On the game domination number of graphs with given minimum degree

Csilla Bujtás*

Department of Computer Science and Systems Technology
University of Pannonia
Veszprém, Hungary

bujtas@dcs.uni-pannon.hu

Submitted: Jun 28, 2014; Accepted: Aug 14, 2015; Published: Aug 28, 2015
Mathematics Subject Classifications: 05C57, 91A43, 05C69

Abstract

In the domination game, introduced by Brešar, Klavžar, and Rall in 2010, Dominator and Staller alternately select a vertex of a graph G. A move is legal if the selected vertex v dominates at least one new vertex – that is, if we have a $u \in N[v]$ for which no vertex from $N[u]$ was chosen up to this point of the game. The game ends when no more legal moves can be made, and its length equals the number of vertices selected. The goal of Dominator is to minimize whilst that of Staller is to maximize the length of the game. The game domination number $\gamma_g(G)$ of G is the length of the domination game in which Dominator starts and both players play optimally. In this paper we establish an upper bound on $\gamma_g(G)$ in terms of the minimum degree δ and the order n of G. Our main result states that for every $\delta \geqslant 4$,

$$\gamma_g(G) \leqslant \frac{15\delta^4 - 28\delta^3 - 129\delta^2 + 354\delta - 216}{45\delta^4 - 195\delta^3 + 174\delta^2 + 174\delta - 216}\, n.$$

Particularly, $\gamma_g(G) < 0.5139\, n$ holds for every graph of minimum degree 4, and $\gamma_g(G) < 0.4803\, n$ if the minimum degree is greater than 4. Additionally, we prove that $\gamma_g(G) < 0.5574\, n$ if $\delta = 3$.

Keywords: domination game; game domination number; 3/5-conjecture; minimum degree.

*Research supported by the European Union and Hungary and co-financed by the European Social Fund through the project TÁMOP-4.2.2.C-11/1/KONV-2012-0004 - National Research Center for Development and Market Introduction of Advanced Information and Communication Technologies.

1 Introduction

In this note, our subject is the domination game introduced by Brešar, Klavžar, and Rall in [4].

1.1 Basic definitions

For a simple undirected graph $G = (V, E)$ and for a vertex $v \in V$, the *open neighborhood* of v is $N_G(v) = \{u : uv \in E\}$, while its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. Then the *degree* $d_G(v)$ of v is just $|N_G(v)|$ and the minimum degree $\min\{d_G(v) : v \in V\}$ is denoted by $\delta(G)$. As usual, we will write $N(v)$, $N[v]$ and $d(v)$ for $N_G(v)$, $N_G[v]$ and $d_G(v)$, respectively, if G is clear from the context.

Each vertex dominates itself and its neighbors, moreover a set $S \subseteq V$ dominates exactly those vertices which are contained in $N[S] = \bigcup_{v \in S} N[v]$. A vertex set $D \subseteq V$ is called a *dominating set* of G if $N[D] = V$. The smallest cardinality of a dominating set is the *domination number* $\gamma(G)$ of G.

The *domination game*, introduced by Brešar, Klavžar, and Rall [4], is played on a simple undirected graph $G = (V, E)$ by two players, named Dominator and Staller, respectively. They take turns choosing a vertex from V such that a vertex v can be chosen only if it dominates at least one new vertex – that is, if we have a $u \in N[v]$ for which no vertex from $N[u]$ was selected up to this turn of the game. The game is over when no more legal moves can be made; equivalently, when the set D of vertices chosen by the two players becomes a dominating set of G. The aim of Dominator is to finish the game as soon as possible, while that of Staller is to delay the end of the game. The *game domination number* $\gamma_g(G)$ is the number of turns in the game when the first turn is Dominator's move and both players play optimally. Analogously, the *Staller-start game domination number* $\gamma_g'(G)$ is the length of the game when Staller begins and the players play optimally.

1.2 Results

Although the subject is quite new, lots of interesting results have been obtained on the domination game (see [2, 3, 4, 5, 6, 7, 9, 13, 14]). Note that also the total version of the domination game was introduced [11] and studied [12] recently.

Concerning our present work, the bounds proved for the game domination number $\gamma_g(G)$ are the most important preliminaries. The following fact was verified in [4] and [13] as well.

$$\gamma(G) \leqslant \gamma_g(G) \leqslant 2\gamma(G) - 1 \tag{1}$$

Upper bounds in terms of the order were inspired by the following "3/5-conjecture" raised by Kinnersley, West, and Zamani [13].

Conjecture 1. If G is an isolate-free graph of order n, then $\gamma_g(G) \leqslant 3n/5$ holds.

Conjecture 1 has been proved for the following graph classes:

- for trees of order $n \leqslant 20$ (Brešar, Klavžar, Košmrlj and Rall [3]);
- for caterpillars – that is, for trees in which the non-leaf vertices induce a path (Kinnersley, West, and Zamani [13]);
- for trees in which no two leaves are at distance four apart (Bujtás [6, 7]).

Moreover, in a manuscript in preparation, Henning and Kinnersley prove Conjecture 1 for graphs of minimum degree at least 2 [10].

On the other hand, upper bounds weaker than $3n/5$ were obtained for some wider graph classes. For trees, the inequality $\gamma_g(G) \leqslant 7n/11$ was established by Kinnersley, West, and Zamani in [13] and it was recently improved to $\gamma_g(G) \leqslant 5n/8$ by the present author in [7]. For the most general case, Kinnersley, West, and Zamani proved [13] that the game domination number of any isolate-free graph G of order n satisfies $\gamma_g(G) \leqslant \lceil 7n/10 \rceil$. In Section 2 we improve this upper bound by establishing the following claim.

Proposition 2. *For any isolate-free graph G of order n,*

$$\gamma_g(G) \leqslant \frac{2n}{3} \qquad \textit{and} \qquad \gamma'_g(G) \leqslant \frac{2n}{3}.$$

In fact, in a manuscript under preparation [8] we will prove the stronger inequality $\gamma_g(G) \leqslant 0.64n$, but the proof of Proposition 2 may be of interest because of its simplicity and gives illustration for the proof technique applied in the later sections.

One of our main results gives an upper bound smaller than $0.5574n$ on the game domination number of graphs with minimum degree 3.

Theorem 3. *For any graph G of order n and with minimum degree 3,*

$$\gamma_g(G) \leqslant \frac{34n}{61} \qquad \textit{and} \qquad \gamma'_g(G) \leqslant \frac{34n-27}{61}.$$

For graphs all of whose vertices are of degree greater than 3, we prove an upper bound in terms of the order and the minimum degree.

Theorem 4. *If G is a graph on n vertices and its minimum degree is $\delta(G) \geqslant d \geqslant 4$, then*

$$\gamma_g(G) \quad \leqslant \quad \frac{15d^4 - 28d^3 - 129d^2 + 354d - 216}{45d^4 - 195d^3 + 174d^2 + 174d - 216}\, n.$$

As the coefficient in this upper bound equals $37/72 < 0.5139$ for $d = 4$, and equals $2102/4377 < 0.4803$ for $d = 5$, the following immediate consequences are obtained.

Corollary 5. (i) *For any graph G of order n and with minimum degree $\delta(G) = 4$, the inequality $\gamma_g(G) \leqslant 37n/72$ holds.*

(ii) *For any graph G of order n and with minimum degree $\delta(G) \geqslant 5$, the inequality $\gamma_g(G) \leqslant 2102n/4377$ holds.*

Particularly, these statements show that the coefficient 3/5 in Conjecture 1 can be significantly improved if only those graphs with $\delta(G) \geqslant 4$ are considered.

On the other hand, note that Theorem 3 and Theorem 4 establish new results only for $3 \leqslant \delta(G) \leqslant 21$. Although it was not mentioned in the earlier papers, the upper bound in (1) together with the well-known theorem (see e.g., [1])

$$\gamma(G) \leqslant \frac{1+\ln(\delta+1)}{\delta+1}\, n$$

clearly yields

$$\gamma_g(G) < 2 \cdot \frac{1+\ln(\delta+1)}{\delta+1}\, n \tag{2}$$

for each $\delta \geqslant 2$. For integers $3 \leqslant \delta(G) = d \leqslant 21$, it is easy to check that our bound is better than the above one in (2).

Our proof technique is based on a value assignment to the vertices where the value of a vertex depends on its current status in the game. We will consider a greedy strategy of Dominator, where the greediness is meant concerning the decrease in the values. Our main goal is to estimate the average decrease in a turn achieved under this assumption. We introduced this type of approach in the conference paper [6] and in the paper [7]. The frame of this technique and the basic observations are contained here in Section 2. Then, in Section 3 and Section 4 we specify the details and prove our Theorem 3 and Theorem 4 respectively. In the last section we make some additional notes concerning the Staller-start version of the game.

2 Preliminaries

Here we introduce the notion of the residual graph, define the color assignment to the vertices and give a general determination for the phases of the game. Then, we make some simple observations which will be used in the later sections.

Colors Consider any moment of the process of a domination game on the graph $G^* = (V, E)$, and denote by D the set of vertices chosen up to this point of the game. As it was introduced in [6] and [7], we distinguish between the following three types of vertices.

- A vertex $v \in V$ is *white* if $v \notin N[D]$.
- A vertex $v \in V$ is *blue* if $v \in N[D]$ but $N[v] \nsubseteq N[D]$.
- A vertex $v \in V$ is *red* if $N[v] \subseteq N[D]$.

Residual graph Clearly, a red vertex v and all its neighbors are already dominated in the game. Hence the choice of v would not be a legal move in the later turns and further, the status of v remains red. So, red vertices do not influence the continuation of the game and they can be deleted. Similarly, edges connecting two blue vertices can be omitted too. This graph, obtained after the deletion of red vertices and edges between two blue vertices, is called the *residual graph*, as it was introduced in [13]. At any point of the game, the set of vertices chosen up to this point is denoted by D and the residual graph is denoted by G. When it is needed, we use the more precise notations D_i and G_i for the current D and G just before the ith turn.

Phases of the game The phases will be defined for the Dominator-start game that is, for each odd integer j the jth turn belongs to Dominator. The Staller-start version will be treated later by introducing a Phase 0 for the starting turn.

In our proofs, nonnegative values $p(v)$ are assigned to the vertices, and the value $p(G)$ of the residual graph is just the sum of the values of the vertices. Also, we assume that Dominator always chooses greedily. More precisely, for each odd j, in the jth turn he plays a vertex which results the possible maximum $p(G_j) - p(G_{j+1})$. This difference is called the decrease in the value of $p(G)$ and also referred to as the gain of the player.

Definition 6. Let $(C1), \dots (C\ell)$ be conditions all of which relate to the jth turn of the game where j is odd. Then, for each $i = 1, \dots \ell$, Phase i of the game is defined as follows.

- (i) Phase 1 begins with the first turn of the game.
- (ii) If Phase i begins with the b_ith turn, it is continued as long as (Ci) is satisfied in each turn of Dominator. That is, Phase i ends right after the e_ith turn where e_i is the smallest even integer with $b_i < e_i$ for which (Ci) is not satisfied in the $(e_i + 1)$st turn.
- (iii) If Phase i ends after the e_ith turn but the game is not over yet, then the $(e_i + 1)$st turn is the beginning of Phase i', where i' is the smallest integer with $i < i'$ such that (Ci') is fulfilled in the $(e_i + 1)$st turn.
- (iv) If Phase i is followed by Phase i' and $i+2 \leqslant i'$ holds, we say that Phases $i+1, \dots i'-1$ are skipped; moreover, their starting and end points are interpreted to be the same as the end of Phase i.

Further notations The colors white, blue and red will be often abbreviated to W, B and R, respectively. For example, a B-neighbor is a blue neighbor, and the notation v: W$\to$B/R means that vertex v changed from white to either blue or red in the turn considered. Moreover, $d^W(v)$ and $d^B(v)$ stand for the number of W-neighbors and B-neighbors of v, respectively.

We cite the following observations (in a slightly modified form) from [7]:

Lemma 7. *The following statements are true for every residual graph G in a domination game started on G^*.*

- (i) *If v is a white vertex in G, then v has the same neighborhood in G as it had in G^*. Thus, $d_G(v) = d_{G^*}(v)$ holds for every W-vertex of G and moreover, $d_G^W(v)+d_G^B(v) = d_{G^*}(v)$.*

(*ii*) *If v is a blue vertex in G, then v has only white neighbors and definitely has at least one. That is, $d_G^W(v) = d_G(v) \geqslant 1$ and $d_G^B(v) = 0$ if v is a B-vertex in G.*

At the end of this section, we provide a simple example for applying the tools introduced above. We prove Proposition 2, which states that for any isolate-free graph G of order n,

$$\gamma_g(G) \leqslant \frac{2n}{3} \qquad \text{and} \qquad \gamma_g'(G) \leqslant \frac{2n}{3}$$

hold.

Proof of Proposition 2. First, we consider the Dominator-start game on $G^* = (V, E)$, which is a simple graph without isolated vertices. In every residual graph G, let the value $p(v)$ of a vertex v be equal to 2, 1 and 0, when v is white, blue and red, respectively. Hence, we start with $p(G^*) = 2n$ and assume that Dominator always selects a vertex which results in a maximum decrease in $p(G)$. The game is divided into two phases, which are determined due to Definition 6 with the following conditions:

($C1$) Dominator gets at least 4 points.

($C2$) Dominator gets at least 1 point.

Phase 1. If Staller selects a W-vertex, then it becomes red and causes at least 2-point decrease in the value of the residual graph. In the other case, Staller selects a B-vertex v which has a W-neighbor u. Then, the changes v: B$\rightarrow$R and u: W$\rightarrow$B/R together result in a decrease of at least $1+1 = 2$. Hence, in each of his turns Staller gets at least 2 points. By condition ($C1$), Dominator always gets at least 4 points. As Dominator begins the phase, the average decrease in $p(G)$ must be at least 3 in a turn.

Phase 2. When Phase 2 starts, Dominator cannot seize 4 or more points by playing any vertex of G_j. This implies the following properties of the residual graph:

- For every W-vertex v, $d^W(v) \leqslant 1$.
  Indeed, if v had two W-neighbors u_1 and u_2, then Dominator could choose v and the changes v: W$\rightarrow$R and u_1, u_2: W$\rightarrow$B/R would give a gain of at least $2 + 2 \cdot 1 = 4$ points, which is a contradiction.

- For every W-vertex v, $d^W(v) = 0$.
  We have seen that $d^W(v) \leqslant 1$. Now, assuming two W-neighbors v and u, the choice of v would result in the changes v, u: W$\rightarrow$R, which give a gain of at least 4 points to the player. This is a contradiction again.

- For every B-vertex v, $d(v) = 1$.
  By Lemma 7(*ii*), $d(v) \geqslant 1$. Now, assume that v has two different W-neighbors u_1 and u_2. As we have shown, $d^W(u_1) = d^W(u_2) = 0$ must hold and consequently, if Dominator plays v, then both u_1 and u_2 turn to red. This gives a gain of at least $1 + 2 \cdot 2 = 5$ points, which cannot be the case at the endpoint of Phase 1.

- Each component of G_j is a P_2 and contains exactly one white and one blue vertex. By the claims above, each component is a star with a white center and blue leaves. If it contained at least two leaves then Dominator could play the center and get at least 4 points.

Therefore, at the beginning of Phase 2, the residual graph consists of components of order 2. As follows, in each turn an entire component becomes red and $p(G)$ decreases by exactly 3 points.

In the game, the value of the residual graph decreased from $2n$ to zero, and the average decrease in a turn was proved to be at least 3. Consequently, the number of turns required is no greater than $2n/3$, which proves $\gamma_g(G^*) \leqslant 2n/3$.

If Staller starts the game, his first move definitely decreases $p(G)$ by at least 3 points as there are no isolated vertices. Then, the game is continued as in the Dominator-start game, and the average decrease remains at least 3 points. Thus, $\gamma_g'(G^*) \leqslant 2n/3$ holds.

3 Graphs of minimum degree 3

In this section we prove the upper bound stated on the game domination number of graphs with minimum degree 3. Also, this proof serves as an introduction to the details of the idea applied in the next section to prove our main theorem.

Proof of Theorem 3. We consider a graph $G = (V, E)$ of minimum degree 3 and define the value assignments of types A1.1, A1.2 and A1.3 as they are given in Table 1.

Table 1: Value assignments used in the proof of Theorem 3

Abbrev.	Type of the vertex	Value in A1.1	Value in A1.2	Value in A1.3
W	white vertex	34	34	34
B_3	blue vertex of degree $\geqslant 3$	16	16	—
B_2	blue vertex of degree 2	16	13	13
B_1	blue vertex of degree 1	16	10	9
R	red vertex	0	0	0

Hence, the game starts with $p(G^*) = 34n$. First, assume that Dominator begins the game and determine Phases 1-4 due to Definition 6 with the following specified conditions:

($C1$) Dominator gets at least 88 points due to the assignment A1.1.

($C2$) Dominator gets at least 91 points due to the assignment A1.2.

($C3$) Dominator gets at least 84 points due to the assignment A1.3.

($C4$) Dominator gets at least 1 point due to the assignment A1.3.

Phase 1. Here, we apply the value assignment A1.1. In each of his turns, Staller either selects a white vertex and gets at least 34 points; or he plays a blue vertex v which has a white neighbor u and then the color changes v: B→R and u: W→B/R give at least 16+18=34 points. By condition $(C1)$, each move of Dominator yields a gain of at least 88 points. As Dominator begins the game, we have the following estimation on the average decrease of $p(G)$ in a turn.

Lemma 8. *In Phase 1, the average decrease of $p(G)$ in a turn is at least 61 points.*

At the end of Phase 1 we have some structural properties which remain valid in the continuation of the game.

Lemma 9. *After the end of Phase 1, throughout the game, each white vertex has at most 2 white neighbors, and each blue vertex has at most 3 white neighbors.*

Proof. By definition, at the end of the first phase Dominator has no possibility to seize 88 or more points by playing a vertex of the residual graph G. Now, assume that there exists a W-vertex v with three W-neighbors u_1, u_2 and u_3 in G. Then, Dominator could choose v and the color changes v: W→R and u_1, u_2, u_3: W→B/R would decrease $p(G)$ by at least $34 + 3 \cdot 18 = 88$ points, which is a contradiction. Similarly, if there exists a B-vertex v with at least four W-neighbors, then Dominator could get at least $16+4 \cdot 18 = 88$ points by playing v, which is a contradiction again.

In the continuation of the game, new white vertices cannot arise, moreover a new blue vertex may arise only by the color change W→B. This implies that the stated properties remain valid throughout all the later phases. □

Phase 2. At the beginning of this phase we change to assignment A1.2. Clearly, the values of the vertices do not increase. As no blue vertex has a degree greater then 3, we observe that each B-vertex v has value $p(v) = 7 + 3d(v)$. Further, assignment A1.2 ensures that when a vertex v is played, the value of every blue vertex from $N[N[v]]$ is decreased.

Lemma 10. *The following statements are true in Phase 2.*

(i) If a W-vertex v with W-degree $d^W(v)$ is played, then $p(G)$ decreases by at least $43 + 24d^W(v)$ points.

(ii) If a B-vertex v with degree $d(v)$ is played, then $p(G)$ decreases by at least $7+24d(v)$ points.

(iii) In each turn $p(G)$ decreases by at least 31 points

Proof. When the degree of a B-vertex is decreased by x, its value decreases by at least $3x$, no matter whether the change is of type $\mathrm{B}_i \to \mathrm{B}_{i-x}$ or $\mathrm{B}_x \to$R. Thus, if a vertex v is played and $N^W[v]$ denotes the set of white vertices in $N[v]$, the sum of the values of B-vertices contained in $N[N[v]] \setminus \{v\}$ is decreased by at least

$$3 \sum_{u \in N[v]} d^B(u) \geqslant 3 \sum_{u \in N^W[v]} (3 - d^W(u))$$

if v is white, and by at least

$$3\sum_{u\in N[v]}(d^B(u)-1)\geqslant 3\sum_{u\in N^W[v]}(2-d^W(u))$$

if v is blue.

First, assume that the played vertex v is white, $d^W(v)=k$ and the W-neighbors of v are $u_1,\dots u_k$. For each $1\leqslant i\leqslant k$, the W-vertex u_i becomes either a B-vertex of degree at most $d^W(u_i)-1$ or an R-vertex. As $1\leqslant d^W(u_i)\leqslant 2$, $p(u_i)$ decreases by at least $34-(7+3d^W(u_i)-3)=30-3d^W(u_i)$ in either case. Then, the decrease in $p(G)$ is not smaller than

$$34+\sum_{i=1}^{k}(30-3d^W(u_i))+3(3-k)+3\sum_{i=1}^{k}(3-d^W(u_i))=43+36k-6\sum_{i=1}^{k}d^W(u_i)\geqslant 43+24k,$$

where $0\leqslant k\leqslant 2$ must hold. This establishes statement (i).

In the other case, v is blue with $d(v)=k$ and its W-neighbors are $u_1,\dots u_k$. As v has only white neighbors and definitely has at least one and no more than 3, $1\leqslant k\leqslant 3$ holds; moreover, $0\leqslant d^W(u_i)\leqslant 2$ is true for all $1\leqslant i\leqslant k$. When v is played, u_i becomes red if $d^W(u_i)=0$, otherwise it will be a blue vertex of degree at most $d^W(u_i)$. Therefore, the decrease in $p(u_i)$ is at least $34-(7+3d^W(u_i))=27-3d^W(u_i)$ and that in $p(v)$ is exactly $7+3k$. Then, the sum of the decreases cannot be smaller than

$$7+3k+\sum_{i=1}^{k}(27-3d^W(u_i))+3\sum_{i=1}^{k}(2-d^W(u_i))=7+36k-6\sum_{i=1}^{k}d^W(u_i)\geqslant 7+24k$$

as stated in (ii).

To prove (iii), it suffices to consider the minimum of $43+24k$ in case (i), which is 43; and that of $7+24k$ in case (ii), which is 31 because of the condition $k\geqslant 1$. □

By Lemma 10(iii), Staller gets at least 31 points, and by Condition $(C2)$, Dominator gets at least 91 points in each of their turns. Hence, we have the following estimation.

Lemma 11. *In Phase 2, the average decrease of* $p(G)$ *in a turn is at least* 61 *points.*

As shown by the next lemma, the W-degrees are more strictly bounded from the end of Phase 2 than earlier.

Lemma 12. *After the end of Phase 2, throughout the game, each white vertex has at most 1 white neighbors, and each blue vertex has at most 2 white neighbors.*

Proof. By condition $(C2)$, at the end of Phase 2 Dominator can seize only less than 91 points by choosing any vertex of G. By Lemma 10(i), the selection of a W-vertex v with $d^W(v)=2$ causes a decrease of at least $43+24\cdot 2=91$ points in $p(G)$. Hence, each W-vertex has either zero or exactly one W-neighbor.

Now, assume that v is a B-vertex with three W-neighbors, say u_1, u_2 and u_3. We have already seen that the inequalities $0 \leqslant d^W(u_i) \leqslant 1$ hold for $i = 1, 2, 3$. Then, as it was shown in the proof of Lemma 10(ii), the choice of v would decrease $p(G)$ by at least

$$7 + 36 \cdot 3 - 6 \sum_{i=1}^{3} d^W(u_i) \geqslant 97,$$

which is a contradiction. □

Phase 3. The phase starts with changing the value assignment A1.2 to A1.3. By Lemma 12, there are no B-vertices of degree 3 or higher, moreover we observe that the change to A1.3 cannot cause increase in the value of G. Also, one can easily check that the value of a B-vertex decreases by at least $4x$ points, if it loses x W-neighbors in a turn.

Lemma 13. *The following statements are true in Phase 3.*

(i) *If a W-vertex v is played, then $p(G)$ decreases by at least 84 points if $d^W(v) = 1$, and $p(G)$ decreases by at least 46 points if $d^W(v) = 0$.*

(ii) *If a B-vertex v is played, then $p(G)$ decreases by at least 67 points if $d(v) = 2$, and $p(G)$ decreases by at least 38 points if $d(v) = 1$.*

(iii) *In each turn $p(G)$ decreases by at least 38 points*

Proof. (i) Consider a W-vertex v whose only W-neighbor is u. By Lemma 12, all the further neighbors of v and u are blue. This implies $d^B(u) + d^B(v) \geqslant 4$. Hence, when v is played, the color changes v, u: W→R decrease $p(G)$ by 68 points, while the sum of the values of B-vertices contained in $N[\{v, u\}]$ decreases by at least $4 \cdot 4 = 16$. Hence, the gain of the player is at least 84 points. In the other case, when v has no W-neighbors, it has at least three B-neighbors. Then, the change v: W→R gives at least 34 points and additionally, the decrease in the degrees of the B-neighbors means at least 12 points. This proves that $p(G)$ decreases by at least 46 points.

(ii) If the played vertex v is blue and has exactly one white neighbor u, then the changes v: B_1 →R and u: W→B_1/R cause a decrease of at least $9 + 25 = 34$ points in $p(G)$. Additionally, u has at least one B-neighbor different from v, whose value is decreased by at least 4 points. Consequently, the total decrease is at least 38 points. Similarly, if v is blue and has two W-neighbors u_1 and u_2, then the total decrease in $p(G)$ is at least $13 + 2 \cdot 25 + 2 \cdot 4 = 71$ points.

(iii) As the four cases above cover all possible moves which can be made in Phase 3, $p(G)$ is decreased by at least 38 points in each turn. □

As consequences of Condition ($C3$) and Lemma 13(iii), Dominator gets at least 84 points and Staller gets at least 38 points in each of his turns. Hence, we have the desired average.

Lemma 14. *In Phase 3, the average decrease of $p(G)$ in a turn is at least 61 points.*

When Dominator cannot get at least 84 points in a turn, the structure of the residual graph must be very simple.

Lemma 15. *At the end of Phase 3, each component of the residual graph is a star of order $k \geqslant 4$ with a white center and $k-1$ blue leaves.*

Proof. Let G_i be the residual graph obtained at the end of Phase 3. Due to Lemma 13(i), the presence of a W-vertex v with $d^W(v) = 1$ provides an opportunity for Dominator to get at least 84 points. Then, the ith turn would belong to Phase 3, which is a contradiction. Consequently, in G_i each W-vertex has only B-neighbors.

Next, assume that we have a B-vertex v which has two W-neighbors u_1 and u_2 in G_i. As we have seen, in G_i $d^W(u_1) = d^W(u_2) = 0$ must hold and moreover, both u_1 and u_2 have at least two B-neighbors. Therefore, if v is selected by Dominator, the changes v: $\mathrm{B}_2 \to$R and u_1, u_2: W$\to$R with the change in the values of B-neighbors, all together yield at least $13 + 2 \cdot 34 + 4 \cdot 4 = 97$ point decrease in $p(G_i)$, which is a contradiction again. Hence, each B-vertex has at most one W-neighbor.

Since each W-vertex v has the same degree in the residual graph G_i as it had in G^*, it has at least three B-neighbors in G_i. In addition, each B-vertex is a leaf in G_i. This implies that at the end of Phase 3 every component is a star with the structure stated. □

Phase 4. By Lemma 15, Phase 4 begins with star-components containing a white center and at least three blue leaves. Then, in each turn a component becomes completely red, no matter whether a white or a blue vertex is played. Thus, each move decreases the value of G by at least $34 + 3 \cdot 9 = 61$ points.

Lemma 16. *In Phase 4, the average decrease of $p(G)$ in a turn is at least 61 points.*

By Lemmas 8, 11, 14 and 16, if Dominator starts the game and he plays the prescribed greedy strategy, then for the number t^* of turns

$$\gamma_g(G^*) \leqslant t^* \leqslant \frac{34}{61}n$$

holds.

Finally, for the Staller-start version of the game we define Phase 0, which contains only the first turn and the values are counted due to A1.1. Observe that Staller's any choice results in at least $34 + 3 \cdot 18 = 88$ point decrease in $p(G^*)$. Then, Phase 1 might be skipped if $(C1)$ is not true for G_1, but otherwise the game continues as in the Dominator-start version and our lemmas remain valid. Therefore, by the 27-point overplus arising in Phase 0, for $\gamma_g'(G^*)$ we obtain a slightly better bound,

$$\gamma_g'(G^*) \leqslant \frac{34n - 27}{61}.$$

This completes the proof of Theorem 3.

4 Graphs with minimum degree greater than 3

Here we prove Theorem 4 and Corollary 5.

Proof of Theorem 4. First, we consider the Dominator-start game on a graph $G^* = (V, E)$ of order n, whose minimum degree is $\delta(G^*) \geqslant d \geqslant 4$.

The proof and the game starts with the value assignment A2.1 to the vertices as shown in Table 2. Later, we use a more subtle distinction between the types of blue vertices due to assignments A2.2, A2.3 and A2.4 (see Table 2). We will see that the value $p(G)$ of the residual graph cannot increase when we change to an assignment with a higher index.

Table 2: Value assignments used in the proof of Theorem 4

Abbrev.	Type of the vertex	A2.1	A2.2	A2.3	A2.4
W	white vertex	a	a	a	a
B_4	blue vertex of degree $\geqslant 4$	b	b	—	—
B_3	blue vertex of degree 3	b	$b - x_1$	$b - x_1$	—
B_2	blue vertex of degree 2	b	$b - 2x_1$	$b - x_1 - x_2$	$b - x_1 - x_2$
B_1	blue vertex of degree 1	b	$b - 3x_1$	$b - x_1 - 2x_2$	$b - x_1 - x_2 - x_3$
R	red vertex	0	0	0	0

The values of a, b, x_1, x_2, x_3 and s are defined in terms of the parameter d. We aim to prove that s is a lower bound on the average decrease of $p(G)$ in a turn, if Dominator follows the prescribed greedy strategy.

$$\begin{aligned}
a &= 30d^4 - 56d^3 - 258d^2 + 708d - 432 \\
b &= 111d^3 - 561d^2 + 888d - 432 \\
x_1 &= 6d^3 - 19d^2 + 15d \\
x_2 &= 15d^3 - 64d^2 + 65d \\
x_3 &= 30d^3 - 144d^2 + 202d - 72 \\
s &= 90d^4 - 390d^3 + 348d^2 + 348d - 432
\end{aligned}$$

Concerning the values above and the change between assignments, we take the following observations.

Lemma 17. *For every fixed integer $d \geqslant 4$:*

(i) $0 < x_1 < x_2 < x_3 < b - x_1 - x_2 - x_3 < b < a$ *and* $x_3 < a - b$.

(ii) For every $1 \leqslant i < j \leqslant 4$ and every residual graph G, $p(G)$ does not increase if the value assignment A2.i is changed to A2.j (assuming that A2.j is defined for G).

Table 3: Values of the differences for the proof of Lemma 17

		$d=4$	$d=5$	$d=6$
x_1	$6d^3-19d^2+15d$	140	350	702
x_2-x_1	$9d^3-45d^2+50d-72$	56	250	624
x_3-x_2	$15d^3-80d^2+137d-288$	156	488	1110
$b-x_1-x_2-2x_3$	$30d^3-190d^2+404d$	208	732	1776
$a-b$	$30d^4-167d^3+303d^2-180d$	1120	4550	12636
$a-b-x_3$	$30d^4-197d^3+447d^2-382d+72$	768	3462	10200

Proof. The proof of (i) is based on a simple counting and estimation. Table 3 shows the differences and their exact values for $d=4,5,6$. The comparison of coefficients verifies our statements for $d \geqslant 7$.

Once (i) is proved, Table 2 shows that no vertex has greater value by A2.j than by A2.i, whenever $j>i$ holds. $\square$

Note that later we will use further relations between a, b, x_1, x_2, x_3 and s but these are equations, which can be verified by simple counting, so the details will be omitted.

The game is divided into five phases due to Definition 6 with the following five conditions:

$(C1)$ Dominator gets at least $5a-4b$ points due to the assignment A2.1.

$(C2)$ Dominator gets at least $4a-3b+(4d-6)x_1$ points due to the assignment A2.2.

$(C3)$ Dominator gets at least $3a-2b+2x_1+(3d-2)x_2$ points due to the assignment A2.3.

$(C4)$ Dominator gets at least $2a+(2d-2)x_3$ points due to the assignment A2.4.

$(C5)$ Dominator gets at least 1 point due to the assignment A2.4.

Thus, the game starts on $G^* = G_1$ with $p(G_1) = a \cdot n$, and ends with a residual graph whose value equals 0. Recall that Dominator plays a purely greedy strategy. Our goal is to prove that the average decrease in $p(G)$ is at least s points in a turn.

Phase 1. In each turn, the player either selects a W-vertex which turns red and hence $p(G)$ decreases by at least a points; or he selects a B-vertex v which has a W-neighbor u. In the latter case the changes v: B→R and u: W→B/R together yield a decrease of at least $b+(a-b)=a$ points. Therefore, Staller gets at least a points in each of his turns in Phase 1. By condition $(C1)$, Dominator seizes at least $5a-4b$ points and therefore, in any two consecutive turns $p(G)$ decreases by at least $6a-4b=2s$ points. As Dominator starts, the following statement follows.

Lemma 18. *In Phase 1, the average decrease of $p(G)$ in a turn is at least s points.*

Concerning the structure of the residual graph obtained at the end of this phase, we prove the following properties.

Lemma 19. *At the end of Phase 1,*

(i) If v is a W-vertex, then $d^W(v) \leqslant 3$.

(ii) If v is a B-vertex, then $d(v) \leqslant 4$.

Proof. At the end of the phase, we have a residual graph G_i in which Dominator cannot get $5a - 4b$ or more points. Assuming a W-vertex v with W-neighbors u_1, u_2, u_3 and u_4, Dominator could play v and the changes v: W→R and u_1, u_2, u_3, u_4: W→B/R would result in a decrease of at least $a + 4(a - b) = 5a - 4b$ points, which is a contradiction. In the other case, the choice of a B-vertex which has five W-neighbors would yield a gain of at least $b + 5(a - b) = 5a - 4b$ points, which is a contradiction again. □

Phase 2. In this phase we apply the value assignment A2.2. By Lemma 19(*ii*), each B-vertex has degree smaller than or equal to 4. Moreover by the definition of A2.2 and by Lemma 17, in the jth turn the value of a B-vertex u decreases by at least $(d_{G_j}(u) - d_{G_{j+1}}(u))x_1$ points.

Lemma 20. *In Phase 2, the average decrease of $p(G)$ in a turn is at least s points.*

Proof. If a W-vertex v is played, each of its neighbors has a decrease of at least x_1 points in its value, no matter whether this change on the neighbor is B_i →B_{i-1} or B_1 →R or W→B_i/R. Then, playing a W-vertex results in at least $a + d \cdot x_1$ point decrease in $p(G)$.

In the other case, when the played vertex v is blue, the decrease in its value is at least $b - 3x_1$. As v has a W-neighbor u, whose W-degree is at most 3, the change u: W→B_i/R ($i \leqslant 3$) yields further at least $a - (b - x_1)$ points gain; and since u has at least $d - 4$ B-neighbors different from v, the total decrease in $p(G)$ is at least $(b - 3x_1) + a - (b - x_1) + (d - 4)x_1 = a + (d - 6)x_1$. This yields that Staller gets at least $a + (d - 6)x_1$ points whenever a white or a blue vertex is played by him.

Complying with ($C2$), each move of Dominator results in a gain of at least $4a - 3b + (4d - 6)x_1$ and consequently, in any two consecutive turns of Phase 2, $p(G)$ decreases by at least $5a - 3b + (5d - 12)x_1 = 2s$ points. This proves the lemma. □

Lemma 21. *At the end of Phase 2,*

(i) If v is a W-vertex, then $d^W(v) \leqslant 2$.

(ii) If v is a B-vertex, then $d(v) \leqslant 3$.

Proof. To prove (*i*), assume that Dominator selects a W-vertex v with W-neighbors u_1, u_2 and u_3. Remark that each u_ℓ may have at most two W-neighbors different from v. Therefore, the changes v: W→R and u_1, u_2, u_3: W→B_i/R ($i \leqslant 2$) give at least $a + 3(a - b + 2x_1)$ points to Dominator. In addition, each of v, u_1, u_2 and u_3 has at least $d-3$ B-neighbors. Hence the total decrease in $p(G)$ is at least $a+3(a-b+2x_1)+4(d-3)x_1 =$

$4a-3b+(4d-6)x_1$ points. In this case, Dominator's turn would belong to Phase 2. Hence for every W-vertex v, $d^W(v) \leqslant 2$ must hold at the end of Phase 2.

Part (ii) can be shown in a similar way but here we can refer to the property (i) proved above. The selection of a B-vertex v which has four W-neighbors, say u_1, u_2, u_3 and u_4, would cause the color changes v: $\mathrm{B}_4 \to$R and u_1, u_2, u_3, u_4: W$\to$B$_i$/R (where $i \leqslant 2$, due to part (i)). Moreover each u_j has at least $d-3$ B-neighbors different from v. These would give a gain of at least

$$b + 4(a-b+2x_1) + 4(d-3)x_1 = 4a-3b+(4d-4)x_1 > 4a-3b+(4d-6)x_1$$

points to Dominator, which is impossible at the end of Phase 2. This verifies part (ii). $\square$

Phase 3. Here we apply the value assignment A2.3. By Lemma 21(ii), each B-vertex v has degree $d(v) \leqslant 3$ and hence, A2.3 is defined for all vertices of the residual graph. We observe concerning this phase that whenever the degree of a B-vertex v is reduced by y, its value decreases by at least yx_2 points.

Lemma 22. *In Phase 3, the average decrease of $p(G)$ in a turn is at least s points.*

Proof. If Staller plays a W-vertex, he gets at least $a + dx_2$ points. In the other case, he plays a B-vertex v which has a W-neighbor u. By Lemma 21, $d^W(u) \leqslant 2$ and hence, u has at least $d-3$ B-neighbors different from v. The changes v: B$_i \to$R and u: W$\to$B$_i$/R (where $i \leqslant 2$), together with the changes on the further B-neighbors of u, yields a decrease of at least

$$(b-x_1-2x_2) + a - (b-x_1-x_2) + (d-3)x_2 = a + (d-4)x_2$$

in $p(G)$. Therefore, Staller gets at least $a + (d-4)x_2$ points in each of his turns. By condition $(C3)$, we have a lower bound on the gain of Dominator as well. These yield the sum

$$4a - 2b + 2x_1 + (4d-6)x_2 = 2s$$

for any two consecutive turns of Phase 3, and we can conclude that the average is at least s indeed. $\square$

Lemma 23. *At the end of Phase 3,*

(i) If v is a W-vertex, then $d^W(v) \leqslant 1$.

(ii) If v is a B-vertex, then $d(v) \leqslant 2$.

Proof. At the end of Phase 3 we have a residual graph G_i, in which the choice of any vertex decreases $p(G_i)$ by strictly less than $3a-2b+2x_1+(3d-2)x_2$ points.
(i) Playing a W-vertex v, which has two W-neighbors say u_1 and u_2, results in the changes v: W$\to$R and u_1, u_2: W$\to$B$_1$/R; additionally $d^B(v)+d^B(u_1)+d^B(u_2) \geqslant 3(d-2)$. This means a decrease of at least

$$a + 2(a-b+x_1+2x_2) + 3(d-2)x_2 = 3a-2b+2x_1+(3d-2)x_2$$

in $p(G_i)$. This cannot be the case; so each W-vertex has either zero or exactly one W-neighbor in G_i.
(ii) Now suppose that a B-vertex v with W-neighbors u_1, u_2 and u_3 is played in G_i. We have already seen that $d^W(u_j) \leqslant 1$ holds for every W-vertex u_j in G_i. Then, we have the changes v: B$_3$ →R and u_1, u_2, u_3: W→B$_1$/R. Further, each vertex from $\{u_1, u_2, u_3\}$ has at least $d-2$ B-neighbors different from v. Hence, the total gain of the player would be at least

$$b - x_1 + 3(a - b + x_1 + 2x_2) + 3(d-2)x_2 > 3a - 2b + 2x_1 + (3d-2)x_2.$$

This contradiction proves (ii). □

Phase 4. First, we change to assignment A2.4. By Lemma 23(ii), in any residual graph of Phase 4, the W-vertices induce a subgraph consisting of isolated vertices and P_2-components; moreover, each blue vertex has at most 2 (white) neighbors. Moreover, by Table 2 and Lemma 17, if a B-vertex loses y W-neighbors in a turn, its value is reduced by at least yx_3 points.

Lemma 24. *In Phase 4, the average decrease of* $p(G)$ *in a turn is at least* s *points.*

Proof. If Staller selects a W-vertex v, each neighbor u of v has a decrease of at least x_3 in its value. Hence, the total decrease in $p(G)$ is not smaller than $a + dx_3$.

If Staller selects a B-vertex v, the change is either v: B$_2$ →R or v: B$_1$ →R, it means at least $(b - x_1 - x_2 - x_3)$-point gain. As $d(v) \geqslant 1$, we necessarily have a W-neighbor u of v whose change is u: W→B$_1$/R. Further, u has at least $d-2$ B-neighbors different from v. Therefore, the decrease in $p(G)$ is at least

$$(b - x_1 - x_2 - x_3) + a - (b - x_1 - x_2 - x_3) + (d-2)x_3 = a + (d-2)x_3.$$

Hence, in any case, Staller gets at least $a + (d-2)x_3$ points in a turn of his own. By $(C4)$, Dominator gets at least $2a + (2d-2)x_3$ points in each of his turns and as follows, the average gain is at least

$$\frac{1}{2}(a + (d-2)x_3 + 2a + (2d-2)x_3) = s + x_3 > s$$

points as stated. □

Lemma 25. *At the end of Phase 4,*

- (i) *Every W-vertex has only B-neighbors.*
- (ii) *Every B-vertex has exactly one W-neighbor.*

Proof. Consider G_i which is the residual graph obtained at the end of Phase 4. As $(C4)$ is not true, Dominator cannot get $2a + (2d-2)x_3$ or more points in the ith turn. By Lemma 23, if (i) is not true, we have a "white-pair" (v, u), where u is the only W-neighbor of v and vice versa. Then, the choice of v would result the changes v, u: W→R. This, together with the fact $d^B(v) + d^B(u) \geqslant 2d-2$, implies that the decrease in $p(G_i)$ is at least $2a + (2d-2)x_3$, which is a contradiction. Thus, (i) is true.

To prove (ii) we suppose for a contradiction that a B-vertex v has two W-neighbors u_1 and u_2. By (i), these neighbors are "single-white" vertices and they turn to red if v is played; in addition both u_1 and u_2 has at least $d-1$ B-neighbors different from v. Hence, selecting v Dominator could seize at least

$$(b - x_1 - x_2) + 2a + 2(d-1)x_3 > 2a + (2d-2)x_3$$

points, which is a contradiction again. □

Phase 5. By Lemma 25(ii), the residual graphs occurring in this phase have simple structure, each of their components is a star of order at least $d+1$ whose center is white and the leaves are blue. Then, in each turn of Phase 5 exactly one such star component becomes completely red, no matter whether a white or a blue vertex is played. Then, the value of the residual graph is decreased by at least $a + d(b - x_1 - x_2 - x_3) = s$ points in each single turn.

Lemma 26. *In each turn of Phase 5, the decrease of $p(G)$ is at least s points.*

By Lemmas 18, 20, 22, 24 and 26, the average decrease per turn in the residual graph is at least s for the entire game. As $p(G_1) = an$ and the changes between assignments nowhere caused increase in $p(G)$, the domination game where Dominator plays the described greedy strategy yields a game with at most an/s turns. This establishes Theorem 4.

5 Concluding remarks on the Staller-start game

In our main theorem, we do not give upper bound on $\gamma_g'(G)$ for graphs with $\delta(G) \geqslant d \geqslant 4$. It is quite clear from the proof that we can establish the same upper bound on $\gamma_g'(G)$ as proved for $\gamma_g(G)$. Moreover, a slight improvement on it is also possible. We close the paper with this complicated formula.

If Staller begins the game, we index this starting turn by zero and take it into Phase 0. Then, from the first turn of Dominator, it continues as in the proof of Theorem 4. In the turn of Phase 0, Staller gets at least

$$a + d(a-b) = s + 30d^5 - 227d^4 + 637d^3 - 786d^2 + 360d$$

points. In later phases, the average decrease remains at least s. This proves that

$$\gamma_g'(G) \leqslant \frac{(30d^4 - 56d^3 - 258d^2 + 708d - 432)n - 30d^5 + 227d^4 - 637d^3 + 786d^2 - 360d}{90d^4 - 390d^3 + 348d^2 + 348d - 432}$$

holds for every $d \geqslant 4$ and for every graph G of minimum degree not smaller than d.

Acknowledgement

I thank the referees for the valuable suggestions.

References

[1] N. Alon and J. H. Spencer. *The Probabilistic Method.* Wiley, 3rd Edition, 2008.

[2] B. Brešar, P. Dorbec, S. Klavžar, and G. Košmrlj. Domination game: Effect of edge- and vertex-removal. *Discrete Math.*, 330:1–10, 2014.

[3] B. Brešar, S. Klavžar, G. Košmrlj, and D. F. Rall. Domination game: extremal families of graphs for the 3/5-conjectures. *Discrete Appl. Math.*, 161:1308–1316, 2013.

[4] B. Brešar, S. Klavžar, and D. F. Rall. Domination game and an imagination strategy. *SIAM J. Discrete Math.*, 24:979–991, 2010.

[5] B. Brešar, S. Klavžar, and D. F. Rall. Domination game played on trees and spanning subgraphs. *Discrete Math.*, 313:915–923, 2013.

[6] Cs. Bujtás. Domination game on trees without leaves at distance four. In *Proceedings of the 8th Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications* (A. Frank, A. Recski, G. Wiener, eds.), June 4–7, 2013, Veszprém, Hungary, pages 73–78, 2013.

[7] Cs. Bujtás. Domination game on forests. *Discrete Math.*, to appear, 2015.

[8] Cs. Bujtás. General upper bound on the game domination number. Manuscript, 2014.

[9] P. Dorbec, G. Košmrlj, and G. Renault. The domination game played on unions of graphs. *Discrete Math.*, 338:71–79, 2015.

[10] M. A. Henning and W. B. Kinnersley. Personal communication, 2014.

[11] M. A. Henning, S. Klavžar, and D. F. Rall. Total Version of the Domination Game. *Graphs Combin.*, doi:10.1007/s00373-014-1470-9, in press.

[12] M. A. Henning, S. Klavžar, and D. F. Rall. The 4/5 Upper Bound on the Game Total Domination Number. *Combinatorica*, to appear.

[13] W. B. Kinnersley, D. B. West, and R. Zamani. Extremal problems for game domination number. *SIAM J. Discrete Math.*, 27:2090–2107, 2013.

[14] G. Košmrlj. Realizations of the game domination number. *J. Comb. Optim.*, 28:447–461, 2014.