# Generalized Stirling permutations and forests: Higher-order Eulerian and Ward numbers

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**Abstract**

We define a new family of generalized Stirling permutations that can be interpreted in terms of ordered trees and forests. We prove that the number of generalized Stirling permutations with a fixed number of ascents is given by a natural three-parameter generalization of the well-known Eulerian numbers. We give the generating function for this new class of numbers and, in the simplest cases, we find closed formulas for them and the corresponding row polynomials. By using a non-trivial involution our generalized Eulerian numbers can be mapped onto a family of generalized Ward numbers, forming a Riordan inverse pair, for which we also provide a combinatorial interpretation.

**Keywords:** generalized Stirling permutations; increasing trees and forests; generalized Eulerian numbers; generalized Ward numbers

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# 1 Introduction

Stirling permutations of order $n$ are permutations of the multiset $\{1^2, 2^2, \ldots, n^2\}$ such that, for each $1 \leqslant r \leqslant n$, the elements appearing between two occurrences of $r$ are at least $r$ [16]. Given a Stirling permutation $\rho = r_1 r_2 \ldots r_{2n}$, the index $i$ will be called an ascent of $\rho$ if $r_i < r_{i+1}$. The number of Stirling permutations of order $n$ with exactly $k$ ascents is given by second-order Eulerian numbers $B_{n,k}$ [16]. Second order Eulerian numbers are closely related to Ward numbers $W_{n,k}$ [30], [22, entry A134991]. They form an *inverse pair* in the sense of Riordan [27] (see[28], [22, entry A008517]):

$$W_{n,k} = \sum_{j=0}^{k} \binom{n-j}{n-k} B_{n,j}\,, \tag{1a}$$

$$B_{n,k} = \sum_{j=0}^{k} (-1)^{k-j} \binom{n-j}{k-j} W_{n,j}\,. \tag{1b}$$

We will use Eq. (1a) to provide a new combinatorial interpretation of Ward numbers in terms of Stirling permutations.

Stirling permutations and Eulerian numbers have been generalized to multisets of the form $\{1^\nu, 2^\nu, \ldots, n^\nu\}$ with $\nu \in \mathbb{N}$ by Gessel [15] (as cited by Park [23]) and Park [23, 24]. Park [25] related the Stirling permutations of these multisets to some generalized Stirling numbers. (See also [19, Theorem 1] and [20, Theorem 2.1].) Brenti considered Stirling permutations of the more general multiset $\{1^{\nu_1}, 2^{\nu_2}, \ldots, n^{\nu_n}\}$ with $\nu_i \in \mathbb{N}$ $(1 \leqslant i \leqslant n)$ in the context of Hilbert polynomials [3]; and in relation to increasing trees by Kuba and Panholzer [20]. The particular case $\{1^\nu, 2^{\nu+2}, \ldots, n^{\nu+2}\}$ was also studied by Janson *et al.* [19]. Stirling permutations also appear in the framework of context-free grammars [7] (see Ref. [8] for more recent literature).

The purpose of the paper is to introduce and study natural generalizations of the Stirling permutations considered by Gessel and Stanley, Park and other authors [16, 23]. We will find bijections between these generalized Stirling permutations and ordered trees [23, 19] and forests, providing a graph-theoretic interpretation of these objects. The combinatorial numbers that count such Stirling permutations with a fixed number of ascents are natural generalizations of the Eulerian numbers. Some particular numbers of this class have been considered in other contexts [4, 12, 5, 20] but, to our knowledge, most of them have not appeared before in the literature. There are indeed other generalizations of the Eulerian numbers that do not fall in the above class: e.g., the $r$–Eulerian numbers [26, 14, 21, 2], or the numbers $A(r, s \mid \alpha, \beta)$ due to Carlitz and Scoville [6].

In all our cases, these Eulerian numbers satisfy two-parameter linear recurrence relations that can be studied in an efficient way by using generating function techniques [1]. With the help of these methods, we define a family of generalized Ward numbers, and get closed expressions for them in terms of generalized Eulerian numbers, in the form of inverse pairs similar to Eqs. (1). These relations provide a simple combinatorial interpretation for the generalized Ward numbers in the present context.

## 2 Generalized Stirling permutations

It is useful for our purposes to introduce several definitions based on the $\nu$–Stirling permutations of order $n$ discussed by Park [23]:

**Definition 1.** Let $\nu$ be a positive integer and $X = \{x_1 < x_2 < \cdots < x_n\}$ be a totally ordered set of cardinality $n$. A $(\nu, X)$–Stirling permutation is a permutation of the multiset $\{x_1^\nu, x_2^\nu, \ldots, x_n^\nu\}$ such that, for each $1 \leqslant j \leqslant n$, the elements occurring between two occurrences of $x_j$ are, at least, $x_j$.

**Remarks.**

1. This definition implies that the elements occurring between two consecutive occurrences of $x_j$ are greater than $x_j$. As a consequence of this, the $\nu$ occurrences of $x_n$ have to go together.

2. If $X = [n]$, then the $(\nu, [n])$–Stirling permutations are equivalent to the canonical $\nu$–Stirling permutations of order $n$. In Definition 5, the $X$ will correspond to different subsets of $[n]$.

3. If $X = \varnothing$ the unique $(\nu, \varnothing)$–Stirling permutation is the empty permutation.

**Definition 2.** Let $\nu, t$ be positive integers, $X = \{x_1 < x_2 < \cdots < x_n\}$ be a totally ordered set of cardinality $n$, and consider $x_0 = 0 < x_1 < x_2 < \cdots < x_n$. A $(\nu, t, X)$–Stirling permutation is a permutation of the multiset $\{0^t, x_1^\nu, x_2^\nu, \ldots, x_n^\nu\}$ such that for each $0 \leqslant j \leqslant n$ the elements occurring between two occurrences of $x_j$ are at least $x_j$.

**Remarks.**

1. If $t = 0$, a $(\nu, 0, X)$–Stirling permutation is just a $(\nu, X)$–Stirling permutation.

2. If $X = \varnothing$ the unique $(\nu, t, \varnothing)$–Stirling permutation is the permutation $0^t$.

3. The number of $(\nu, t, X)$–Stirling permutations is $\displaystyle\prod_{k=0}^{|X|-1} (k\nu + t + 1)$.

In order to count generalized Stirling permutations with a fixed number of ascents, we introduce a three-parameter generalization of the standard Eulerian numbers, that we will refer to as the *$\nu$-order $(s,t)$-Eulerian numbers*:

**Definition 3.** Let $\nu, s \geqslant 1$ and $t \geqslant 0$ be integers. The $\nu$-order $(s,t)$-Eulerian numbers $\left\langle{n \atop k}\right\rangle^{(\nu)}_{(s,t)}$ are defined as those satisfying the recurrence

$$\left\langle{n \atop k}\right\rangle^{(\nu)}_{(s,t)} = (k+s)\left\langle{n-1 \atop k}\right\rangle^{(\nu)}_{(s,t)} + (\nu n - k + t + 1 - \nu)\left\langle{n-1 \atop k-1}\right\rangle^{(\nu)}_{(s,t)} + \delta_{k0}\delta_{n0}\,, \qquad (2)$$

with the additional conditions $\left\langle{n \atop k}\right\rangle^{(\nu)}_{(s,t)} = 0$ if $n < 0$ or $k < 0$.

**Remark.** The values of $\nu, s, t$ do not have to be integers as $\left\langle {n \atop k} \right\rangle_{(s,t)}^{(\nu)}$ is obviously a polynomial in these three parameters. However, we have restricted their ranges to make contact with their combinatorial interpretation.

**Proposition 4.** *The number of* $(\nu, t, [n])$*–Stirling permutations with* $k$ *ascents is equal to* $\left\langle {n \atop k} \right\rangle_{(1,t)}^{(\nu)}$.

*Proof.* This is just a generalization of the proof of Eq. (6.1) in [12]. Let $J_{\nu,t}(n,k)$ be the number of $(\nu, t, [n])$–Stirling permutations with $k$ ascents. We want to show that $J_{\nu,t}(n,k) = \left\langle {n \atop k} \right\rangle_{(1,t)}^{(\nu)}$ by induction on $n$.

The case $n = 0$ is trivial: $J_{\nu,t}(0,k) = \delta_{0,k} = \left\langle {0 \atop k} \right\rangle_{(1,t)}^{(\nu)}$, as there is a unique permutation of this type (the empty permutation).

Let us assume that $J_{\nu,t}(n-1,k) = \left\langle {n-1 \atop k} \right\rangle_{(1,t)}^{(\nu)}$ for all $0 \leqslant k \leqslant n-1$. We have to insert now the block $n^\nu$. This will leave the number of ascents unchanged, or increase it by one unit. We have then only two choices: (1) start from a $(\nu, t, [n-1])$–Stirling permutation with $k$ ascents, or (2) start from a $(\nu, t, [n-1])$–Stirling permutation with $k-1$ ascents. In the first case, we can place the block $n^\nu$ at the beginning of the permutation or insert it at any of the $k$ ascents. In the second case, we can insert the block $n^\nu$ at any of the $\nu(n-1) + t - (k-1)$ non-ascent places. Hence

$$J_{\nu,t}(n,k) \;=\; (k+1) J_{\nu,t}(n-1,k) + (\nu n - k + t + 1 - \nu) J_{\nu,t}(n-1,k-1)\,.$$

This equation completes the proof. ☐

**Remarks.**

1. If $(s,t) = (1,0)$, these numbers reduce to the ordinary Eulerian numbers for $\nu = 1$, to the second-order Eulerian numbers for $\nu = 2$ [16], and to the third-order Eulerian numbers for $\nu = 3$ [22, entry A219512].

2. If $\nu = 2$ and $(s,t) = (1,t)$, these numbers correspond to the generalization by Carlitz [4, 5] and Dillon and Roselle [12].

**Definition 5.** Let us fix integers $\nu \geqslant 1$ and $t \geqslant 0$, and a generalized ordered partition $\boldsymbol{t} = (t_1, \ldots, t_s)$ of $t$ with $s \geqslant 1$ parts (and $t_i \geqslant 0$). A $(\nu, \boldsymbol{t}, n)$–Stirling permutation is a sequence $\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots, \rho_s)$, of length $s$, such that each entry $\rho_i$ is a $(\nu, t_i, X_i)$–Stirling permutation for some generalized ordered partition $(X_1, X_2, \ldots, X_s)$ of $[n]$ (where we allow that some of the $X_i$ are the empty set).

**Remarks.**

1. If $\boldsymbol{t} = (t)$ (i.e., $s = 1$), the $(\nu, \boldsymbol{t}, n)$–Stirling permutations reduce to the $(\nu, t, n)$–Stirling permutations.

2. If $n = 0$, there is a single $(\nu, \boldsymbol{t}, 0)$–Stirling permutation: $(0^{t_1}, 0^{t_2}, \ldots, 0^{t_s})$, where in the cases with $t_i = 0$ we have an empty entry.

**Theorem 6.** *The number of $(\nu, \boldsymbol{t}, n)$–Stirling permutations with $k$ ascents is equal to* $\left\langle {n \atop k} \right\rangle_{(s,t)}^{(\nu)}$.

*Proof.* Let $J_{\nu,\boldsymbol{t}}(n,k)$ be the number of $(\nu, \boldsymbol{t}, n)$–Stirling permutations with $k$ ascents. We want to show that $J_{\nu,\boldsymbol{t}}(n,k) = \left\langle {n \atop k} \right\rangle_{(s,t)}^{(\nu)}$ by induction on $n$.

The case $n = 0$ is trivial: $J_{\nu,\boldsymbol{t}}(0,k) = \delta_{0,k} = \left\langle {0 \atop k} \right\rangle_{(s,t)}^{(\nu)}$, as there is a unique permutation of this type: $(0^{t_1}, 0^{t_2}, \ldots, 0^{t_s})$.

Let us assume that $J_{\nu,\boldsymbol{t}}(n-1,k) = \left\langle {n-1 \atop k} \right\rangle_{(s,t)}^{(\nu)}$ for all $0 \leqslant k \leqslant n-1$. Then, as explained in the proof of Proposition 4, we have two choices to insert the block $n^\nu$ in a $(\nu, \boldsymbol{t}, n-1)$–Stirling permutation with $k$ ascents: (1) start from a $(\nu, \boldsymbol{t}, n-1)$–Stirling permutation with $k$ ascents and insert the block at the beginning of the $s$ entries or at any of the $k$ ascents; or (2) start from a $(\nu, \boldsymbol{t}, n-1)$–Stirling permutation with $k-1$ ascents, and insert the block at any of the $\nu(n-1) + t - (k-1)$ non-ascent places. Then,

$$J_{\nu,\boldsymbol{t}}(n,k) \;=\; (k+s) J_{\nu,\boldsymbol{t}}(n-1,k) + (\nu n - k + t + 1 - \nu) J_{\nu,\boldsymbol{t}}(n-1,k-1)\,.$$

This completes the proof. □

**Remark.** The number of $(\nu, \boldsymbol{t}, n)$–Stirling permutations with $k$ ascents does depend on $\boldsymbol{t}$ but only through $t$ and $s$. This is also true for the number of $(\nu, \boldsymbol{t}, n)$–Stirling permutations that is given by

$$\prod_{k=0}^{n-1} (k\nu + t + s)\,. \tag{3}$$

## 3 Increasing trees and forests

Gessel [15], Park [23], and Janson, Kuba and Panholzer [19] discussed the bijection between $\nu$–Stirling permutations and the class of increasing trees. In this section we discuss generalizations of these results to the class of $(\nu, t, [n])$–Stirling permutations introduced above. These latter ones are a particular case of the Stirling permutations of the multiset $\{1^{\nu_1}, 2^{\nu_2}, \ldots, n^{\nu_n}\}$ discussed by Kuba and Panholzer [20]. For the $(\nu, \boldsymbol{t}, n)$–Stirling permutations, we introduce a similar construction in terms of forests.

**Definition 7.** Let $X = \{x_1 < \cdots < x_n\}$ be a totally ordered set. An increasing $X$–tree is a rooted tree with the internal vertices labelled by the elements of $X$ in such a way that the node labelled $x_1$ is distinguished as the root and such that, for each $2 \leqslant i \leqslant n$, the labels of the nodes in the unique path from the root to the node labelled $x_i$ form an increasing sequence. A generalized increasing $X$–tree is an increasing $X_0$–tree with $|X|+1$ internal vertices labelled by the elements of the set $X_0 = \{x_0 = 0 < x_1 < x_2 < \cdots < x_n\}$.

**Remark.** The family of generalized increasing $X$–trees is bijective with the family of increasing $[|X|+1]$–trees.

**Definition 8.** For an integer $d \geqslant 2$, $d$-ary increasing $X$–trees are increasing $X$–trees where each internal node has $d$ labelled positions for children. Equivalently, for integers $d \geqslant 2$, $d_0 \geqslant 1$, $(d, d_0)$-ary increasing $X$–trees are generalized increasing $X$–trees where the root $x_0 = 0$ has $d_0$ labelled positions for children, and any non-root internal node $x_i$ $(1 \leqslant i \leqslant n)$ has $d$ labelled positions for children.

**Remarks.**

1. A $d$-ary increasing $X$–tree has $d|X|$ edges, $|X|$ internal nodes with outdegree equal to $d$, and $(d-1)|X|+1$ external nodes.

2. A $(d, d_0)$-ary increasing $X$–tree has $d|X| + d_0$ edges, a root with outdegree equal to $d_0$, $|X|$ internal nodes with outdegree equal to $d$, and $(d-1)|X|+d_0$ external nodes.

3. The family of $(d, 1)$-ary increasing $X$–trees is bijective with the $d$-ary increasing $[|X|]$–trees. The family of $(d, d)$-ary increasing $X$–trees is bijective with the $d$-ary increasing $[|X| + 1]$–trees.

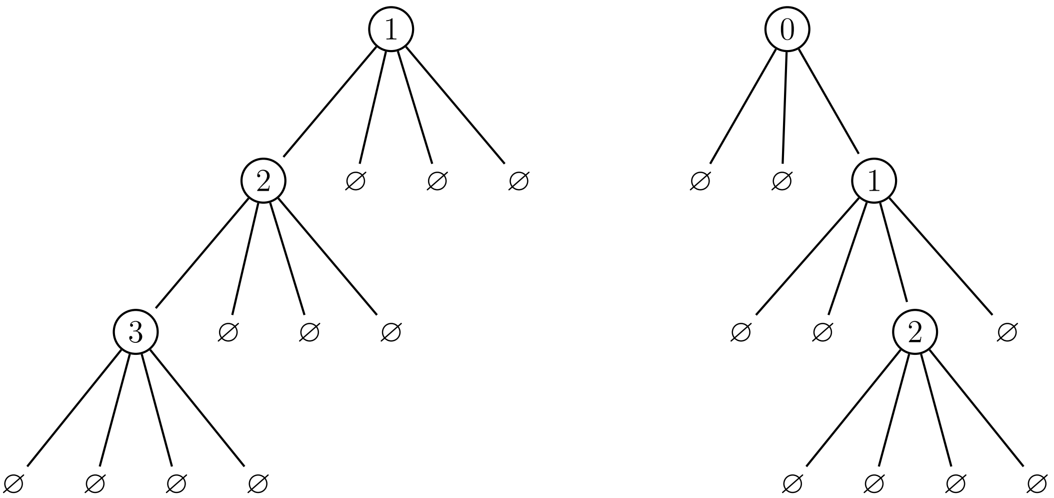
The following theorem relates the family of $(\nu+1, t+1)$–ary increasing $[n]$–trees and the $(\nu, t, [n])$–Stirling permutations. The authors independently derived this result, and later discovered that this result was already proved in Ref. [20, Theorem 2.1]. See also [19, Theorem 1] for a detailed proof of a related statement.

**Theorem 9.** *Let $\nu \geqslant 1$, $t \geqslant 0$ be integers. The family of $(\nu+1, t+1)$–ary increasing $[n]$–trees is in natural bijection with $(\nu, t, [n])$–Stirling permutations.*

*Proof.* Our proof is a generalization of Gessel's theorem (see [23]) that relies on the argument presented in [29] for ordinary permutations. Let $\rho$ be any word on the alphabet $\{x_0 < x_1 < \cdots < x_n\}$ with possible repeated letters. Let us define a planar tree $T(\rho)$ as follows: If $\rho = \varnothing$, then $T(\rho) = \varnothing$; if $\rho \neq \varnothing$, then $\rho$ can be factorized uniquely in the form $\rho = \rho_1 i \rho_2 i \cdots i \rho_{\nu_i+1}$ where $i$ is the least element (letter) of $\rho$ and $\nu_i$ its multiplicity. Let $i$ be the root of $T(\rho)$ and $T(\rho_1)$, $T(\rho_2)$,. . . , $T(\rho_{\nu_i+1})$ the subtrees (from left to right) obtained by removing $i$. This yields an inductive definition of $T(\rho)$. Notice that the outdegree of an internal vertex $i$ is $\nu_i+1$. Notice also that when $\rho$ corresponds to a generalized Stirling permutation, if $j$ is a letter of $\rho_k$, then $j$ does not belong to any $\rho_l$ for $l \neq k$. □

**Remark.** See Figure 1 for some simple examples of the Stirling permutations and their associated trees.

**Definition 10.** Let $n \geqslant 0$, $s \geqslant 1$ be integers. An $(s, [n])$–forest is an ordered forest $\boldsymbol{F} = (T_1, \ldots, T_s)$ composed by $s$ labelled generalized increasing $X_i$–trees $T_i$, for some generalized ordered partition $(X_1, \ldots, X_n)$ of $[n]$ (where we allow $X_i = \varnothing$). Given $\mathbf{u} = (u_1, \ldots, u_s)$ with $u_i \geqslant 1$ integers, a $(d, \mathbf{u})$–ary increasing $(s, [n])$–forest $\boldsymbol{F} = (T_1, \ldots, T_s)$ is an $(s, [n])$–forest such each $T_i$ is a $(d, u_i)$-ary increasing $X_i$-tree, for some generalized partition $(X_1, \ldots, X_s)$ of $[n]$.



(a) (b)

Figure 1: (a) The $(3,[3])$–Stirling permutation 333222111 and its corresponding 4–ary increasing $[3]$–tree. (b) The $(3,2,[2])$–Stirling permutation $0\underline{0}1\underline{1}2221$ and its corresponding $(4,3)$–ary increasing $[2]$–tree. This permutation has two ascents at indices 2 and 4. These ascents are underlined in the permutation for clarity.

**Theorem 11.** *Let $\nu, s \geqslant 1$, $t \geqslant 0$ be integers, $\boldsymbol{t} = (t_1, \ldots, t_s)$ a generalized ordered partition of $t$ ($t_i \geqslant 0$), and $\mathbf{1} = (1, \ldots, 1)$. The family of $(\nu+1, \boldsymbol{t}+\mathbf{1})$–ary increasing $(s,[n])$–forests is in natural bijection with the class of $(\nu, \boldsymbol{t}, n)$–Stirling permutations.*

*Proof.* This is a straightforward generalization of Theorem 9. See Figure 2 for a concrete example of this class of forests. ☐

**Remarks.**

1. It is important to stress that $k$ ascents in a Stirling permutation correspond to $k$ "non-leftmost" internal nodes in the corresponding tree/forest representation. See Figures 1 and 2. Hereafter we will say that a tree/forest has an ascent if the corresponding generalized Stirling permutation has an ascent.

2. Park [23] gives two bijections for the class of $(\nu,[n])$–Stirling permutations: one in terms of $(\nu+1)$-ary increasing trees, and another one in terms of (ordered) forests of increasing trees. We have adapted the former for the class of $(\nu, t, [n])$–Stirling permutations, but we will not use the latter in the present paper.

## 4 The $\nu$-order $(s,t)$-Eulerian numbers

We study in detail some properties of the $\nu$-order $(s,t)$-Eulerian numbers introduced in Definition 3, and whose combinatorial interpretations have been discussed in Theorems 6

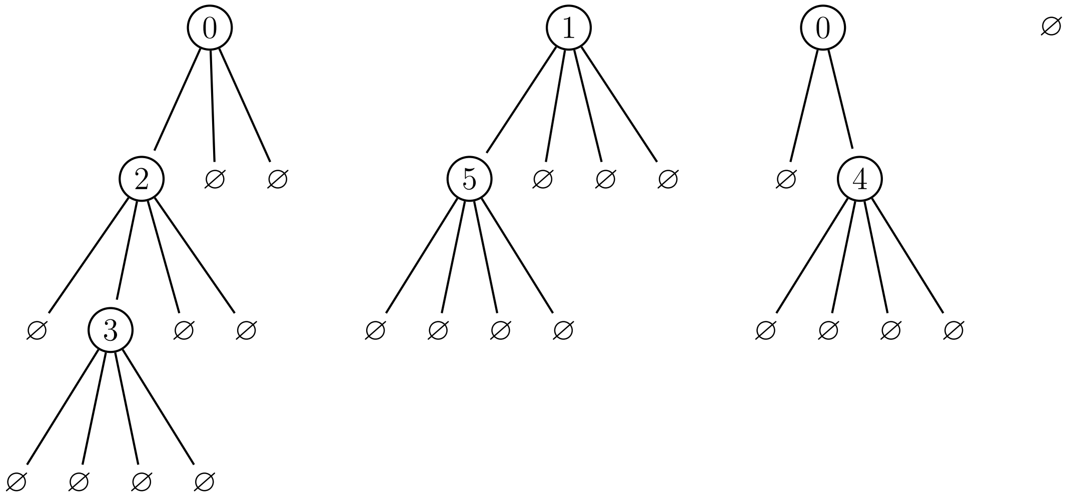


Figure 2: The $(3, \boldsymbol{t}, 5)$–Stirling permutation $(\underline{2}3332200, 555111, \underline{0}444, \varnothing)$ corresponding to $\boldsymbol{t} = (2, 0, 1, 0)$ and the generalized partition $(\{2, 3\}, \{1, 5\}, \{4\}, \varnothing)$ of $[5]$. We show the corresponding $(4, \boldsymbol{t} + \mathbf{1})$–ary $(4, [5])$–forest $\boldsymbol{F} = (T_1, T_2, T_3, T_4)$.

and 11 (see Remark 1 after Theorem 11). In this section, $s, t$ will be considered indeterminate parameters. These numbers satisfy the recurrence relation (2) which is a particular case of the one analyzed in [1]. By using a generating-function approach, that yields a first-order linear partial differential equation which is solved with the method of characteristics [1], we obtain the exponential generating function (EGF)

$$F^{(\nu)}(x, y; s, t) = \sum_{n,k \geqslant 0} \left\langle {n \atop k} \right\rangle^{(\nu)}_{(s,t)} x^k \frac{y^n}{n!} \tag{4}$$

for the $\nu$-order $(s, t)$-Eulerian numbers is given by [1, Section A.1.5]:

$$F^{(\nu)}(x, y; s, t) \;=\; \left( \frac{T_\nu \left( e^{y(1-x)^\nu} \, T_\nu^{-1}(x) \right)}{x} \right)^{s} \left( \frac{1-x}{1 - T_\nu \left( e^{y(1-x)^\nu} T_\nu^{-1}(x) \right)} \right)^{s+t}, \tag{5}$$

where $T_\nu$ ($\nu \in \mathbb{N}$) is a one-parameter family of functions given by

$$T_\nu^{-1}(z) \;=\; z\, e^{Q_\nu(z)}, \quad \text{where} \quad Q_\nu(z) \;=\; \sum_{k=1}^{\nu-1} \binom{\nu-1}{k} \frac{(-z)^k}{k} \,. \tag{6}$$

For $\nu = 1$, $T_1 = \mathbf{1}$ is the identity function, and for $\nu = 2$, $T_2$ is the tree function $T_2 = T$ [10, 11].

The *$\nu$-order $(s, t)$-Eulerian polynomials* are defined as:

$$P_n^{(\nu)}(x; s, t) \;=\; \sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle^{(\nu)}_{(s,t)} x^k \tag{7}$$

and satisfy that $P_n^{(\nu)}(1;s,t)$ is given by the number of $(\nu,t,n)$-Stirling permutations stated in (3). They can be computed by using Theorem 4.1 and Eq. (4.4) of Ref. [1]:

$$P_n^{(\nu)}(x;s,t) = \frac{(1-x)^{s+t+\nu n}}{x^s}\,\frac{n!}{2\pi i}\int_C \frac{z^{s-1}}{(1-z)^{s+t+1-\nu}}\left[\log\frac{ze^{Q_\nu(z)}}{xe^{Q_\nu(x)}}\right]^{-n-1} dz \tag{8a}$$

$$= \frac{(1-x)^{s+t+\nu n}}{x^s}\,\lim_{z\to x}\frac{\partial^n}{\partial z^n}\left(\frac{z^{s-1}(z-x)^{n+1}}{(1-z)^{s+t+1-\nu}}\left[\log\frac{ze^{Q_\nu(z)}}{xe^{Q_\nu(x)}}\right]^{-n-1}\right), \tag{8b}$$

where $C$ is a closed simple curve of index $+1$ surrounding only the singularity at $z=x$ in the complex $z$-plane.

A Rodrigues-like formula for the $\nu$-order $(s,t)$-Eulerian polynomials can also be obtained from the integral (8a) by performing the change of variables $ze^{Q_\nu(z)} = e^u$ and $xe^{Q_\nu(x)} = e^v$. Therefore, $z = T_\nu(e^u)$ and $x = T_\nu(e^v)$. We immediately obtain from (8a):

$$P_n^{(\nu)}(T_\nu(e^v);s,t) \;=\; \frac{(1-T_\nu(e^v))^{s+t+\nu n}}{T_\nu(e^v)^s}\,\frac{d^n}{dv^n}\frac{T_\nu(e^v)^s}{(1-T_\nu(e^v))^{s+t}}, \tag{9}$$

where actual computations are facilitated by the fact that the derivative of $T_\nu(x)$ is given in closed form by the expression

$$T_\nu'(x) \;=\; \frac{T_\nu(x)}{x\,(1-T_\nu(x))^{\nu-1}}. \tag{10}$$

An equivalent representation of $P_n^{(\nu)}(x;s,t)$ can be obtained directly from the EGF (5) after performing the change of variables $y \mapsto u = (1-x)^\nu y$:

$$P_n^{(\nu)}(x;s,t) \;=\; n!\,\frac{(1-x)^{s+t+\nu n}}{x^s}\,[u^n]\frac{\left(T_\nu\left(e^u\,T_\nu^{-1}(x)\right)\right)^s}{\left(1-T_\nu\left(e^u\,T_\nu^{-1}(x)\right)\right)^{s+t}}. \tag{11}$$

We now illustrate the use of the previous results to derive explicit expressions for the $(s,t)$-Eulerian numbers and the second order $(s,t)$-Eulerian numbers. In fact, one can use similar techniques to obtain formulas for higher-order $(s,t)$-Eulerian numbers, although these computations are more involved.

## 5 The $(s,t)$-Eulerian numbers

When $\nu = 1$, we will employ the traditional notation $A_n^{(s,t)}(x) = P_n^{(1)}(x;s,t)$. By using (11), we immediately get

$$A_n^{(s,t)}(x) \;=\; (1-x)^{s+t+n}\,n!\,[u^n]\frac{e^{s\,u}}{(1-x\,e^u)^{s+t}}. \tag{12}$$

This formula allows us to obtain the following closed expressions for $A_n^{(s,t)}$:

$$A_n^{(s,t)}(x) \;=\; (1-x)^{s+t+n} \sum_{j\geqslant 0} \frac{(s+t)^{\overline{j}}}{j!}\,(s+j)^n\,x^j \tag{13}$$

$$\;=\; \sum_{k\geqslant 0} x^k \sum_{j=0}^{k} (-1)^{k-j}\,\frac{(n+s+t)^{\underline{k-j}}}{j!\,(k-j)!}\,(s+t)^{\overline{j}}\,(s+j)^n\,, \tag{14}$$

where $x^{\overline{j}} = x(x+1)\cdots(x+j-1)$ and $x^{\underline{j}} = x(x-1)\cdots(x-j+1)$ are the raising and falling factorials, respectively. From (13), we easily obtain

**Proposition 12.** *The $(s,t)$-Eulerian polynomials $A_n^{(s,t)}$ satisfy the relation*

$$\frac{x\,A_n^{(s,t)}(x)}{(1-x)^{n+s+t}} \;=\; \sum_{k\geqslant 1} \frac{(s+t)^{\overline{k-1}}}{(k-1)!}\,(k+s-1)^n\,x^k\,, \tag{15}$$

*for any $n \geqslant 0$ and arbitrary parameters $s,t$.*

This proposition generalizes the well-known formulas for the ordinary Eulerian polynomials $A_n = A_n^{(1,0)}$:

$$\frac{x\,A_n(x)}{(1-x)^{n+1}} \;=\; \sum_{k\geqslant 1} k^n\,x^k\,, \tag{16}$$

and for the Eulerian polynomials with the traditional indexing $A_n^{(0,1)}$ [2, Theorem 1.21]:

$$\frac{A_n^{(0,1)}(x)}{(1-x)^{n+1}} \;=\; \sum_{k\geqslant 0} k^n\,x^k\,. \tag{17}$$

A closed expression for $\left\langle {n \atop k} \right\rangle_{(s,t)}$ can be obtained from (14) to conclude that

**Theorem 13.** *The generalized $(s,t)$-Eulerian numbers are equal to*

$$\left\langle {n \atop k} \right\rangle_{(s,t)} \;=\; \left\langle {n \atop k} \right\rangle_{(s,t)}^{(1)} \;=\; \frac{1}{k!}\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (n+s+t)^{\underline{k-j}}\,(s+t)^{\overline{j}}\,(s+j)^n \tag{18}$$

*for $n \geqslant 0$ and $0 \leqslant k \leqslant n$.*

**Remarks.**

1. It is obvious in Eq. (18) that the numbers $\left\langle {n \atop k} \right\rangle_{(s,t)}$ are polynomials in both parameters $s,t$.

2. The ordinary Eulerian numbers with the standard [17, Eq. (6.38)] and the traditional [9] ordering are respectively given by

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop k} \right\rangle_{(1,0)} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j+1)^n , \tag{19a}$$

$$A(n,k) = \left\langle {n \atop k} \right\rangle_{(0,1)} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j)^n = \left\langle {n \atop k-1} \right\rangle . \tag{19b}$$

3. The *shifted* $r$-Eulerian numbers corresponding to $(s,t) = (r,0)$ are a natural generalization of the $r$-Eulerian numbers [26, p. 215], [14, Chapter II, p. 17], [21], [2, Problems 17 and 18, p. 38] that fit in the framework of the problem discussed in Ref. [1].

4. Notice that the $(s,-s)$-Eulerian numbers take the simple form (cf. (5)):

$$\left\langle {n \atop k} \right\rangle_{(s,-s)} = (-1)^k \binom{n}{k} s^n . \tag{20}$$

# 6 The second order $(s,t)$-Eulerian numbers

When $\nu = 2$, it is customary to write $B_n^{(s,t)}(x) = P_n^{(2)}(x;s,t)$. By using (11) we immediately get

$$B_n^{(s,t)}(x) = n! \, \frac{(1-x)^{s+t+2n}}{x^s} \, [u^n] \frac{\left(T\left(T^{-1}(x)\, e^u\right)\right)^s}{\left(1 - T\left(T^{-1}(x)\, e^u\right)\right)^{s+t}} , \tag{21}$$

where $T$ is the tree function [10, 11]. For $|z| < e^{-1}$, this function is given by the power series:

$$T(z) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} \, z^n . \tag{22}$$

It satisfies that $T(z)\exp(-T(z)) = z$ (or equivalently, $T^{-1}(z) = ze^{-z}$), and it is closely related to the Lambert $W$ function [10, 11]: $T(z) = -W(-z)$.

Using (21), it is not very difficult to obtain an explicit closed form for both the second-order $(s,t)$-Eulerian polynomials and the second-order $(s,t)$-Eulerian numbers. By expanding $(1 - T(\xi))^{-(s+t)}$ in powers of $T(\xi)$, where $\xi = T^{-1}(x)\, e^u = x\, e^{-x+u}$, we get:

$$B_n^{(s,t)}(x) = n! \, \frac{(1-x)^{s+t+2n}}{x^s} \sum_{j=0}^{\infty} \frac{(s+t)^{\overline{j}}}{j!} \, [u^n] T(\xi)^{s+j} . \tag{23}$$

One important property of the tree function (22) is that the Taylor expansion at $z = 0$ of its powers can be computed in closed form [11, Eq. (10)]:

$$T(z)^s = \sum_{k=0}^{\infty} \frac{s\,(k+s)^{k-1}}{k!} \, z^{s+k} . \tag{24}$$

Using this expression in (23) we obtain

$$B_n^{(s,t)}(x) \;=\; (1-x)^{s+t+2n} \sum_{p=0}^{\infty} \frac{x^p}{p!} e^{-x(p+s)} \times \sum_{j=0}^{p} \binom{p}{j} (s+t)^{\overline{j}} \, (s+j) \, (p+s)^{n+p-j-1} \,. \quad (25)$$

From this equation we easily obtain the following proposition, which resembles Proposition 12 for the $(s,t)$-Eulerian polynomials $A_n^{(s,t)}$:

**Proposition 14.** *The second-order $(s,t)$-Eulerian polynomials $B_n^{(s,t)}$ satisfy for any $n \geqslant 0$ and arbitrary parameters $s, t$ the relation*

$$\frac{x e^{x(s-1)} \, B_n^{(s,t)}(x)}{(1-x)^{2n+s+t}} \;=\; \sum_{k \geqslant 1} \frac{\left(x e^{-x}\right)^k}{(k-1)!} \times \sum_{j=0}^{k-1} \binom{k-1}{j} (s+t)^{\overline{j}} \, (s+j) \, (k+s-1)^{n+k-j-2} \,. \quad (26)$$

When $(s,t) = (1,0)$, we get the following relation for the ordinary second-order Eulerian polynomials $B_n(x) = B_n^{(1,0)}(x)$:

$$\frac{x \, B_n(x)}{(1-x)^{2n+1}} \;=\; \sum_{k \geqslant 1} \frac{k^{n+k-1}}{(k-1)!} \left(x e^{-x}\right)^k \,, \quad (27)$$

that resembles Eq. (16) for the ordinary Eulerian polynomials $A_n(x)$. The proof of these results makes use of the following combinatorial identities:

$$1 \;=\; \sum_{j=0}^{n} \binom{n}{j} j! \, j \, \frac{1}{n^{j+1}} \;=\; \sum_{j=0}^{n} \binom{n}{j} (j+1)! \, \frac{1}{(n+1)^{j+1}} \,. \quad (28)$$

A closed expression for the second-order generalized $(s,t)$-Eulerian numbers can be obtained by writing (25) in the form

$$B_n^{(s,t)}(x) \;=\; \sum_{k \geqslant 0} \frac{x^k}{k!} \sum_{r=0}^{k} \binom{k}{r} (s+t+2n)^{\underline{k-r}} \sum_{p=0}^{r} \binom{r}{p} (-1)^{k-p} \times \sum_{j=0}^{p} \binom{p}{j} (s+t)^{\overline{j}} \, (s+j) \, (p+s)^{n+r-j-1} \,, \quad (29)$$

to conclude

**Theorem 15.** *The second-order generalized* $(s,t)$*-Eulerian numbers are equal to*

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle_{(s,t)} = \left\langle {n \atop k} \right\rangle^{(2)}_{(s,t)} = \frac{1}{k!} \sum_{r=0}^{k} \binom{k}{r} (s+t+2n)^{\underline{k-r}} \sum_{p=0}^{r} \binom{r}{p} (-1)^{k-p} \times \sum_{j=0}^{p} \binom{p}{j} (s+t)^{\overline{j}} (s+j)(p+s)^{n+r-j-1} \qquad (30)$$

*for* $n \geqslant 0$ *and* $0 \leqslant k \leqslant n$.

**Remarks.**

1. Again, from Eq. (30) we see that the numbers $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle_{(s,t)}$ are polynomials in both parameters $s,t$.

2. The ordinary second-order Eulerian numbers with the standard [17, Eq. (6.38)] and the traditional [9] ordering are respectively given by

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle_{(1,0)} = \sum_{r=0}^{k} (-1)^{k-r} \binom{1+2n}{k-r} \left\{ {n+r+1 \atop r+1} \right\}, \qquad (31a)$$

$$B_{n,k} = \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle_{(0,1)} = \sum_{r=0}^{k} (-1)^{k-r} \binom{1+2n}{k-r} \left\{ {n+r \atop r} \right\} = \left\langle\!\!\left\langle {n \atop k-1} \right\rangle\!\!\right\rangle, \qquad (31b)$$

   where the numbers $\left\{ {n \atop k} \right\}$ are the standard Stirling subset numbers [17]. The inverse relation of Eq. (31a) is given in [17, Eq. (6.43)].

3. The second-order $(s,-s)$-Eulerian numbers take the form

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle_{(s,-s)} = s \sum_{r=0}^{k} \frac{1}{r!} \binom{2n}{k-r} \sum_{p=0}^{r} \binom{r}{p} (-1)^{k-p} (p+s)^{n+r-1}. \qquad (32)$$

## 7 The $\nu$-order generalized $(s,t)$-Ward numbers

The standard Ward numbers and the second-order Eulerian numbers form an inverse Riordan pair (1b). A natural question is how to generalize these Ward numbers such that they form a Riordan inverse pair with the $\nu$-order $(s,t)$-Eulerian numbers. To achieve this goal, we start by defining:

**Definition 16.** Let $\nu, s \geqslant 1$ and $t \geqslant 0$ be integers. The $\nu$-order generalized $(s,t)$-Ward numbers $W^{(\nu)}(n,k;s,t)$ are defined as those satisfying the recurrence

$$W^{(\nu)}(n,k;s,t) = (k+s)\, W^{(\nu)}(n-1,k;s,t) + (\nu n + k + s + t - 1 - \nu)\, W^{(\nu)}(n-1,k-1;s,t) + \delta_{k0}\delta_{n0}, \qquad (33)$$

with the additional conditions $W^{(\nu)}(n,k;s,t) = 0$ if $n < 0$ or $k < 0$.

The family of $\nu$-order generalized $(s,t)$-Ward numbers is related to the $\nu$-order $(s,t)$-Eulerian numbers by a *non-trivial* involution $F \to \widehat{F}$ that can be derived from the following:

**Proposition 17.** *Let $F(x,y) = F(x,y;\boldsymbol{\mu})$ be the solution of*

$$-(\beta+\beta' x)\, x\, \frac{\partial F}{\partial x} + (1-\alpha\, y - \alpha'\, x\, y)\, \frac{\partial F}{\partial y} \;=\; (\alpha+\gamma+(\alpha'+\beta'+\gamma')\, x)\, F\,, \tag{34}$$

*with parameters $\boldsymbol{\mu} = (\alpha,\beta,\gamma;\alpha',\beta',\gamma')$, $\beta \neq 0$, and initial condition $F(x,0) = F(x,0;\boldsymbol{\mu})$ $= 1$. Then,*

$$\widehat{F}(x,y) \;=\; \widehat{F}(x,y;\widehat{\boldsymbol{\mu}}) \;=\; F\left(\frac{\beta\, x}{\beta-\beta'\, x}, y\,\frac{\beta-\beta'\, x}{\beta};\boldsymbol{\mu}\right)\,, \tag{35}$$

*is a solution of Eq.* (34) *with parameters*

$$\widehat{\boldsymbol{\mu}} \;=\; \left(\alpha,\beta,\gamma;\alpha'+\beta'-\frac{\alpha\,\beta'}{\beta}, -\beta', \gamma'+\beta'-\frac{\gamma\,\beta'}{\beta}\right)\,, \tag{36}$$

*and initial condition $\widehat{F}(x,0) = \widehat{F}(x,0;\widehat{\boldsymbol{\mu}}) = 1$.*

The straightforward proof relies on making the appropriate change of variables in Eq. (34), and then regrouping the resulting terms. Proposition 17 implies

**Corollary 18.** *If $\left|{n \atop k}\right|$ (resp. $\widehat{\left|{n \atop k}\right|}$) is the solution of*

$$\left|{n \atop k}\right| = (\alpha n + \beta k + \gamma)\left|{n-1 \atop k}\right| + (\alpha' n + \beta' k + \gamma')\left|{n-1 \atop k-1}\right| \;+\; \delta_{n0}\delta_{k0} \tag{37}$$

*with parameters $\boldsymbol{\mu}$ (resp. $\widehat{\boldsymbol{\mu}}$), then*

$$\left|{n \atop k}\right| \;=\; \sum_{j=0}^{k} \widehat{\left|{n \atop j}\right|} \binom{n-j}{n-k} \left(\frac{\beta'}{\beta}\right)^{k-j}\,, \tag{38a}$$

$$\widehat{\left|{n \atop k}\right|} \;=\; \sum_{j=0}^{k} \left|{n \atop j}\right| \binom{n-j}{n-k} \left(-\frac{\beta'}{\beta}\right)^{k-j}\,. \tag{38b}$$

**Remark.** Notice that when $\beta\beta' \neq 0$ the pair

$$a_k \;=\; \sum_{j=0}^{k} \widehat{a}_j \binom{n-j}{n-k} \left(\frac{\beta'}{\beta}\right)^{k-j}\,, \tag{39a}$$

$$\widehat{a}_k \;=\; \sum_{j=0}^{k} a_j \binom{n-j}{n-k} \left(-\frac{\beta'}{\beta}\right)^{k-j}\,, \tag{39b}$$

is an *inverse pair* in the sense of Riordan [27] (see also [18]), and it generates the combinatorial identity

$$\sum_{i=j}^{k}(-1)^{i+j}\binom{n-i}{n-k}\binom{n-j}{n-i} \;=\; \delta_{kj}\,. \tag{40}$$

According to the results presented in Sections A.15 and A.1.6 of Ref. [1], the EGF for the $\nu$-order $(s,t)$-Ward numbers $F_W(x,y;\widehat{\boldsymbol{\mu}})$ with $\widehat{\boldsymbol{\mu}}=(0,1,s;\nu,1,t+s-\nu-1)$ and the EGF for the $(\nu+1)$-order $(s,t)$-Eulerian numbers $F_E(x,y;\boldsymbol{\mu})$ with $\boldsymbol{\mu}=(0,1,s;\nu+1,-1,t-\nu)$ are related by (cf. (35)):

$$F_W(x,y;\widehat{\boldsymbol{\mu}}) \;=\; F_E\left(\frac{x}{1+x},y\,(1+x);\boldsymbol{\mu}\right), \tag{41a}$$

$$F_E(x,y;\boldsymbol{\mu}) \;=\; F_W\left(\frac{x}{1-x},y\,(1-x);\widehat{\boldsymbol{\mu}}\right). \tag{41b}$$

If we use (5), we obtain from (41a) the EGF for the $\nu$-order $(s,t)$-Ward numbers [1, Section A.1.6]:

$$F_W(x,y)=\frac{T_{\nu+1}\left(e^{y\,(1+x)^{-\nu}}\,T_{\nu+1}^{-1}\left(\frac{x}{1+x}\right)\right)^s}{\left[1-T_{\nu+1}\left(e^{y\,(1+x)^{-\nu}}\,T_{\nu+1}^{-1}\left(\frac{x}{1+x}\right)\right)\right]^{s+t}}\,\frac{1}{x^s\,(1+x)^t}\,. \tag{42}$$

Finally, using (38) we obtain the following

**Corollary 19.** *The numbers* $\left\langle{n \atop k}\right\rangle^{(\nu)}_{(s,t)}$ *and* $W^{(\nu)}(n,k;s,t)$ *are related by the equations*

$$W^{(\nu)}(n,k;s,t) \;=\; \sum_{j=0}^{k}\left\langle{n \atop j}\right\rangle^{(\nu+1)}_{(s,t)}\binom{n-j}{n-k}, \tag{43a}$$

$$\left\langle{n \atop k}\right\rangle^{(\nu+1)}_{(s,t)} \;=\; \sum_{j=0}^{k}(-1)^{k-j}\,W^{(\nu)}(n,j;s,t)\binom{n-j}{n-k}. \tag{43b}$$

Notice that when $(\nu,s,t)=(1,0,1)$ we recover the Ward numbers [22, entry A134991] $W^{(1)}(n,k;0,1)=W(n,k)=\left\{\!\!\left\{{n+k \atop k}\right\}\!\!\right\}$, corresponding to $\boldsymbol{\mu}=(0,1,0;1,1,-1)$. The numbers $\left\{\!\!\left\{{n \atop k}\right\}\!\!\right\}$ are the associated Stirling subset numbers [13], [22, entry A008299]. Eq. (43) relates these numbers with the second-order $(0,1)$-Eulerian numbers (i.e., the second-order Eulerian numbers with the traditional indexing $\left\langle\!\!\left\langle{n \atop k}\right\rangle\!\!\right\rangle_{(0,1)}=B_{n,k}$) in the form mentioned in Eq. (1):

$$\left\{\!\!\left\{{n+k \atop k}\right\}\!\!\right\} \;=\; \sum_{j=0}^{k}B_{n,j}\binom{n-j}{n-k}, \tag{44a}$$

$$B_{n,k} \;=\; \sum_{j=0}^{k}(-1)^{k-j}\left\{\!\!\left\{{n+j \atop j}\right\}\!\!\right\}\binom{n-j}{k-j}. \tag{44b}$$

As $B_{n,k} = \left\langle\!\left\langle {n \atop k-1} \right\rangle\!\right\rangle$ for $n \geqslant 1$ and $1 \leqslant k \leqslant n$, we can substitute this expression into (44) and, after some algebraic manipulations, we arrive at the formulas [28, Corollaries 5 and 4]:

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = \sum_{j=0}^{k} (-1)^{k-j} \left\{\!\!\left\{ {n+j+1 \atop j+1} \right\}\!\!\right\} \binom{n-j-1}{k-j}, \tag{45a}$$

$$\left\{\!\!\left\{ {n+k \atop k} \right\}\!\!\right\} = \sum_{j=0}^{k} \left\langle\!\!\left\langle {n \atop j} \right\rangle\!\!\right\rangle \binom{n-j-1}{k-j-1}. \tag{45b}$$

## 8 Combinatorial interpretation of the generalized Ward numbers

In this section, we will give a combinatorial interpretation of the $\nu$-order generalized $(s,t)$–Ward numbers (cf. (33)) based on the identity (43a).

For fixed values of $n$, $k$, $s$, $t$, and a given generalized partition $\boldsymbol{t}$ of $t$ with $s$ parts, the interpretation relies on the fact that, to obtain $W^{(\nu)}(n,k;s,t)$, we sum over the number of $(\nu+2,\boldsymbol{t}+\boldsymbol{1})$–ary increasing $(s,[n])$–forests $\boldsymbol{F}$ with $j$ ascents times $\binom{n-j}{n-k}$. (Recall Remark 1 after Theorem 11.) In this latter factor, $n-j$ admits a simple interpretation in terms of the set $\mathcal{E}(\boldsymbol{F})$ of internal nodes of $\boldsymbol{F}$ that are the first (leftmost) children of their respective parents. For a tree $T$ with $n$ internal nodes, the cardinality of this set is denoted by $D_{n,1} = |\mathcal{E}(T)|$ by Janson *et al.* [19]. We will see that $\binom{n-j}{n-k}$ is closely related to the number of ways of marking $n-k$ nodes of the set $\mathcal{E}(\boldsymbol{F})$.

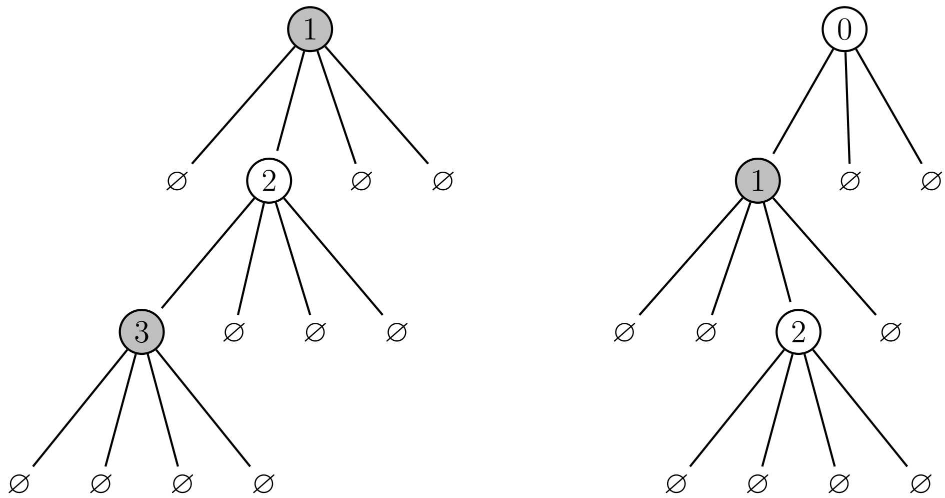
Let us start with the simplest case $s=1$ by considering the class of $(\nu+2,t+1)$–ary increasing $[n]$–trees. Then, for any tree $T$ of this class with $j$ ascents, it is easy to prove that (see [19, Theorem 2]):

$$n - j = |\mathcal{E}(T)| + \delta_{t,0}\,. \tag{46}$$

When $t>0$, we can choose the $n-k$ distinguished nodes from the set $\mathcal{D}_t(T) = \mathcal{E}(T)$; when $t=0$, we make the choice from the set $\mathcal{D}_0(T)$ which is now the union of the root node and the set $\mathcal{E}(T)$. (Notice that our definition of ascent slightly differs from that of Ref. [19].) See Figure 3 for two examples with $t=0$ (a) and $t>0$ (b). In this figure, distinguished nodes are depicted in gray. Putting all together, we can conclude that:

**Theorem 20.** *Let us fix integers $n,t \geqslant 0$, $\nu \geqslant 1$, and $0 \leqslant k \leqslant n$. Then, $W^{(\nu)}(n,k;1,t)$ counts the number of $(\nu+2,t+1)$–ary increasing $[n]$–trees $T$ with at most $k$ ascents and $n-k$ distinguished nodes from the set $\mathcal{D}_t(T)$.*

Let us now consider the extension of Theorem 20 for $s \geqslant 2$. In this case, our basic objects are indeed the $(\nu+2,\boldsymbol{t}+\boldsymbol{1})$–ary increasing $(s,[n])$-forests $\boldsymbol{F}$ with $j$ ascents. Each connected component $T_i$ of the forest $\boldsymbol{F} = (T_1,\ldots,T_s)$ is a $(\nu+2,t_i+1)$–ary increasing $X_i$–tree, where $(X_1,X_2,\ldots,X_s)$ is a generalized partition of $[n]$. Let the $j_i$ be the number



(a) (b)

Figure 3: (a) A 4–ary increasing [3]–tree $T_{\mathrm{a}}$, which is equivalent to the $(3,[3])$-Stirling permutation $\underline{\mathbf{1}}\mathbf{333}222\mathbf{11}$ with one ascent at index 1 (which is underlined) and two distinguished nodes (in boldface): the root and the one labelled 3. These nodes are depicted in gray. Note that $\mathcal{D}_0(T_{\mathrm{a}}) = \{①,③\}$ and $|\mathcal{D}_0(T_{\mathrm{a}})| = D_{3,1}+1 = 2$ for this tree. (b) A $(4,3)$–ary increasing [2]–tree $T_{\mathrm{b}}$ equivalent to the $(3,2,[2])$–Stirling permutation $\mathbf{1}\underline{\mathbf{1}}222\mathbf{1}00$ with one ascent at index 2, and one distinguished node (labelled 1). In this case, $\mathcal{D}_1(T_{\mathrm{b}}) = \{①\}$ and $|\mathcal{D}_1(T_{\mathrm{b}})| = D_{2,1} = 1$. In both examples, all possible distinguishable nodes are actually chosen.

of ascents of $T_i$, then $|X_i| - j_i = |\mathcal{D}_{t_i}(T_i)|$. If we define the set

$$\mathcal{D}_{\boldsymbol{t}}(\boldsymbol{F}) \;=\; \bigcup_{i=1}^{s} \mathcal{D}_{t_i}(T_i)\,, \tag{47}$$

we get that, irrespectively of the partition $(X_1, X_2, \ldots, X_s)$, for any $(\nu+2, \boldsymbol{t}+\mathbf{1})$–ary increasing $(s,[n])$-forest $\boldsymbol{F}$ with $j$ ascents

$$|\mathcal{D}_{\boldsymbol{t}}(\boldsymbol{F})| \;=\; \sum_{i=1}^{s} (|X_i| - j_i) = n - j\,. \tag{48}$$

This fact allows us to generalize Theorem 20 when $s \geqslant 2$:

**Theorem 21.** *Let us fix integers $n, t \geqslant 0$, $\nu, s \geqslant 1$, and $0 \leqslant k \leqslant n$. Given any generalized ordered partition $\mathbf{t} = (t_1, \ldots, t_s)$ of $t$, $W^{(\nu)}(n,k;s,t)$ counts the number of $(\nu+2, \boldsymbol{t}+\mathbf{1})$–ary increasing $(s,[n])$–forests $\boldsymbol{F}$ with at most $k$ ascents and $n-k$ distinguished nodes from the set $\mathcal{D}_{\boldsymbol{t}}(\boldsymbol{F})$ defined in* (47).

This theorem completes the combinatorial interpretation of the $\nu$–order generalized $(s,t)$–Ward numbers for $\nu, s \geqslant 1$ and $t \geqslant 0$. Figure 4 shows an example of a $(4, \boldsymbol{t}+\mathbf{1})$–ary

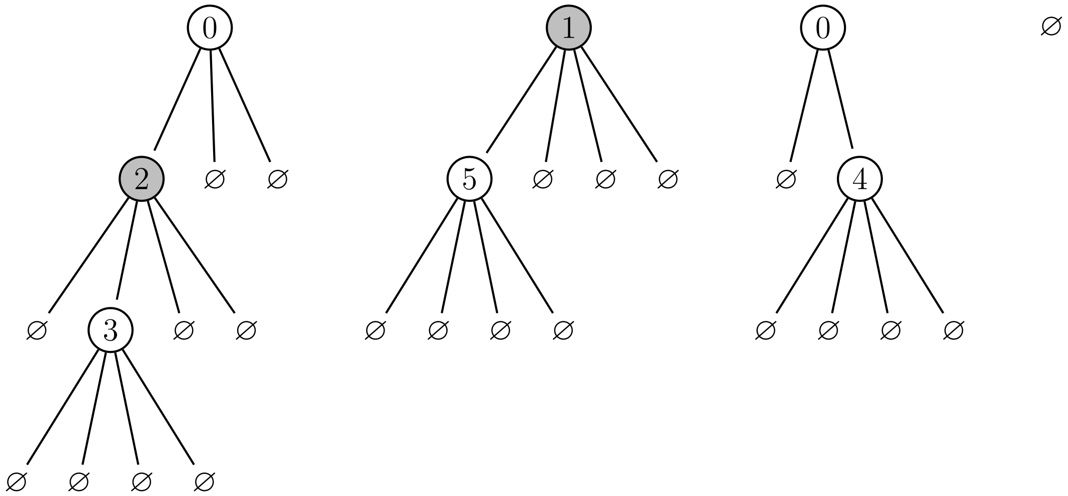


Figure 4: A $(4,\boldsymbol{t}+\mathbf{1})$–ary increasing $(4,[5])$–forest $\boldsymbol{F}=(T_1,T_2,T_3,T_4)$ with $\boldsymbol{t}=(2,0,1,0)$ and the generalized partition $(\{2,3\},\{1,5\},\{4\},\varnothing)$ of $[5]$. It corresponds to the $(3,\boldsymbol{t},5)$–Stirling permutation $(\underline{\mathbf{2}}333\mathbf{22}00, 555\mathbf{111}, \underline{0}444, \varnothing)$ with 2 ascents and two distinguished nodes (labelled 1 and 2) out of the three possible ones. From left to right, the first tree $T_1$ has one ascent at index 1 and one distinguished node out of $|\mathcal{D}_2(T_1)|=1$; the second tree has no ascents and one distinguished node out of $|\mathcal{D}_0(T_2)|=2$; the third tree has one ascent at index 1 and no distinguished nodes ($\mathcal{D}_1(T_3)=\varnothing$); and the last one, $T_4$, is the trivial empty tree.

increasing $(4,[5])$–forest $\boldsymbol{F}=(T_1,T_2,T_3,T_4)$ with $\boldsymbol{t}=(2,0,1,0)$, and the generalized partition $(\{2,3\},\{1,5\},\{4\},\varnothing)$ of $[5]$. This forest has two ascents and two distinguished nodes out of the three possible ones $\mathcal{D}_{\boldsymbol{t}}(\boldsymbol{F})=\{①,②,⑤\}$.

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## References

[1] J.F. Barbero G., J. Salas, and E.J.S. Villaseñor, Bivariate generating functions for a class of linear recurrences: General structure, J. Combin. Theory A **125** (2014) 146–165, arXiv:1307.2010 [math.CO].

[2] M. Bóna, *Combinatorics of Permutations*, Second Edition (Chapman & Hall/CRC, Boca Raton, Florida, 2012).

[3] F. Brenti, Hilbert polynomials in combinatorics, J. Algebraic Combin. **7** (1998) 127–156.

[4] L. Carlitz, Eulerian numbers and polynomials of higher order, Duke Math. J. **27** (1960) 401–424.

[5] L. Carlitz, Enumeration of permutations by rises and cycle structure, J. Reine Angew. Math. **262/263** (1973) 220–233.

[6] L. Carlitz and R. Scoville, Generalized Eulerian numbers: combinatorial applications, J. Reine Angew. Math. **265** (1974) 110–137.

[7] W.Y.C. Chen, Context-free grammars, differential operators and formal power series, Theoret. Comput. Sci. **117** (1993) 113-129.

[8] W.Y.C. Chen, R.X.J. Hao, and H.R.L. Yang, Context-free grammars and multivariate stable polynomials over Stirling permutations, preprint, arXiv:1208.1420 [math.CO].

[9] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions* (Reidel, Dordrecht–Boston, 1974).

[10] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, and D.E. Knuth, On the Lambert $W$ function, Adv. Comput. Math. **5** (1996) 329–359.

[11] R.M. Corless, D.J. Jeffrey, and D.E. Knuth, A sequence of series for the Lambert $W$ function, in *Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation* (Association for Computing Machinery, New York, 1997), pp. 197–204 (electronic, available on-line at http://dl.acm.org).

[12] J.F. Dillon and D.P. Roselle, Eulerian numbers of higher order, Duke Math. J. **35** (1968) 247–256.

[13] A.E. Fekete, Apropos "Two notes on notation", Amer. Math. Monthly **101** (1994) 771–778.

[14] D. Foata and M.-P. Schützenberger, *Théorie Géométrique des Polynômes Eulériens*, Lecture Notes in Mathematics #138 (Springer-Verlag, Berlin–Heidelberg–New York, 1970).

[15] I. Gessel, A note on Stirling permutations, manuscript, August, 1978.

[16] I. Gessel and R.P. Stanley, Stirling polynomials, J. Combin. Theory A **24** (1978) 24–33.

[17] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd ed. (Addison-Wesley, Reading, MA, 1994).

[18] T.-X. He, L.C. Hsu, and P.J.-S. Shiue, The Sheffer group and the Riordan group, Discrete Appl. Math. **155** (2007) 1895–1909.

[19] S. Janson, M. Kuba, and A. Panholzer, Generalized Stirling permutations, families of increasing trees and urn models, J. Combin. Theory A **118** (2011) 94–114, arXiv:0805.4084 [math.CO].

[20] M. Kuba and A. Panholzer, Analysis of statistics for generalized Stirling permutations, Combin. Probab. Comput. **20** (2011) 875–910.

[21] D. Magagnosc, Recurrences and formulae in an extension of the Eulerian numbers, Discrete Math. **30** (1980) 265–268.

[22] The OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, http://oeis.org/.

[23] S. Park, The $r$–multipermutations, J. Combin. Theory A **67** (1994) 44–71.

[24] S. Park, Inverse descents of $r$-multipermutations, Discrete Math. **132** (1994) 215–229.

[25] S. Park, $P$-partitions and $q$-Stirling numbers, J. Combin. Theory A **68** (1994) 33–52.

[26] J. Riordan, *An Introduction to Combinatorial Analysis* (Wiley, New York, 1958).

[27] J. Riordan, *Combinatorial Identities* (Wiley, New York, 1968).

[28] L.M. Smiley, Completion of a rational function sequence of Carlitz, preprint, arXiv:math/0006106 [math.CO].

[29] R.P. Stanley, *Enumerative Combinatorics. Vol. 1* (Wadsworth & Brooks/Cole, Monterey, CA, 1986).

[30] M. Ward, Representation of Stirling's numbers and Stirling's polynomials as sums of factorials, Amer. J. Math. **34** (1934) 87–95.