-vertices. Here, the four 2-vertices are split $2 + 1 + 1$ or $2 + 2 + 0$ in $\partial(f)$ by 4^+-vertices, which means that any 4^+-vertex in $\partial(f)$ is actually a 6^+-vertex since $w_3 \geqslant 10$, and we are done.

Subcase 2.2. f is incident with three 2-vertices, a 3-vertex, and three 4^+-vertices. Now f has 1 to share among its three 4^+-vertices, so we are done unless f is incident with precisely three vertices of degree between 4 and 5.

Clearly, this cannot happen if two 2-vertices in $\partial(f)$ are adjacent due to $w_3 \geqslant 10$. Note that then there are two 2-vertices in $\partial(f)$ at distance precisely two from each other (since there are only four 3^+-vertices separating the three 2-vertices). However, a vertex with two neighbors of degree 2 cannot be a 5^--vertex, a contradiction.

CASE 3. $d(f) > 7$ (Induction Step). As said in Remark 9, f is assumed to be any 7^+-face with the property that each 3-path P_3 in $\partial(f)$ satisfies $w_3(P_3) \geqslant 10$, and $(2, 2, \infty, 2)$-paths in $\partial(f)$ are allowed.

Suppose v_2 is a vertex of the smallest degree in $\partial(f) = v_1 \dots v_{d(f)}$. Let a face f' be defined by dropping v_2 from $\partial(f)$; namely, we put $\partial(f') = v_1 v_3 \dots v_{d(f)}$.

Clearly, $\partial(f')$ satisfies $w_3(P_3) \geqslant 10$ for each 3-path P_3 in $\partial(f')$. Indeed, every P_3 that lies in $\partial(f')$ but not in $\partial(f)$ satisfies either $P_3 = v_1 v_3 v_4$ or $P_3 = v_3 v_1 v_{d(f)}$. However, by the choice of v_2, we have $w_3(P_3) \geqslant d(v_1) + d(v_2) + d(v_3) \geqslant 10$, as desired.

Note that $d(f') = d(f) - 1$ and $\mu(f') = \mu(f) - 2$. By the inductive assumption, $\mu'(f') \geqslant 0$. Since f and f' give the same charge by the rule $\mathbf{R}$ to every vertex in $\partial(f')$, while f gives at most 2 to v_2 by $\mathbf{R}$, we have $\mu'(f) \geqslant \mu'(f') + 2 - 2 \geqslant 0$. □

3.2 Completing the proof of Theorems 6 and 7

It remains to prove that each 5-vertex v satisfies $\mu'(v) \geqslant 0$. Indeed, then we have $\mu'(x) \geqslant 0$ for every $x \in V \cup F$, which contradicts (1):

$$0 \leqslant \sum_{v \in V} \mu'(v) = \sum_{v \in V} \mu(v) = -12.$$

First suppose $g \geqslant 7$. In view of Remark 9, we actually proved in Lemma 8 that every face gives $\frac{1}{2}$ to each incident 5-vertex v, regardless of the presence of $(2,2,\infty,2)$-paths in our M, which means that $\mu'(v) \geqslant 5 - 6 + 5 \times \frac{1}{2} > 0$.

This completes the proof of Theorem 7.

Finally, suppose $g = 6$. By Lemma 8, v receives at least $\frac{1}{4}$ from an incident face f, unless f is a $(2,2,3,3,\ldots)$-face with $d(f) = 6$ (see Fig. 2). So if v is not incident with such a face, then we have $\mu'(v) \geqslant 5 - 6 + 5 \times \frac{1}{4} > 0$.

Now suppose $f_1 = v_1vv_2xyz$ is a $(2,2,3,3,\ldots)$-face with $\partial(f) = v_1vv_2xyz$ (see Fig. 3). Since $d(v) = 5$ and $w_3 \geqslant 10$, we have two possibilities to consider.

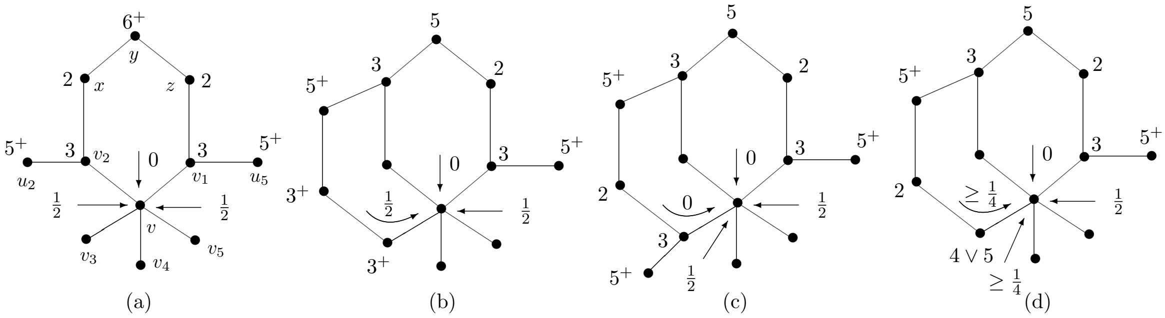

Figure 3: Every 5-vertex receiving 0 from an incident 6-face receives at least 1 in total.

CASE A. $d(v_1) = d(v_2) = 3$ and $d(x) = d(z) = 2$ (see Fig. 3a). Here, each of the faces $f_2 = v_3vv_2u_2\ldots$ and $f_5 = v_5vv_1u_5\ldots$ gives $\frac{1}{2}$ to v due to Lemma 8. Indeed, f_5, say, contains a path vv_1u_5 with $d(u_5) \geqslant 5$ (because of the path zv_1u_5), $d(v_1) = 3$, and $d(v) = 5$, but such a path does not appear in faces giving less than $\frac{1}{2}$ to a 5-vertex, which are depicted in Fig. 2.

CASE B. $d(v_1) = d(x) = 3$ and $d(v_2) = d(z) = 2$ (see Fig. 3(b–d)). As observed in Case A, the face $f_5 = v_5vv_1u_5\ldots$ gives $\frac{1}{2}$ to v. Thus v has to collect at least $\frac{1}{2}$ or $\frac{1}{4} + \frac{1}{4}$ from other faces.

We first look at the face $f_2 = \ldots v_2vv_3u_2$. If $d(u_2) \geqslant 3$ (see Fig. 3b), then f_2 brings another $\frac{1}{2}$ to v by Lemma 8 since $d(v_3) \geqslant 3$ due to $w_3 \geqslant 10$, and we are done.

So suppose $d(u_2) = 2$ (see Fig. 3(c–d)). Now if f_2 gives 0 to v, which is equivalent to say that $d(v_3) = 3$ (see Fig. 3c and Fig.2), then the face $f_3 = v_4vv_3u_3\ldots$ gives $\frac{1}{2}$ to v since $d(u_3) \geqslant 5$ because of the path $u_2v_3u_3$, as desired.

It remains to assume that $4 \leqslant d(v_3) \leqslant 5$ (see Fig. 3d), for otherwise f_2 already gives $\frac{1}{2}$ to v, and we are done. Now each of the faces f_2 and $f_3 = v_3vv_4$ gives at least $\frac{1}{4}$ to v (see Fig. 2 for the faces giving 0 to v), as required.

This completes the proof of Theorem 6. □

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