

Vizing's conjecture for graphs with domination number 3 - a new proof

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Abstract

Vizing's conjecture from 1968 asserts that the domination number of the Cartesian product of two graphs is at least as large as the product of their domination numbers. In this note we use a new, transparent approach to prove Vizing's conjecture for graphs with domination number 3; that is, we prove that for any graph G with $\gamma(G) = 3$ and an arbitrary graph H , $\gamma(G \square H) \geq 3\gamma(H)$.

Keywords: Cartesian product; domination; Vizing's conjecture

1 Introduction

The following long-standing conjecture

Conjecture 1. ([13]) For every pair of finite graphs G and H ,

$$\gamma(G \square H) \geq \gamma(G) \gamma(H), \quad (1)$$

is arguably the most important open problem in graph domination. (As usually, γ stands for the domination number, and $G \square H$ is the standard notation for the Cartesian product of graphs G and H .) The first extensive survey on Vizing's conjecture was published already in 1998 [9], and for a recent survey see [3]. Although many partial results are known that prove inequality (1) (or a weaker form of it) for several classes of graphs, the conjecture remains unsolved. As a side-effect of these approaches some new methods and concepts in domination theory have been developed. In particular, the conjecture

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motivated the study of different domination parameters in Cartesian products of graphs, and was one of the main motivations for the introduction of the rainbow domination [4] and of the domination game [5].

One of the first partial results proving that inequality (1) holds for a very special class of graphs was obtained by Barcalkin and German [2]. The proof in a clever way uses the simple idea of projecting the vertices in a dominating set of $G \square H$ to $\gamma(G)$ copies of the graph H in such a way that to each copy at least $\gamma(H)$ vertices are projected. A somewhat more sophisticated idea of the so-called double projection has been used by Clark and Suen [6] in their proof of the weaker version of (1), where the right side of the inequality is multiplied by $1/2$; cf. [14, 11] for recent improvements of this result, and [7] for another very recent approach to the conjecture. The best result so far in terms of domination numbers of factors due to Liang Sun [12] states that the conjectured inequality holds if one of the factors has domination number (at most) 3 and the other factor is arbitrary. By saying that a graph G *satisfies Vizing's conjecture* we mean that for any other graph H the conjectured inequality $\gamma(G \square H) \geq \gamma(G) \gamma(H)$ holds; hence in this terminology, Liang Sun's result proves that all graph with domination number (at most) 3 satisfy Vizing's conjecture. However, the proof in [12] is very technical and a need for a more transparent proof of this result has been expressed for instance in [8].

In this paper we present a new proof of the result that graphs with domination number 3 satisfy Vizing's conjecture, see Theorem 2. The proof is based on the simple idea of projecting a dominating set of the product $G \square H$ to three copies of H , so that projected vertices in each copy induce a dominating set of H (where G is a graph with domination number 3). The idea works because of a special rule that we impose on the projection maps. In the proof several cases appear, one of which needs somewhat deeper analysis. Nevertheless, we believe that the overall approach of the proof is transparent and not hard to follow, making it convenient for possible further generalizations and extensions.

In the rest of this section we establish the notation, and present some preliminary observations needed in the proof of the main result, while Section 2 contains its proof.

Let $G = (V(G), E(G))$ be a finite, simple graph. For subsets A and B of vertices of G , we say that B *dominates* A (or A *is dominated by* B) if $A \subseteq N[B]$; that is, if each vertex of A is in B or is adjacent to some vertex of B . The *domination number* of G is the smallest cardinality, denoted $\gamma(G)$, of a set that dominates $V(G)$. If D dominates $V(G)$, we will also say that D dominates the graph G and that D is a *dominating set* of G .

The *Cartesian product* $G \square H$ of graphs G and H is the graph whose vertex set is $V(G) \times V(H)$. Two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G \square H$ if either $g_1 = g_2$ and $h_1 h_2$ is an edge in H or $h_1 = h_2$ and $g_1 g_2$ is an edge in G . For a vertex g of G , the subgraph of $G \square H$ induced by the set $\{(g, h) \mid h \in H\}$ is called an *H-fiber* and is denoted by ${}^g H$. Similarly, for $h \in H$, the *G-fiber*, G^h , is the subgraph induced by $\{(g, h) \mid g \in G\}$. We will make use of projection maps from the Cartesian product $G \square H$ to one of the factors G or H . The *projection to H* is the map $p_H : V(G \square H) \rightarrow V(H)$ defined by $p_H(g, h) = h$. When D is a dominating set of the product $G \square H$, we will often say that $(g, h) \in D$ is projected to h in H_i , by which we will indicate that h is added to a dominating set that is being built in the copy H_i of H .

In this note we prove the following result.

Theorem 2. *Let G and H be finite graphs, $\gamma(G) = 3$, and H is an arbitrary graph. Then*

$$\gamma(G \square H) \geq 3\gamma(H). \quad (2)$$

We will use the following well-known observation.

Observation 3. *Let G be a graph that satisfies Vizing's conjecture, and let G' be a spanning subgraph of G such that $\gamma(G') = \gamma(G)$. Then G' also satisfies Vizing's conjecture.*

Indeed, if H is any graph, then

$$\gamma(G')\gamma(H) = \gamma(G)\gamma(H) \leq \gamma(G \square H) \leq \gamma(G' \square H).$$

The last inequality holds since $G' \square H$ is a spanning subgraph of $G \square H$. Hence, to prove that Vizing's conjecture is satisfied by graphs G with domination number 3 we may restrict in the proof only to graphs G that are *edge-maximal* with respect to domination number (edge-maximal means that by adding any edge between two non-adjacent vertices in G the domination number drops).

Observation 4. *Let G be a graph with $\gamma(G) = 3$, which is edge-maximal with respect to domination number. Then there exists a dominating set $D = \{a, b, c\}$ of G such that $N[a] \cup N[b] = V(G) \setminus \{c\}$.*

To see that Observation 4 is correct, note that by adding an edge between two non-adjacent vertices a and c in G , the domination number of the resulting graph $G + ac$ is 2, and either a or c must be in a minimum dominating set of $G + ac$. Let $\{a, b\}$ be a dominating set of $G + ac$. But then in G vertices a and b dominate every vertex except c .

In the proof we will also use the following classical theorem from set theory (known as *Hall's marriage theorem*). Recall that a family of (nonempty) finite sets $\{S_1, \dots, S_n\}$ has a *system of distinct representatives*, if there exists a set of their elements $\{x_1, \dots, x_n\}$ such that $x_i \in S_i$ for all $i \in [n]$ (as usually, $[n]$ denotes the set $\{1, \dots, n\}$).

Theorem 5. *Let $\mathcal{S} = \{S_1, \dots, S_n\}$ be a family of (nonempty) finite sets. There exists a system of distinct representatives of the sets from \mathcal{S} if and only if for each subfamily S_{i_1}, \dots, S_{i_k} of \mathcal{S} the following (Hall's condition) holds:*

$$|S_{i_1} \cup \dots \cup S_{i_k}| \geq k.$$

2 Proof of Theorem 2

Let G be a (connected) graph with $\gamma(G) = 3$, and let $I = \{v_1, v_2, v_3\}$ be a dominating set of G . By Observations 3 and 4 we may assume that G is edge-maximal and that $N[v_1] \cup N[v_2] = V(G) \setminus \{v_3\}$.

For any $i \in [3] = \{1, 2, 3\}$, let P_i denote the set of external private neighbors of v_i in G with respect to I , that is

$$P_i = \{x \in V(G) \setminus I : N(x) \cap I = v_i\}.$$

Note that $P_3 = \emptyset$ and set $P = P_1 \cup P_2$. Let $Q_i = P_i \cup \{v_i\}$, and $Q = Q_1 \cup Q_2 \cup Q_3$. Next, for any $S \subseteq [3]$, $|S| \geq 2$,

$$P_S = \{x \in V(G) \setminus Q : N(x) \cap I = \{v_i \mid i \in S\}\}.$$

For instance, $P_{\{1,3\}}$ is the set of vertices that are adjacent to v_1 and v_3 and are not adjacent to v_2 .

Let D be a dominating set of $G \square H$. We will project vertices of D to three copies of H , denoted by H_1, H_2, H_3 . Each vertex of D is allowed to project to only one vertex in only one copy of H ; the set of projected vertices in the copy H_i is denoted by D_i . We will prove that projections can be performed in such a way that each D_i is a dominating set in H_i . If this is done, it clearly implies that $|D| \geq 3\gamma(H)$. The projections will be done in three steps (the third step is optional, depending on the structure of G and of the dominating set of $G \square H$, and makes some corrections to projections made in the first step).

In the **first step** we project the vertices $(x, h) \in D$, where $x \in Q$ in the following natural way: if $x \in Q_i$ then we project (x, h) to h in the copy H_i (Note that by *projecting* (x, h) to h in H_i we mean that we put h to the set D_i of H_i .)

After the first step, we examine each fiber G^h for $h \in V(H)$, and consider the copies of h in all H_i that are not yet dominated by the set D_i , constructed so far. We say that the pair $\{i, h\}$ is *vertically undominated* if $(Q_i \times N_H[h]) \cap D = \emptyset$. Obviously in each fiber G^h there are between zero and three such vertically undominated pairs $\{i, h\}$.

In the **second step** we project the vertices $(x, h) \in D$, where $x \in V(G) \setminus Q$, in such a way that the following rule is obeyed:

(R) if $x \in P_S$, where $S \subseteq [3]$, then there exists an $i \in S$ and a vertex (y, h) from $D \cap G^h$, such that (y, h) is projected to h in the copy H_i .

We will say a vertex (x, h) from $D \cap G^h$ is *free* if there exists another vertex $(y, h) \in D$, which is projected to h in H_i , and $i \in S$, where $x \in P_S$.

We distinguish 7 cases with respect to the following properties of the fibers G^h , for which not all copies of h in H_i are dominated by D_i . By **Case k/m** , where $1 \leq m \leq k \leq 3$ we mean that in G^h there are k pairs $\{i, h\}$ that are vertically undominated, and that there are m vertices in $((V(G) \setminus Q) \times \{h\}) \cap D$. As it turns out, it can also happen that there is one pair $\{i, h\}$ that is vertically undominated, yet there are no vertices in $((V(G) \setminus Q) \times \{h\}) \cap D$. This case will be denoted by **Case 1/0**. It will also become clear that the cases when $m > k$ can be easily translated to the case when $m = k$.

In each of the cases we argue how the projection of the vertices from $G^h \cap D$ is performed in the second step, so that for each vertically undominated pair $\{i, h\}$, the vertex h becomes dominated in H_i , and at the same time the rule (R) is obeyed for the

projected vertices. The only exception, where the projections fail to achieve the desired goal, may appear in Case 2/1. This case is dealt with at the end, after all the projections of the second step are done in all the G -fibers, which fall under other cases. If such a problem in Case 2/1 appears, it leads to the third step in which some modifications of projections from the first step are performed.

Case 1/0. As there are no vertices in $((V(G) \setminus Q) \times \{h\}) \cap D$, it is clear that $\{3, h\}$ cannot be vertically undominated, because no vertex from $(P_1 \cup P_2) \times \{h\}$ is adjacent to (v_3, h) . (Note that from the same reason, cases 3/0 and 2/0 are not possible.) Hence we may assume without loss of generality that $\{1, h\}$ is vertically undominated. Then (v_1, h) and all (u, h) for $u \in P_1$ must be dominated from $G^h \cap D$. In particular, the only way to dominate (v_1, h) is from $I \times \{h\}$; more precisely, we infer that $(v_2, h) \in D$. In addition, to dominate a vertex (u, h) for $u \in P_1$ there must exist a vertex $(v, h) \in D$ where $v \in P_2$. Now, as both vertices (v_2, h) and (v, h) are projected in the first step to the same vertex h in H_2 , we find that one of them is a free vertex. We project it to h in H_1 , and so h is dominated in each of the three copies H_i .

Case 1/1. In this case exactly one $\{i, h\}$, $i \in [3]$ is vertically undominated, and there is one vertex from $(x, h) \in D$ with $x \notin Q$. If x is adjacent to v_i , then we can project (x, h) to h in H_i by which the rule (R) is obeyed, and the proof in this case is done. On the other hand, if x is not adjacent to v_i , then $i \in \{1, 2\}$ because (v_i, h) must be dominated within G^h and v_3 is not adjacent to any vertex from Q . We may assume without loss of generality that (v_1, h) is vertically undominated. This implies that $(v_2, h) \in D$, v_2 is adjacent to v_1 , and $x \in P_{\{2,3\}}$. Hence the rule (R) is already obeyed for (x, h) (with the vertex (v_2, h) projected to h in H_2) and so (x, h) is free. We project it to h in H_1 , and so h becomes dominated in H_1 .

Case 2/2. Let $\{i, h\}$ and $\{j, h\}$ be vertically undominated, let k be the unique index in $[3] \setminus \{i, j\}$, and let $(x, h), (y, h) \in D$ such that $x, y \notin Q$. If none of (v_i, h) and (v_j, h) is dominated by (v_k, h) , then it is easy to see, that we can project (x, h) and (y, h) to the copies of h in H_i and H_j in such a way that (R) is obeyed. Now, assume that at least one of (v_i, h) , (v_j, h) is dominated by (v_k, h) , which is thus in D . Then each of the vertices (x, h) and (y, h) is either free (when it is adjacent to (v_k, h)), or it is in $P_{\{i,j\}}$. In either case we can project them to the copies of h in H_i and H_j , by which the proof of this case is complete.

Case 3/1. In this case $\{1, h\}, \{2, h\}$ and $\{3, h\}$ are vertically undominated, and let (x, h) be the unique vertex in $G^h \cap D$ (clearly $x \notin Q$). We claim that in each of $P_{\{1,2\}}, P_{\{1,3\}}$, and $P_{\{2,3\}}$ there exists a vertex that is not adjacent to x . Indeed, if every vertex in, say $P_{\{1,2\}}$, were adjacent to x , then $\{x, v_3\}$ would be a dominating set of G , which is a contradiction with $\gamma(G) = 3$. Let $u \in P_{\{1,2\}}, v \in P_{\{1,3\}}$, and $w \in P_{\{2,3\}}$ be vertices not adjacent to x . Hence $(u, h), (v, h), (w, h)$ are (vertically) dominated by some vertices $(u, h'), (v, h''), (w, h''') \in D$, where $h', h'', h''' \in N_H(h)$. By (R) these vertices are projected to h', h'', h''' , respectively, in one of the copies H_1, H_2 or H_3 (more precisely, (u, h') in H_1 or H_2 , (v, h'') in H_1 or H_3 and (w, h''') in H_2 or H_3). It is easy to see that in at most one of the copies H_1, H_2, H_3 the vertex h is not vertically dominated after these three

projections are performed. If H_i , $i \in \{1, 2, 3\}$, is such a copy, we let (x, h) project to h in H_i , otherwise (x, h) is projected in an arbitrary copy of H .

Case 3/2. Again assume without loss of generality that $\{1, h\}$, $\{2, h\}$ and $\{3, h\}$ are vertically undominated, and let $(x, h), (y, h) \in D$. Since $\{x, y\}$ is not a dominating set of G , there exists a vertex $(z, h) \in G^h$ that is not adjacent to these two vertices. Moreover, $z \notin Q$, and (z, h) is vertically dominated by some vertex $(z, h') \in D$, where $h' \in N_H(h)$. The projection of (z, h') to the copy of h' in some H_i , dominates the copy of h in H_i . For the other two copies of h we use projection of $(x, h), (y, h)$ to h in these copies. Note that such projections can be done, by obeying rule (R), because each of the vertices x and y is adjacent to at least two vertices from I and $\{x, y\}$ dominates I .

Case 3/3. Let $\{1, h\}$, $\{2, h\}$ and $\{3, h\}$ be vertically undominated, and let $(x_1, h), (x_2, h)$ and (x_3, h) be the vertices from D . For $i = 1, 2, 3$, let P_{S_i} be the set in which x_i lies. Note that the sets S_1, S_2, S_3 enjoy Hall's condition, and so there exists a system of distinct representatives of these sets. That is, each set has a unique representative from $\{1, 2, 3\}$, say S_i is represented by $\pi(i)$, where π is the corresponding permutation. By projecting (x_i, h) to h in $H_{\pi(i)}$, the rule (R) is obeyed and for each $i \in [3]$ the copy of h in H_i is dominated by h itself.

Case 2/1. Let $\{i, h\}$ and $\{j, h\}$ be vertically undominated, let k be the unique index in $[3] \setminus \{i, j\}$, and let (x, h) be the unique vertex in $((V(G) \setminus Q) \times \{h\}) \cap D$.

First assume that $|G^h \cap D| = 1$. Clearly, the vertex (x, h) then dominates $(Q_i \cup Q_j) \times \{h\}$. Since $\{x, v_k\}$ is not a dominating set of G (as $\gamma(G) = 3$), there exists a vertex $(z, h) \in P_{\{i, j\}}$, which is not adjacent to (x, h) . Thus (z, h) must be vertically dominated, i.e. there exists $h' \in N_H(h)$ such that $(z, h') \in D$. By the rule (R), (z, h') is projected to h' in H_i or in H_j , and so the copy of the vertex h is dominated by h' either in H_i or in H_j ; we may assume without loss of generality that the copy of h in H_i is dominated by $h' \in D_i$. Since x is adjacent to v_j , we can project (x, h) to the copy of h in H_j , by which (R) is obeyed, and so all three copies of h are dominated.

Next, if $|G^h \cap D| \geq 3$, this readily implies that $|(Q_k \times \{h\}) \cap D| \geq 2$. Since in the first step the (at least two) vertices from $Q_k \times \{h\}$ project to the same vertex h in H_k , at least one of these vertices is free. We project it to h in H_i , and project (x, h) to h in H_j , making all three copies of h dominated.

In the remainder of this case, we can assume that $|G^h \cap D| = 2$. Now, suppose that $(v_k, h) \in D$. Since $x \in V(G) \setminus Q$, it is either in $P_{i, j}$ or (x, h) is a free vertex, but in any case, (x, h) can be projected to any copy of h , without violating (R). We proceed analogously as in the case $|G^h \cap D| = 1$. Notably, since $\{x, v_k\}$ is not a dominating set of G , there exists a vertex $(z, h) \in P_{\{i, j\}}$, which is not adjacent to (x, h) . Thus there exists $h' \in N_H(h)$ such that $(z, h') \in D$. By (R), (z, h') is projected to h' in H_i or in H_j , and so the copy of the vertex h is dominated by h' either in H_i or in H_j ; we project (x, h) to the other copy of h , and so all three copies of h are dominated.

Finally, let $(v, h) \in D$, where $v \in P_k$, be the only vertex in $D \cap G^v$ beside (x, h) . At this point we assume that the first and the second step of projections to the copies of H have already been performed for all vertices of D in all G -fibers in which this was possible. That is, for all G^h , which fall under any of the above cases (including the preceding subcases of

Case 2/1), the projections of the second step were also performed. The projections can be done in any order; by starting in an arbitrary G -fiber G^h and project its vertices of $((V(G) \setminus Q) \times \{h\}) \cap D$ according to the rule (R), which in turn determines the projections of the neighboring G -fibers, and so on. The only eventual remaining G -fibers are the ones that fall under the subcase of Case 2/1, in which we are now, and which will be described in what follows.

Suppose that there is a neighboring G -fiber $G^{h'}$ of G^h , in which the projections of the second step have also not been performed yet. This means that $G^{h'}$, where $h' \in N_H(h)$, also falls under the last subcase of Case 2/1. This implies that there exists a vertex (x', h') , which is unique in $((V(G) \setminus Q) \times \{h'\}) \cap D$, and there is only one other vertex of $D \cap G^{h'}$. Now, since $\{i, h\}$ and $\{j, h\}$ are vertically undominated, this implies that the remaining vertex of $D \cap G^{h'}$ lies in $Q_k \times \{h'\}$. Hence, we find that (x', h') can be projected to h' in any copy of H (in the same way, as we deduced it for (x, h)). We project (x, h) to h in H_i and (x', h') to h' in H_j , by which each of h and h' are dominated in all three copies of H . More generally, we can clearly resolve the projections in the second step for any nontrivial connected subgraph \tilde{H} of H , where G^h falls under this subcase of Case 2/1 for all $h \in \tilde{H}$. The final remaining case is that the mentioned subgraph is trivial, i.e. for all the neighboring G -fibers of G^h the projections fall under previous cases of the second step (which have already been performed).

Suppose that there is a G -fiber $G^{h'}$, where $h' \in N_H(h)$, such that a vertex $(u, h') \in D$ is projected in the second step to h in H_i (or H_j resp.). Then we can clearly conclude the proof of this case, by projecting (x, h) to h in H_j (or H_i resp.), and so all copies of h are dominated. Hence, we may assume that for each $h' \in N_H(h)$, the vertices from $((V(G) \setminus Q) \times \{h'\}) \cap D$ are projected to h' in H_k . If this is the case, then we arrive at the **third step**.

In the third step we make a nonstandard re-projection, by declaring that (v, h) is projected to one of H_i or H_j instead of H_k (that is, we delete h from D_k and put it in D_i or D_j). At the same time we project (x, h) to h in H_j (or H_i resp., taking care that (R) is obeyed), by which h is dominated in the copies H_i and H_j . We claim that by this re-projection h remains dominated also in the copy H_k , and that all neighbors $h' \in N_{H_k}(h)$ remain dominated. That is, we claim that $h \in D_k$ is not needed in the constructed dominating set of H_k ; i.e. it remains a dominating set of H_k also after we remove h from it.

Since $\{x, v\}$ is not a dominating set of G , there exists a vertex $y \in V(G)$, which is not dominated by these two vertices. Note that $y \in P_k \cup (V(G) \setminus Q)$. Now, this implies that (y, h) must be vertically dominated, i.e. there exists $(y, \bar{h}) \in D$, where $\bar{h} \in N_H(h)$. In any case (y being either in P_k or in $V(G) \setminus Q$), the vertex (y, \bar{h}) is projected to \bar{h} in H_k . Thus h is dominated by $\bar{h} \in D_k$.

Suppose that there is a vertex $h' \in N_H(h)$ such that $h' \in V(H_k)$ was dominated only by h before the third step. In particular, this implies that (v_k, h') is not vertically dominated. Since (v_j, h') and (v_i, h') are not in D (because $\{i, h\}$ and $\{j, h\}$ are vertically undominated), we infer that (v_k, h') is dominated by a vertex $(y, h') \in D$, where y is in $V(G) \setminus Q$. Since (y, h') is projected in the second step to h' in H_k we derive a contradiction

with the assumption that $h' \in V(H_k)$ was not dominated by $D_k \setminus \{h\}$ after the first two steps. This contradiction implies that after the re-projection of h from D_k to D_i or D_j , the set $D_k \setminus \{h\}$ remains a dominating set of H_k , and so the proof is complete.

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