The Covering Problem in Rosenbloom-Tsfasman Spaces

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Abstract

We investigate the covering problem in RT spaces induced by the Rosenbloom-Tsfasman metric, extending the classical covering problem in Hamming spaces. Some connections between coverings in RT spaces and coverings in Hamming spaces are derived. Several lower and upper bounds are established for the smallest cardinality of a covering code in an RT space, generalizing results by Carnielli, Chen and Honkala, Brualdi et al., Yildiz et al. A new construction of MDS codes in RT spaces is obtained. Upper bounds are given on the basis of MDS codes, generalizing well-known results due to Stanton et al., Blokhuis and Lam, and Carnielli. Tables of lower and upper bounds are presented too.

Keywords: Generalization of Hamming metric, Covering code, Bound, MDS code

1 Introduction

Rosenbloom and Tsfasman [22] introduced a new metric on linear spaces over finite fields, motivated by possible applications to interference in parallel channels of communication systems (see also [24]). Nowadays this metric is known as Rosenbloom-Tsfasman (RT) metric (or ρ metric). A concept similar to RT metric was implicity posed by Niederreiter [16], and later Brualdi et al. [4] generalized it by introducing a new family of metrics called poset metrics.

These seminal papers have shed new light on the subject. Since RT metric generalizes Hamming metric, central concepts on codes in Hamming spaces have been investigated in

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RT environmental: perfect code, MDS code, linear code, distribution, according to [4, 11, 17, 13, 20, 22, 24] for instance. In particular, Quistorff [17] presented the combinatorial foundations of the packing problem for RT spaces in a more general setting, extending the famous packing problem in Hamming spaces.

Both packing and covering problems play a central role in combinatorial coding theory due to many reasons. One remarkable feature is the wide use of links and tools from many fields of mathematics, computer science, information theory. Several applications have motivated the research, for instance: compression with distortion, decoding of errors and erasures, cellular telecommunication. See [8] for an overview on covering codes.

Since packing problem in RT spaces has been studied [17, 22, 24], it seems interesting to investigate the covering problem in RT space. However the literature on this subject has remained rather poor: the only result deals with the computation of a particular class of RT spaces (when the poset is a chain), according to [4, 27].

In this paper we explore this gap, by studying the covering problem for an arbitrary RT space. We focus on the minimal cardinality of covering codes in RT spaces, mainly upper bounds on such covering codes. Sharp bounds are given in Theorems 8 and 13, and Corollaries 19 and 22.

This paper is organized as follows. We review some concepts on posets and poset metric in Section 2. In Section 3 we introduce covering in RT spaces and present preliminary results which are needed in our investigation, including a sharp bound in [4, 27]. A connection between covering codes in Hamming and RT spaces is established in Section 4. Inductive relations are discussed in Section 5, generalizing several known bounds. Such relationships combined with well-known results on covering codes yield some optimal classes. In Section 6 we show a new method to construct MDS codes in RT spaces. In Section 7 bounds arising from MDS codes are obtained, extending constructions due to Stanton et al. [25], Blokhuis and Lam [3], and Carnielli [5]. We conclude this work with some tables of lower and upper bounds for small instances.

2 Preliminaries

2.1 Poset

Our results are based on the perspective given by Brualdi et al. [4], where a codeword is viewed as a vector and the RT metric is associated to a suitable poset metric. To present this approach, we briefly describe a few concepts and properties on partially ordered set, henceforth abbreviated *poset*. We refer to [9] for an overview on poset.

Let P be a finite poset whose partial order relation is denoted by $\preceq$. A poset is a *chain* when any two elements are comparable; a poset is an *anti-chain* when no two distinct elements are comparable.

A subset I of P is an *ideal* of P when the following property holds: if $b \in I$ and $a \preceq b$, then $a \in I$. The *ideal generated* by a subset A of P is the ideal of smaller cardinality which contain A, denoted by $\langle A \rangle$. An element $a \in I$ is *maximal in I* if $a \preceq b$ implies that $b = a$. Analogously, an element $a \in I$ is *minimal in I* if $b \preceq a$ implies that $b = a$. The complement of a subset I of P is denoted by I^c.

Let m and s be positive integers. Consider the set $[m \times s] := \{1, \ldots, ms\}$ partitioned into m pairwise disjoint subsets $\{(i-1)s+1, \ldots, is\}$ of size s, for $i = 1, \ldots, m$. Each one of these parts is a chain whose the elements are ordered as $(i-1)s+1 \preceq \cdots \preceq is$. Hence the set $[m \times s]$ has the structure of a poset: it is the union of m disjoint chains with s elements each. We call this poset as *Rosembloom-Tsfasman poset* $[m \times s]$, and briefly by RT poset $[m \times s]$.

Let $\mathcal{I}^k(RT)$ be the set of all ideals with cardinality k of the RT poset $[m \times s]$. A basic property on $\mathcal{I}^k(RT)$ is presented in the next result.

Proposition 1. *[13, Proposition 1.1] Given $I \in \mathcal{I}^k(RT)$, for each $k \leqslant r \leqslant ms$ there exists $J \in \mathcal{I}^r(RT)$ such that $I \subseteq J$.*

Denote by $\Omega_j(i)$ the number of ideals of the RT poset $[m \times s]$ whose cardinality is i and with exactly j maximal elements.

Example 2. The RT poset $[2 \times 3]$ is represented in Figure 1.

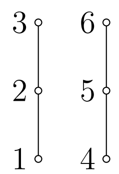

Figure 1: RT poset $[2 \times 3]$

Figure 2 presents all ideals of cardinality 4 of the RT poset $[2 \times 3]$.

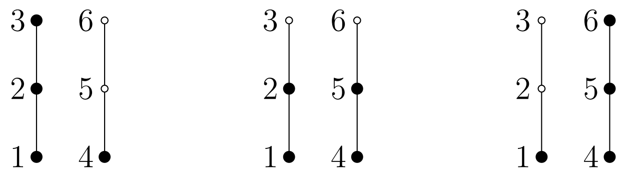

Figure 2: Set $\mathcal{I}^4(RT)$

Table 1 below displays all the parameters $\Omega_j(i)$ of the RT poset $[2 \times 3]$.

$j \backslash i$	1	2	3	4	5	6
1	2	2	2	0	0	0
2	0	1	2	3	2	1

Table 1: Parameters $\Omega_j(i)$

2.2 RT metric

We review now the RT metric as a poset metric, according to [4]. Assume that the n positions of the coordinates of a vector in $\mathbb{Z}_q^n$ are in standard bijection with the elements of the set $[n]$. A vector $z = (z_1, \dots, z_n) \in \mathbb{Z}_q^n$ can be represented briefly by $z = (z_1 \dots z_n)$. As usual, the *support* of a vector z is denoted by $supp(z) = \{i : z_i \neq 0\}$.

Given $x, y \in \mathbb{Z}_q^{ms}$ and the RT poset $[m \times s]$, the *RT distance* between x and y is

$$d_{RT}(x, y) = |\langle supp(x - y)\rangle|.$$

The set $\mathbb{Z}_q^{ms}$ endowed with the RT distance is a *Rosenbloom-Tsfasman space*, or simply, *RT space*.

The additive group $\mathbb{Z}_q$ can be replaced by an additive group G with $|G| = q$ in the definition of RT distance above. The RT distance can also be defined over an arbitrary set Q with $|Q| = q$ without a structure of an additive group, see for instance [11, 13, 22].

Example 3. Consider the RT poset $[2 \times 3]$ presented in Example 2. Given $x = (010100)$ and $y = (100110)$ in $\mathbb{Z}_2^6$, the RT distance between x and y is

$$d_{RT}(x, y) = |\langle\{1, 2, 5\}\rangle| = |\{1, 2, 4, 5\}| = 4.$$

In the RT space $\mathbb{Z}_q^{ms}$, the RT ball centered at x of radius R is the set

$$B^{RT}(x, R) = \{y \in \mathbb{Z}_q^{ms} : d_{RT}(x, y) \leqslant R\},$$

with cardinality

$$V_q^{RT}(m, s, R) = 1 + \sum_{i=1}^{R} \sum_{j=1}^{\min\{m,i\}} q^{i-j}(q-1)^j \Omega_j(i), \tag{1}$$

according to [4]. Since any RT ball of radius ms is the set $\mathbb{Z}_q^{ms}$, Eq (1) yields

$$1 + \sum_{i=1}^{ms} \sum_{j=1}^{\min\{m,i\}} q^{i-j}(q-1)^j = q^{ms}.$$

Thus one can determinate $V_q^{RT}(m, s, R)$ by making the difference between the number of vectors in $\mathbb{Z}_q^{ms}$ and the number of vectors in the complementary set of a ball $B^{RT}(x, R)$, that is,

$$V_q^{RT}(m, s, R) = q^{ms} - \sum_{i=R+1}^{ms} \sum_{j=1}^{\min\{m,i\}} q^{i-j}(q-1)^j \Omega_j(i). \tag{2}$$

3 Covering codes in RT space

Covering codes in Hamming spaces are extended naturally to an arbitrary RT space, as described below.

Definition 4. Given an RT poset $[m \times s]$, let C be a subset of $\mathbb{Z}_q^{ms}$. The code C is an *R-covering* of the RT space $\mathbb{Z}_q^{ms}$ when satisfies the property: for every $x \in \mathbb{Z}_q^{ms}$, there is a codeword $c \in C$ such that $d_{RT}(x, c) \leqslant R$, that is,

$$\bigcup_{c \in C} B^{RT}(c, R) = \mathbb{Z}_q^{ms}.$$

The number $K_q^{RT}(m, s, R)$ denotes the smallest cardinality of an R-covering of the RT space $\mathbb{Z}_q^{ms}$.

It is worth mentioning that an RT metric associated to an anti-chain ($s = 1$) of length m is equivalent to the well-known Hamming distance. Therefore, the number $K_q^{RT}(m, s, R)$ generalizes the classical number $K_q(m, R) = K_q^{RT}(m, 1, R)$.

Remark 5. Trivial values are easily computed for both extremal radius. The set $C = \mathbb{Z}_q^{ms}$ is the only 0-covering of the RT space $\mathbb{Z}_q^{ms}$, thus $K_q^{RT}(m, s, 0) = q^{ms}$. Given an arbitrary vector $c \in \mathbb{Z}_q^{ms}$, the code $C = \{c\}$ is an ms-covering of the RT space $\mathbb{Z}_q^{ms}$. Therefore $K_q^{RT}(m, s, ms) = 1$.

We begin our results with a general upper bound.

Proposition 6. *(Trivial upper bound) For every m, s, $q \geqslant 2$ and R such that $0 \leqslant R \leqslant ms$,*

$$K_q^{RT}(m, s, R) \leqslant q^{ms-R}.$$

Proof. Proposition 1 implies that there is an ideal I of the RT poset $[m \times s]$ with R elements. The bound follows from the fact that the code

$$C = \{(c_1, \ldots, c_{ms}) \in \mathbb{Z}_q^{ms} : c_i = 0 \text{ for all } i \in I\}$$

is an R-covering of the RT space $\mathbb{Z}_q^{ms}$. □

A closer look on the proof above reveals that every subset I with size R ensures an R-covering C of the Hamming space $\mathbb{Z}_q^m$. In contrast, the restriction of I being an ideal is essential in RT spaces. We go back to Example 2 to illustrate this fact. The set $I = \{2, 3, 5, 6\}$ (which is not an ideal of the RT poset $[2 \times 3]$) induces the subspace

$$C = \{000000, 000100, 100000, 100100\},$$

which is not a 4-covering of the RT space $\mathbb{Z}_2^6$, because $d_{RT}(c, 111111) = 6$ for all $c \in C$.

A simple but important lower bound on covering codes in RT spaces is stated below, extending the classical ball covering code.

Proposition 7. *(Ball covering bound) For every m, s, $q \geqslant 2$ and R such that $0 \leqslant R \leqslant ms$,*

$$K_q^{RT}(m, s, R) \geqslant \frac{q^{ms}}{V_q^{RT}(m, s, R)}.$$

Proof. The proof is straightforward. □

Let C a subset of $\mathbb{Z}_q^{ms}$. The code C is *perfect* (with respect to the RT poset $[m \times s]$) provided that there exists an integer R such that the balls of radius R with centers at the codewords of C are pairwise disjoint and their union is the space $\mathbb{Z}_q^{ms}$.

Let C be an R-covering of the RT space $\mathbb{Z}_q^{ms}$. The code C is perfect if and only if the equality holds in Proposition 7.

The codes mentioned in Remark 5 are trivial perfect codes. Particularly interesting, a classification of linear perfect codes with respect to a chain is established in [4, Theorem 2.1], whose proof yields implicitly the next result (see also [27, Theorem 2.3]). Characterizations of such codes in more general setting are presented in [1, Proposition 3.1] and [19, Theorem 7].

Theorem 8. *[4, 27] For any positive integer s,*

$$K_q^{RT}(1, s, R) = q^{s-R}.$$

Proof. An alternative proof is presented here. An RT ball of radius R in the RT space $\mathbb{Z}_q^s$ has cardinality q^R. Indeed, a look on the chain $[1 \times s]$ reveals that $\Omega_1(i) = 1$ and $\Omega_j(i) = 0$ for every $1 < i \leqslant s$. Eq. (1) produces

$$V_q^{RT}(1, s, R) = 1 + \sum_{i=1}^{R} q^{i-1}(q-1) = 1 + (q-1)\sum_{i=1}^{R} q^{i-1} = q^R.$$

Hence the ball covering bound yields $K_q^{RT}(1, s, R) \geqslant q^{s-R}$. On the other hand, a simple application of the trivial upper bound concludes the argument. □

Both Propositions 6 and 7 are sharp at least for $R = 0$, $R = ms$ and for a chain (when $m = 1$). Nevertheless, it is expected that these bounds can be far from the exact value for most instances, like in the classical covering codes. A careful analysis on the coordinates can be a tool to find improvements.

Let $I = \{i_1, \ldots, i_k\}$ be a subset of $[n] = \{1, \ldots, n\}$. Given a vector $x = (x_1, \ldots, x_n) \in \mathbb{Z}_q^n$, the projection of x with respect to I is the vector $\pi_I(x) = (x_{i_1}, \ldots, x_{i_k}) \in \mathbb{Z}_q^k$. More generally, for a non-empty subset A of $\mathbb{Z}_q^n$, the projection of A with respect to I is the set $\pi_I(A) = \{\pi_I(a) : a \in A\}$.

Example 9. We claim $K_2^{RT}(4, 2, 5) \leqslant 3$, which improves significantly the trivial upper bound 8. For this purpose, choose the subset $C = \{c_1, c_2, c_3\}$ of $\mathbb{Z}_2^8$, where

$$c_1 = (00000000), c_2 = (01010101), c_3 = (11111111).$$

The set C is a 5-covering of the RT space $\mathbb{Z}_2^8$. Indeed, let I be the set formed by all maximal elements of the RT poset $[4 \times 2]$ (so $I = \{2, 4, 6, 8\}$). Given an arbitrary vector $x \in \mathbb{Z}_2^8$, we analyze a few cases:

Case 1: If $\pi_I(x) \in \{(0000), (0001), (0010), (0100), (1000)\}$, then a simple inspection reveals that x is covered by c_1.

Case 2: If $\pi_I(x) \in \{(1111), (1110), (1101), (1011), (0111)\}$, then x is covered by c_3.

Case 3: Otherwise, $\pi_I(x) \in \{(1100), (0011), (1001), (0110), (1010), (0101)\}$. The analysis is divided into the following subcases:

(a) If $\pi_J(x) = (000)$ for some $J = \{i-1, i, j\}$ with $i, j \in I$. Note that c_1 covers x.

(b) If $\pi_J(x) = (111)$ for some $J = \{i-1, i, j\}$ with $i, j \in I$. Here the word c_3 covers x.

(c) Otherwise, if the subcases above do not hold for any $i \in I$, then $\pi_{\{i-1\}}(x) \neq \pi_{\{i\}}(x)$. A closer look shows us that x is covered by c_2.

Example 10. The optimal bound $K_2^{RT}(2, 2, 2) = 3$ holds. For the upper bound, choose $C = \{(0000), (0101), (1111)\}$. A similar argument used in Example 9 yields that C is a 2-covering of the RT space $\mathbb{Z}_2^4$, by using the ideals $I = \{1, 2\}$ and $J = \{3, 4\}$ of the RT poset $[2 \times 2]$. On the other hand, suppose by a contradiction that there is a 2-covering C' with $|C'| = 2$. Since $V_2^{RT}(2, 2, 2) = 8$, the set C' is a perfect code. Given distinct $x, y \in \mathbb{Z}_2^4$, let $w = (\pi_I(x), \pi_J(y))$. Since $d_{RT}(x, w) \leqslant 2$ and $d_{RT}(y, w) \leqslant 2$, we obtain $B^{RT}(x, 2) \cap B^{RT}(y, 2) \neq \varnothing$. This statement is a contradiction with the fact that C' is a perfect 2-covering.

4 A Relationship between coverings in Hamming and RT spaces

A connection between the Hamming and RT metrics is established in this section, and produces a systematic way for finding new lower bounds on $K_q^{RT}(m, s, R)$ from known values of the classical covering codes.

Proposition 11. *For every $q \geqslant 2$,*

$$K_q(ms, R) \leqslant K_q^{RT}(m, s, R) \leqslant K_q^{RT}(1, ms, R).$$

Proof. Let C be an R-covering of the RT space $\mathbb{Z}_q^{ms}$ with $|C| = K_q^{RT}(m, s, R)$. Given $x \in \mathbb{Z}_q^{ms}$, there is $c \in C$ such that $d_{RT}(x, c) \leqslant R$. Since $supp(x - c)$ is a subset of $\langle supp(x - c)\rangle$, it follows that

$$d_H(x, c) = |supp(x - c)| \leqslant |\langle supp(x - c)\rangle| = d_{RT}(x, c) \leqslant R.$$

Hence C is also an R-covering of the Hamming space $\mathbb{Z}_q^{ms}$, and this proves that $K_q(ms, R) \leqslant K_q^{RT}(m, s, R)$.

The upper bound on $K_q^{RT}(m, s, R)$ is straightforward from Proposition 6 and Theorem 8. $\square$

Rodemich [21, Theorem 2] proved the classical bound $K_q(n, n-2) \geqslant q^2/(n-1)$. Hass et al. [12, Theorem 8] reached $K_q(n, n-2) \geqslant 3q - 2n + 2$, improving the previous bound when $5 \leqslant n < q \leqslant 2n - 4$. These facts and Proposition 11 yield immediately:

Corollary 12. *The following bounds hold:*

1. *If $q > 10$, then $K_q^{RT}(3, 2, 4) \geqslant q^2/5$.*
2. *If $q \in \{7, 8\}$, then $K_q^{RT}(3, 2, 4) \geqslant 3q - 10$.*

Both lower bounds in Corollary 12 improve those from the ball covering bound.

Theorem 13. *For every $m \geqslant (t-1)q + 1$, $K_q^{RT}(m, s, ms - t) = q$ holds. In particular, $K_q^{RT}(m, s, ms - 1) = q$.*

Proof. The repetition code

$$C = \{(0, \ldots, 0), (1, \ldots, 1), \ldots, (q-1, \ldots, q-1)\}$$

is an $(ms - t)$-covering of the RT space $\mathbb{Z}_q^{ms}$. Given an arbitrary vector $x \in \mathbb{Z}_q^{ms}$, choose $(t-1)q+1$ coordinates that are maximal elements of the RT poset $[m \times s]$. The pigeonhole principle states that there is a symbol, say y, that appears at least t times in these coordinates. Take $c = (y, \ldots, y) \in C$, and notice that $d_{RT}(x, c) = |\langle supp(x-c) \rangle| \leqslant ms - t$. Hence $K_q^{RT}(m, s, ms - t) \leqslant |C| = q$.

For the lower bound, suppose by contradiction that C is an $(ms - t)$-covering of $\mathbb{Z}_q^{ms}$ whose cardinality is less than q. For each $i \in [m \times s]$, choose $x_i \in \mathbb{Z}_q$ such that $c_i \neq x_i$ for all $c \in C$. The vector $x = (x_1, \ldots, x_{ms})$ satisfies

$$d_{RT}(x, c) = |\langle supp(x - c) \rangle| = |\{1, \ldots, ms\}| = ms$$

for any $c \in C$, that is, the vector x is not covered by C. Hence $K_q^{RT}(m, s, ms - t) \geqslant q$. $\square$

Theorem 13 generalizes the results:

- $K_q(2, 1) = q$, due to Kalbfleisch and Stanton [14];
- $K_q(m, 1, m - t) = q$ for $m \geqslant (t-1)q + 1$, by Carnielli [5, Theorem 9], Chen and Honkala [7, Theorem 6].

The numbers $K_q^{RT}(m, s, ms - 1) = K_q(ms, ms - 1) = q$ reveal that Proposition 11 is optimal at least for a class of parameters.

5 Inductive Relations

Inductive relations between parameters play a central role in the literature on covering codes. In this section we focus on the behavior of inductive relations in RT spaces. All the results in the present section deal with upper bounds for covering codes in RT spaces. Sharp bounds are given in Corollaries 19 and 22.

Proposition 14. *(Trivial relations) For every m, s, and R such that $0 \leqslant R \leqslant ms$, the following bounds hold:*

1. $K_q^{RT}(m, s, R+1) \leqslant K_q^{RT}(m, s, R)$.
2. $K_q^{RT}(m, s, R) \leqslant K_{q+1}^{RT}(m, s, R)$.

The result below extends the very useful relation $K_q(n_1 + n_2, R_1 + R_2) \leqslant K_q(n_1, R_1) K_q(n_2, R_2)$, see [8] for instance.

Theorem 15. *(Directed sum) For $R_1 \leqslant m_1 s$ and $R_2 \leqslant m_2 s$,*

$$K_q^{RT}(m_1 + m_2, s, R_1 + R_2) \leqslant K_q^{RT}(m_1, s, R_1) K_q^{RT}(m_2, s, R_2).$$

Proof. The RT poset $[(m_1 + m_2) \times s]$ can be viewed as a disjoint union of the RT poset $[m_1 \times s]$ and the RT poset $[m_2 \times s]$. This simple remark is the key of the proof. For $i = 1, 2$, let C_i be an optimal R_i-covering of the RT space $\mathbb{Z}_q^{m_i s}$. The set

$$C_1 \oplus C_2 = \{(c_1, c_2) \in \mathbb{Z}_q^{(m_1+m_2)s} : c_1 \in C_1 \text{ and } c_2 \in C_2\}$$

is an $(R_1 + R_2)$-covering of the RT space $\mathbb{Z}_q^{(m_1+m_2)s}$. For a vector $(x, y) \in \mathbb{Z}_q^{(m_1+m_2)s}$, notice that $d_{RT}((x, y), C_1 \oplus C_2) = d_{RT}(x, C_1) + d_{RT}(y, C_2)$ and the result follows. □

Corollary 16. *If $n \leqslant m$ and $R \leqslant ns$, then*

$$K_q^{RT}(m, s, R) \leqslant q^{(m-n)s} K_q^{RT}(n, s, R).$$

Proof. Apply Theorem 15 with $m_1 = m - n$, $m_2 = n$, $R_1 = 0$, $R_2 = R$ and use the trivial number $K_q^{RT}(m - n, s, 0) = q^{(m-n)s}$. □

In a similar spirit, known results on covering codes can be adapted to RT spaces.

Proposition 17. *For $n \leqslant m$ and $ns \leqslant R$,*

$$K_q^{RT}(m, s, R) \leqslant K_q^{RT}(m - n, s, R - ns).$$

Proof. Let C' be an $(R - ns)$-covering of the RT space $\mathbb{Z}_q^{(m-n)s}$. Consider the ideal $I = \{1, \dots, ns\}$ of the RT poset $[m \times s]$. Take the set

$$C = \{c \in \mathbb{Z}_q^{ms} : \pi_I(c) = \mathbf{0} \in \mathbb{Z}_q^{ns} \text{ and } \pi_{I^c}(c) \in C'\}.$$

Note that $|C| = |C'|$. It remains to prove that C is an R-covering of the RT space $\mathbb{Z}_q^{ms}$. Given an arbitrary $x \in \mathbb{Z}_q^{ms}$, there is $c' \in C'$ such that $d_{RT}(\pi_{I^c}(x), c') \leqslant R - ns$. Choose $c \in C$ such that $\pi_{I^c}(c) = c'$. Thus

$$d_{RT}(x, c) \leqslant ns + d_{RT}(\pi_{I^c}(x), c') \leqslant ns + R - ns = R,$$

concluding the proof. □

Given an RT poset $[m \times s]$ and a positive integer $r < s$, the notation below denotes the ideal

$$I(r) = \langle \{r, s + r, \ldots, (m-2)s + r, (m-1)s + r\} \rangle. \tag{3}$$

Proposition 18. *For every $r < s$ and $R \leqslant mr$,*

$$K_q^{RT}(m, s, R) \leqslant q^{m(s-r)} K_q^{RT}(m, r, R).$$

Proof. Let C' be an optimal R-covering of the RT space $\mathbb{Z}_q^{mr}$ and let $I = I(r)$ denote the ideal in Eq. (3). Consider the following subset of $\mathbb{Z}_q^{ms}$

$$C = \{c \in \mathbb{Z}_q^{ms} : \pi_I(c) \in C'\}.$$

We claim that C is an R-covering of the RT space $\mathbb{Z}_q^{ms}$. Indeed, given $x \in \mathbb{Z}_q^{ms}$, note that $\pi_I(x) \in \mathbb{Z}_q^{mr}$, hence there is a codeword $c' \in C'$ such that $d_{RT}(\pi_I(x), c') \leqslant R$. Take now $c \in C$ with $\pi_I(c) = c'$ and $\pi_{I^c}(c) = \pi_{I^c}(x)$. It is clear that $d_{RT}(x, c) = d_{RT}(\pi_I(x), \pi_I(c)) \leqslant R$. Since $|C| = q^{m(s-r)} K_q^{RT}(m, r, R)$, the statement is proved. $\square$

Corollary 19. *For any $s \geqslant 2$, $K_2^{RT}(3, s, 1) = 2^{3s-2}$.*

Proof. The lower bound is derived from the ball covering bound. A combination of Proposition 18 ($r = 1$) with $K_2(3, 1) = 2$ implies the upper bound. $\square$

A simple analysis on the construction in Proposition 18 reveals that the code C is linear if and only if C' is linear. By beginning with the code $C' = \{001, 110\}$, Corollary 19 produces a class of nonlinear perfect codes.

Example 20. We discuss a comparative analysis between Corollary 16 and Proposition 17. For this purpose, we need the bounds: $K_2^{RT}(3, 2, 3) \leqslant 8$ (trivial upper bound), $K_2^{RT}(4, 2, 5) \leqslant 3$ (Example 9), and $K_2^{RT}(3, 2, 1) = 16$ (Corollary 19).

1. Corollary 16 produces $K_2^{RT}(4, 2, 3) \leqslant 2^2 K_2^{RT}(3, 2, 3) \leqslant 32$. However, an application of Proposition 17 when $n = 1$ shows us $K_2^{RT}(4, 2, 3) \leqslant K_2^{RT}(3, 2, 1) = 16$, which improves significantly the previous bound 32.

2. In contrast, $K_2^{RT}(5, 2, 5) \leqslant K_2^{RT}(3, 2, 1) = 16$ follows from Proposition 17 when $n = 2$. But Corollary 16 implies $K_2^{RT}(5, 2, 5) \leqslant 2^2 K_2^{RT}(4, 2, 5) \leqslant 12$, which is better than the previous bound 16.

Proposition 21. *For every $r < s$ and $mr \leqslant R$,*

$$K_q^{RT}(m, s, R) \leqslant K_q^{RT}(m, s - r, R - mr).$$

Proof. Let C' be an optimal $(R - mr)$-covering of the RT space $\mathbb{Z}_q^{m(s-r)}$. Again $I = I(r)$ denotes the ideal from Eq. (3). The subset

$$C = \{c \in \mathbb{Z}_q^{ms} : \pi_I(c) = \mathbf{0} \in \mathbb{Z}_q^{mr} \text{ and } \pi_{I^c}(c) \in C'\}$$

of $\mathbb{Z}_q^{ms}$ is an R-covering of the RT space $\mathbb{Z}_q^{ms}$. The proof of this statement resembles strongly that presented in Proposition 18. $\square$

The result above is imperceptible in Hamming spaces, because Proposition 21 applied to the case $r = 0$ is collapsed into the innocuous bound $K_q^{RT}(m, s, R) \leqslant K_q^{RT}(m, s, R)$.

An exact class is derived from the previous proposition, more specifically:

Corollary 22. *For any $s \geqslant 2$, $K_2^{RT}(2, s, 2s - 2) = 3$.*

Proof. We firstly show that $V_2^{RT}(2, s, 2s - 2) = 2^{2s-1}$. Indeed, Eq. (2) implies

$$V_2^{RT}(2, s, 2s - 2) = 2^{2s} - \sum_{i=2s-1}^{2s} \sum_{j=1}^{2} 2^{i-j}\Omega_j(i).$$

If $i > s$, then $\Omega_1(i) = 0$ in the RT poset $[2 \times s]$. It is easy to see that $\Omega_2(2s - 1) = 2$ and $\Omega_2(2s) = 1$. Hence $V_2^{RT}(2, s, 2s - 2) = 2^{2s-1}$.

Suppose by a contradiction that there is a $(2s - 2)$-covering C of the RT space $\mathbb{Z}_2^{2s}$ with $|C| = 2$. The ball covering bound implies $K_2^{RT}(2, s, 2s - 2) \geqslant 2$. Since $V_2^{RT}(2, s, 2s - 2) = 2^{2s-1}$, the covering C is perfect. However, given distinct $x, y \in \mathbb{Z}_2^{2s}$, note that $B^{RT}(x, 2s-2) \cap B^{RT}(y, 2s-2) \neq \varnothing$. Indeed, write $x = (x_1, \ldots, x_s, x_{s+1}, \ldots, x_{2s})$ and $y = (y_1, \ldots, y_s, y_{s+1}, \ldots, y_{2s})$. Take $z = (x_1, \ldots, x_s, y_{s+1}, \ldots, y_{2s})$. A simple inspection shows that $d_{RT}(x, z) \leqslant s \leqslant 2s - 2$ and $d_{RT}(y, z) \leqslant s \leqslant 2s - 2$, which implies $z \in B^{RT}(x, 2s - 2) \cap B^{RT}(y, 2s - 2)$. This contradicts the fact that C is a perfect $(2s - 2)$-covering.

On the other hand, note that $K_2^{RT}(2, 2, 2) = 3$ from Example 10. By applying Proposition 21 when $r = s - 2$, we have $K_2^{RT}(2, s, 2s - 2) \leqslant K_2^{RT}(2, 2, 2) = 3$. Therefore the optimal value is proved. □

Covering codes in Hamming spaces can be very useful to improve certain upper bounds on covering codes in RT spaces, as described in the next result.

Theorem 23. *Let $m, s, q \geqslant 1$ and p, k, n, r integers such that $0 < p \leqslant s$, $0 < k \leqslant m$ and $m = nr$. Then the following inequality holds:*

$$K_q^{RT}(m, s, ks + (m - k)p) \leqslant K_a^{RT}(n, r, k),$$

where $a = K_q^{RT}(1, s, p) = q^{s-p}$ (see Theorem 8).

Proof. Let H_1 be an optimal p-covering of the RT space $\mathbb{Z}_q^s$. Then $|H_1| = K_q^{RT}(1, s, p) = a$, and a bijection identify the sets H_1 and $\mathbb{Z}_a$. Consider H_2 an optimal k-covering of the RT space $\mathbb{Z}_a^m$, where the symbols are viewed as elements in H_1.

The set H_2 yields a $(ks + (m - k)p)$-covering of the RT space $\mathbb{Z}_q^{ms}$. Indeed, given $x \in \mathbb{Z}_q^{ms}$, write $x = (y_0, \ldots, y_{m-1})$, where $y_i = (x_{is+1}, \ldots, x_{(i+1)s}) \in \mathbb{Z}_q^m$ for $i = 0, 1, \ldots, m - 1$. For each $y_i \in \mathbb{Z}_q^s$, there exists $z_i \in H_1$ such that $d_{RT}(y_i, z_i) \leqslant p$ (in respect to RT poset $[1 \times s]$). Define $z = (z_0, \ldots, z_{m-1}) \in \mathbb{Z}_a^m$. Then there exists a vector $w = (w_0, \ldots, w_{m-1}) \in H_2$ such that $d_{RT}(z, w) \leqslant k$ (in respect to RT poset $[n \times r]$). Thus $d_{RT}(z, w) \leqslant ks$ (in respect to RT poset $[m \times s]$). We claim that the vector $w \in H_2$ covers x in the RT space $\mathbb{Z}_q^{ms}$. In the coordinates where w and z differs, $d_{RT}(x, w)$ is at most ks. Moreover, in the $m-k$ coordinates where w and z coincide, $d_{RT}(y_i, z_i) \leqslant p$ holds, then $d_{RT}(x, w)$ is at most $(m-k)p$. Hence, $d_{RT}(x, w) \leqslant ks + (m-k)p$, that is, we prove that H_2 is a $(ks + (m-k)p)$-covering of the RT space $\mathbb{Z}_q^{ms}$ with $|H_2| = K_a^{RT}(n, r, k)$. The proof is complete. $\square$

Theorem 23 above generalizes a bound by Carnielli [5, Theorem 5].

Remark 24. Proposition 11 provides $K_a(nr, k) \leqslant K_a^{RT}(n, r, k)$. Since $m = nr$ in Theorem 23, a closer look reveals that the best choice to reduce the upper bound is $n = m$ and $r = 1$. Therefore Theorem 23 can be rewritten as $K_q^{RT}(m, s, ks + (m-k)p) \leqslant K_a(m, k)$.

Appropriate parameters m, s, q, k and p sometimes provide more than one bound on $K_q^{RT}(m, s, R)$. As an illustration, take the parameters $m = 4$, $s = 4$, $q = 2$, $k = 1$ and $p = 2$ in Theorem 23. Then $K_2^{RT}(4, 4, 10) \leqslant K_4(4, 1) = 24$. On the other hand, take $k = 2$ and $p = 1$. Theorem 23 states that $K_2^{RT}(4, 4, 10) \leqslant K_8(4, 2) \leqslant 23$, which is better than 24.

Example 25. Let us now compare Theorem 23 and the inductive relations presented in this section. Take $m = 3$, $s = 4$, $q = 2$, $k = 1$ and $p = 1$. Let $a = K_2^{RT}(1, 4, 1) = 8$ (Theorem 8) and $K_8(3, 1) = 32$ ([15]). Theorem 23 produces $K_2^{RT}(3, 4, 6) \leqslant K_8(3, 1) = 32$, which improves the bound 48 from Corollary 16.

On the other hand, the sharp bound $a = K_2^{RT}(1, 4, 2) = 4$ holds from Theorem 8. In view of $K_4(3, 1) = 8$ ([15]), Theorem 23 yields $K_2^{RT}(3, 4, 8) \leqslant K_4(3, 1) = 8$. This bound is better than 12, according to Proposition 17.

6 A new class of MDS codes in RT spaces

The famous Singleton bound was extended to an RT linear space over a finite field in [22]. The papers [11, 13, 17] consider this bound in a slightly more general setting, as follows. Let C be a code in the RT space $\mathbb{Z}_q^{ms}$ with cardinality q^k, length ms, and minimum distance $d = d_{RT}(C)$. The parameters of C with respect to RT metric are denoted by $[m, s, k, d]_q$. The Singleton bound states that

$$d_{RT}(C) \leqslant ms + 1 - k. \tag{4}$$

A code C meeting the Singleton bound (Eq. 4) is an MDS (maximum distance separable) code. A research problem arises naturally: the existence of such MDS codes. In this direction, Quistorff [17] presented a classification for the binary alphabet, and two classes of MDS codes in RT spaces are built in [22] (constructions using hyperderivative of a polynomial over a finite field are showed by Skriganov [24]).

Theorem 26. *[22] Given a prime power q, for any s and a such that $s \leqslant q$ and $0 \leqslant a \leqslant qs$, there exist MDS linear codes with parameters $[q, s, a+1, qs-a]_q$ and $[q+1, s, a+1, (q+1)s-a]_q$.*

In this section we aim to establish a new class of MDS codes in RT spaces. Under some conditions we construct an MDS code with parameters $[t, n, 2, tn-1]_q$.

Theorem 27. *Suppose that there is an MDS code C in the Hamming space $\mathbb{Z}_q^n$ with $d_H(C) = n-1$. For each $t \in \{2,\ldots,n\}$ there is an MDS code C_t in the RT space $\mathbb{Z}_q^{tn}$ (RT poset $[t \times n]$) with $d_{RT}(C_t) = tn-1$.*

Proof. Let σ be the cyclic permutation $\sigma = (1\ 2\ \ldots\ n)$. Given $t \in \{2,\ldots,n\}$, define the code C_t according to the rule: each $c = (c_1,\ldots,c_n)$ in C is associated to the vector

$$\phi(c) = (c_1,\ldots,c_n, c_{\sigma(1)},\ldots,c_{\sigma(n)},\ldots,c_{\sigma^{t-1}(1)},\ldots,c_{\sigma^{t-1}(n)})$$

in $\mathbb{Z}_q^{tn}$. Let $C_t = \phi(C)$. Since $|C_t| = |C| = q^2$, Singleton bound implies $d_{RT}(C_t) \leqslant tn-1$. It remains to prove that $d_{RT}(C_t) \geqslant tn-1$. For this purpose, we describe properties related to codewords in C_t as follows. Given $a = (a_1,\ldots,a_n) \in C$ and $b = (b_1,\ldots,b_n) \in C$, note that a and b coincide at most in one coordinate, since $d_H(C) = n-1$. Therefore, for every $i \in \{1,\ldots,n-1\}$ the vectors $(a_{\sigma^i(1)},\ldots,a_{\sigma^i(n)})$ and $(b_{\sigma^i(1)},\ldots,b_{\sigma^i(n)})$ also coincide at most in one coordinate. Moreover, the statements hold:

(1) If $a_n = b_n$, then $a_{\sigma^i(n)} \neq b_{\sigma^i(n)}$ for every $i \in \{1,\ldots,n-1\}$.

(2) If $a_n \neq b_n$, then there is at most one index $i \in \{1,\ldots,n-1\}$ such that $a_{\sigma^i(n)} = b_{\sigma^i(n)}$ and $a_{\sigma^i(n-1)} \neq b_{\sigma^i(n-1)}$.

Given distinct codewords $\phi(a)$ and $\phi(b)$ in C_t, we have

$$\begin{aligned}\phi(a) &= (a_1,\ldots,a_n, a_{\sigma(1)},\ldots,a_{\sigma(n)},\ldots,a_{\sigma^{t-1}(1)},\ldots,a_{\sigma^{t-1}(n)})\\ \phi(b) &= (b_1,\ldots,b_n,\, b_{\sigma(1)},\ldots,b_{\sigma(n)},\,\ldots,b_{\sigma^{t-1}(1)},\ldots,b_{\sigma^{t-1}(n)}).\end{aligned}$$

The statements (1) and (2) imply that $d_{RT}(\phi(a),\phi(b)) \geqslant tn-1$. Therefore, the code C_t meets the Singleton bound. ☐

Given a positive integer q, $N(q)$ denotes the maximum cardinality of a set of mutually orthogonal Latin squares of order q. An MDS code (in a Hamming space) with cardinality q^2, length n, and minimum distance $n-1$ is equivalent to a set of $n-2$ mutually orthogonal Latin squares of order q ($n-2 \leqslant N(q)$), according to [23, Theorem 3]. The following result is an immediate consequence of Theorem 27.

Corollary 28. *Given positive integers q, t and n such that $3 \leqslant n \leqslant N(q)+2$ and $2 \leqslant t \leqslant n$, there exists an MDS code with parameters $[t,n,2,tn-1]_q$.*

Since $N(q) = q-1$ for a prime power q, the class of MDS codes with parameters $[q+1,1,2,q]_q$ is also derived from Theorem 26. While Theorem 26 produces always linear codes, Theorem 27 generates nonlinear codes over an arbitrary alphabet. For instance, because $N(12) \geqslant 5$ there is an MDS code of type $[t,7,2,7t-1]_{12}$ for any $2 \leqslant t \leqslant 7$ by the previous corollary.

Example 29. The nonlinear code $C = \{(100),(111),(001),(010)\}$ is an MDS in the Hamming space $\mathbb{Z}_2^3$ with minimum distance 2. Let σ be the cyclic permutation $\sigma = (1\,2\,3)$. Each $c = (c_1,c_2,c_3) \in C$ is associated to

$$\begin{aligned}\phi(c) &= (c_1,c_2,c_3,c_{\sigma(1)},c_{\sigma(2)},c_{\sigma(3)},c_{\sigma^2(1)},c_{\sigma^2(2)},c_{\sigma^2(3)})\\ &= (c_1,c_2,c_3,c_2,c_3,c_1,c_3,c_1,c_2)\end{aligned}$$

The construction above produces

$$C_3 = \phi(C) = \{(100001010), (111111111), (001010100), (010100001)\}.$$

Theorem 27 states that C_3 is an MDS code in the RT space $\mathbb{Z}_2^9$ with minimum distance 8 (with respect to the RT poset $[3 \times 3]$). In contrast, C_3 is not an MDS code in the Hamming space $\mathbb{Z}_2^9$.

7 Upper bounds from MDS codes

MDS codes have been a tool to construct covering codes in Hamming spaces. Such codes have been applied to covering codes in [3, 5, 6, 18, 25] as well as to closely related concepts [10]. We refer to [8] for additional contributions on this topic.

In this section we explore several upper bounds for covering codes in RT spaces arising from MDS codes, extending a well-known result in the literature.

Theorem 30. *Suppose that there is an MDS code in the RT space $\mathbb{Z}_q^{ms}$ with minimum distance $d+1$. For every $r \geqslant 2$,*

$$K_{qr}^{RT}(m, s, d) \leqslant q^{ms-d} K_r^{RT}(m, s, d).$$

Proof. Throughout this proof, the set $\mathbb{Z}_{qr}$ is regarded as the set $\mathbb{Z}_{qr} = \mathbb{Z}_q \times \mathbb{Z}_r$ by the bijection $x+ry \to (x, y)$. This strategy allows us to analyze information on the coordinates x and y separately.

Let H be an optimal d-covering of the RT space $\mathbb{Z}_r^{ms}$, and let C be an MDS code with $d_{RT}(C) = d+1$ in the RT space $\mathbb{Z}_q^{ms}$. The set

$$G = \{((c_1, h_1), \ldots, (c_{ms}, h_{ms})) \in \mathbb{Z}_{qr}^{ms} : (c_1, \ldots, c_{ms}) \in C,\ (h_1, \ldots, h_{ms}) \in H\}$$

is a d-covering of the RT space $\mathbb{Z}_{qr}^{ms}$.

Indeed, given an arbitrary $z = ((x_1, y_1), \ldots, (x_{ms}, y_{ms})) \in \mathbb{Z}_{qr}^{ms}$, take $x = (x_1, \ldots, x_{ms})$ and $y = (y_1, \ldots, y_{ms})$. Clearly, $x \in \mathbb{Z}_q^{ms}$ and $y \in \mathbb{Z}_r^{ms}$.

Since H is a d-covering of the RT space $\mathbb{Z}_r^{ms}$, for $y \in \mathbb{Z}_r^{ms}$ there is $h = (h_1, \ldots, h_{ms}) \in H$ such that $d_{RT}(y, h) \leqslant d$. Consider the ideal $J = \langle supp(y-h) \rangle$. By Proposition 1, there is an ideal $I \in \mathcal{I}^d(RT)$ of the RT poset $[m \times s]$ such that $J \subseteq I$. Because $I \in \mathcal{I}^d(RT)$ and C is an MDS code with $d_{RT}(C) = d+1$, there is a codeword $c = (c_1, \ldots, c_{ms})$ in C such that x and c coincide in all coordinates of I^c. Thus $supp(x-c) \subseteq I$.

Choose now $g = ((c_1, h_1), \ldots, (c_{ms}, h_{ms})) \in G$. By construction, z and g coincide in all coordinates of I^c. Hence $d_{RT}(z, g) = |\langle supp(z-g) \rangle| \leqslant |I| = d$, and this completes the argument. Therefore $K_{qr}^{RT}(m, s, d) \leqslant |G| = |C||H| = q^{ms-d} K_r^{RT}(m, s, d)$. $\square$

An MDS code with minimum distance $d+1$ is also known as a $d+1$-Latin code, because its connection with Latin squares. Therefore Theorem 30 generalizes [5, Theorem 6]. See also [8, Theorem 3.7.10].

The impact of Theorem 30 is discussed now. Classical MDS codes in Hamming spaces (see [23]) can be applied in our investigation.

Corollary 31. *The following statements hold:*

1. $K_{qr}^{RT}(m,s,1) \leqslant q^{ms-1} K_r^{RT}(m,s,1)$.

2. *For q a prime power and $ms = q+1$,* $K_{qr}^{RT}(m,s,2) \leqslant q^{q-1} K_r^{RT}(m,s,2)$.

Proof. Let C be a subset of $\mathbb{Z}_q^{ms}$. If C is an MDS code in the Hamming space $\mathbb{Z}_q^{ms}$, then C is also an MDS code in the RT space $\mathbb{Z}_q^{ms}$, according to [17, Theorem 12] (or [13, Corollary 4.2]). This simple but import fact allows us to apply Theorem 30 combined with the well-known classes of MDS codes below:

1. For $n \geqslant 3$, the set $C = \{(c_1, \ldots, c_n) \in \mathbb{Z}_q^n : c_1 + \ldots + c_n \equiv 0 \ (mod\ q)\}$ is an MDS code in the Hamming space $\mathbb{Z}_q^n$ with minimum distance 2.

2. Let C be a Hamming code with length $q+1$, dimension $q-1$, and minimum distance 3. We refer to [23] for the construction of this code.

□

For $s = 1$, Theorem 30 yields the classical relation $K_{qr}(m,1) \leqslant q^{m-1} K_r(m,1)$, obtained initially by Stanton et al. [25] and rediscovered by Blokhuis and Lam [3, Theorem 3.1].

Rosenbloom and Tsfasman [22] built the first classes of MDS codes in RT spaces which are not MDS in Hamming spaces. These codes are known as Reed-Solomon m-codes, since the construction is fully inspired by classical Reed-Solomon codes. Theorems 26 and 30 imply the following bounds.

Corollary 32. *Given q a prime power and positive integer such that $s \leqslant q$, and $a \leqslant qs$,*

1. $K_{qr}^{RT}(q, s, qs-(a+1)) \leqslant q^{a+1} K_r^{RT}(q, s, qs-(a+1))$.

2. $K_{qr}^{RT}(q+1, s, (q+1)s-(a+1)) \leqslant q^{a+1} K_r^{RT}(q+1, s, (q+1)s-(a+1))$.

Example 33. Given $q = 3$, $r = 2$, and $a = 2$, Corollary 32.2 and Example 9 produce $K_6^{RT}(4,2,5) \leqslant 27 K_2^{RT}(4,2,5) \leqslant 81$, improving significantly the trivial upper bound 216.

We conclude our results with an immediate application of Theorem 30 and Theorem 27.

Corollary 34. *Suppose that there exists an MDS code C in the Hamming space $\mathbb{Z}_q^s$ with $d_H(C) = s-1$. For any $2 \leqslant m \leqslant s$,*

$$K_{qr}^{RT}(m, s, ms-2) \leqslant q^2 K_r^{RT}(m, s, ms-2).$$

Example 35. Let us illustrate this bound:

1. Example 29 states that there exists an MDS code in the RT space $\mathbb{Z}_2^9$ with minimum distance 8. Theorem 13 implies $K_2^{RT}(3,3,7) = 2$, therefore $K_4^{RT}(3,3,7) \leqslant 2^{9-7} K_2^{RT}(3,3,7) = 8$, which is better than the trivial upper bound 16.

2. There exists an MDS code C in the Hamming space $\mathbb{Z}_6^3$ with minimum distance $d_H(C) = 2$, according to [23, Theorem 3]. Corollary 34 implies $K_{12}^{RT}(3,3,7) \leqslant 6^{9-7} K_2^{RT}(3,3,7) = 72$, since $K_2^{RT}(3,3,7) = 2$ by Theorem 13. This bound improves the trivial upper bound 144.

As mentioned in this section, MDS codes may give good bounds for covering codes in RT spaces. Some classes of codes with distance properties close to MDS codes have been studied, see near MDS code in [2] for instance. It would be interesting to investigate how these classes can be applied to covering codes in RT spaces.

8 Tables

We finally present tables of lower and upper bounds on $K_2^{RT}(m,s,R)$ for "small" values of m, s and R. We do not take account the case $s = 1$ because this class corresponds to the classical numbers $K_q(m,R)$. The case $m = 1$ is omitted too, since its numbers are completely determined in Theorem 8. Updated tables by Kéri [15] are very useful for constructing some columns of our tables.

A few conventions are adopted. In the tables, the unmarked lower and upper bounds are derived from Proposition 7 or Proposition 6, respectively. When the bound is sharp, a capital letter at the right side explains the reason. When an upper bound is improved, we use a lower case letter at the right side of the upper bound, according to the keys in Table 2.

Table 2: Keys

a	Example 9	f	Theorem 23
b	Corollary 16	A	Example 10
c	Proposition 17	B	Theorem 13
d	Proposition 18	C	Corollary 19
e	Proposition 21	D	Corollary 22

Table 3: $K_2^{RT}(2,s,R)$

$s\backslash R$	1	2	3	4	5	6	7	8
2	6-8	3 A	2	1				
3	22-32	8-12 d	4-8	3 D	2	1		
4	86-128	32-48 d	13-32	6-12 d	4-8	3 D	2	1
5	342-512	128-192 d	52-128	22-48 d	10-32	6-12 d	4-8	3 D

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Table 4: $K_2^{RT}(3, s, R)$

$s\backslash R$	1	2	3	4	5	6	7
2	16 C	5-12 b	3-8	2 B	2	1	
3	128 C	40-96 b	14-64	7-16 d	4-8 f	2-8	2 B
4	1024 C	316-768 b	108-512	40-128 d	19-96 d	10-32 f	5-16 d

Table 5: $K_2^{RT}(4, s, R)$ for $R = 1, \ldots, 5$

$s\backslash R$	1	2	3	4	5
2	52-64 b	14-32 d	6-16 c	3-8 b	2-3 a
3	820-1024 b	216-512 d	66-256 d	26-128 b	12-48 d
4	13108-16384 b	3450-8192 d	1041-4096 d	342-2048 b	135-768 d

Table 6: $K_2^{RT}(4, s, R)$ for $R = 6, \ldots, 13$

$s\backslash R$	6	7	8	9	10	11	12	13
2	2 B	2	1					
3	6-24 f	4-16 b	3-7 f	2-3 e	2 B	2	1	
4	58-384 d	27-192 f	14-112 d	8-48 d	5-23 f	3-16 c	2-7 f	2-3 e

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