Iterative properties of birational rowmotion II: rectangles and triangles

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Abstract

Birational rowmotion – a birational map associated to any finite poset P – has been introduced by Einstein and Propp as a far-reaching generalization of the (well-studied) classical rowmotion map on the set of order ideals of P. Continuing our exploration of this birational rowmotion, we prove that it has order $p+q$ on the (p, q)-rectangle poset (i.e., on the product of a p-element chain with a q-element chain); we also compute its orders on some triangle-shaped posets. In all cases mentioned, it turns out to have finite (and explicitly computable) order, a property it does not exhibit for general finite posets (unlike classical rowmotion, which is a permutation of a finite set). Our proof in the case of the rectangle poset uses an idea introduced by Volkov (arXiv:hep-th/0606094) to prove the AA case of the Zamolodchikov periodicity conjecture; in fact, the finite order of birational rowmotion on many posets can be considered an analogue to Zamolodchikov periodicity. We comment on suspected, but so far enigmatic, connections to the theory of root posets.

Keywords: rowmotion; posets; order ideals; Zamolodchikov periodicity; root systems; promotion; graded posets; Grassmannian; tropicalization.

Contents

Introduction

The present paper continues our study of periodicity of the birational rowmotion map on finite ranked posets. While in our first paper [GrRo14a] we consider the case of "skeletal posets", which generalize the class of graded forests, here posets of rectangular and triangular shape are the primary focus. Our main motivation for proving periodicity of rectangles (Theorem 30) came from the work of David Einstein and James Propp's [EiPr13, EiPr14], where they consider the lifting of the combinatorial rowmotion operator on a poset and the "homomesy" phenomenon (well-understood for products of chains, i.e., rectangles) to piecewise-linear and birational settings. For the notion of homomesy used therein, the maps considered need to have finite order, a fact which is no longer obvious when rowmotion operates on infinite sets. Our paper can nevertheless be read independently of their work or our earlier paper.

A shorter exposition of the main points of this work and [GrRo14a] appears in a 12-page extended abstract for FPSAC 2014 [GrRo13]. A more detailed exposition is available on the ArXiv [GrRo14b] and is updated more frequently on the first author's website.[1]

Let P be a finite poset, and $J(P)$ the set of the order ideals of P. (See [Stan11, Chapter 3] for poset basics.) *Rowmotion* is a classical map $J(P) \to J(P)$ which can be defined in various ways, one of which is as follows: For every $v \in P$, let $\mathbf{t}_v : J(P) \to J(P)$ be the map sending every order ideal $S \in J(P)$ to

$$
\begin{cases}
S \cup \{v\}, & \text{if } v \notin S \text{ and } S \cup \{v\} \in J(P); \\
S \setminus \{v\}, & \text{if } v \in S \text{ and } S \setminus \{v\} \in J(P); \\
S, & \text{otherwise.}
\end{cases}
$$

These maps $\mathbf{t}_v$ are called *(classical) toggles*, since all they do is "toggle" an element into or out of an order ideal. Let $(v_1, v_2, \ldots, v_m)$ be a linear extension of P. Then, (classical) rowmotion is defined as the composition $\mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ \ldots \circ \mathbf{t}_{v_m}$ (which, as can be seen, does not depend on the choice of the particular linear extension $(v_1, v_2, \ldots, v_m)$). This rowmotion map has been studied from various perspectives; in particular, it is isomorphic to the map f of Fon-der-Flaass [Flaa93], the map F^{-1} of Brouwer and Schrijver [BrSchr74], and the map f^{-1} of Cameron and Fon-der-Flaass [CaFl95]. More recently, it has been studied (and christened "rowmotion") in Striker and Williams [StWi11], where further sources and context are also given. We have also covered the case of P being a "skeletal poset" (such as a graded forest with all leaves on the same rank) in [GrRo14a].

Among the questions that have been posed about rowmotion, the most prevalent was probably that of its order: While it clearly has finite order (being a bijective map from the finite set $J(P)$ to itself), it turns out to have a much smaller order than one would naively expect when the poset P has certain "special" forms (e.g., a rectangle, a root poset, a product of a rectangle with a 2-chain, or – as shown in [GrRo14a] – a graded forest). Most strikingly, when P is the product of two chains $[p] \times [q]$ (denoted $\mathrm{Rect}(p, q)$ in Definition 27), then the $(p+q)$-th power of the rowmotion operator is the identity map. This is proven in [BrSchr74, Theorem 3.6] and [Flaa93, Theorem 2] (and a proof can also be constructed from the ideas given in [PrRo13, §3.3.1]).

In [EiPr13], David Einstein and James Propp (inspired by work of Arkady Berenstein and Anatol Kirillov) have lifted the rowmotion map from the set $J(P)$ of order ideals to the progressively more general setups of:

(a) the order polytope $\mathcal{O}(P) := \{\text{order-preserving maps } f : P \to [0, 1]\}$ of the poset P [Stan11, Example 4.6.17] or [Stan86, Definition 1.1], and

(b) even more generally, the affine variety of $\mathbb{K}$-labellings of P for $\mathbb{K}$ an arbitrary infinite field.

In case **(a)**, order ideals of P are replaced by points in the order polytope $\mathcal{O}(P)$, and the role of the map $\mathbf{t}_v$ (for a given $v \in P$) is assumed by the map which reflects the

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[1] The top of the source files for the latter contain a simple flag that can be set to create an even longer compiled version (running roughly 300 pages at present), which provides certain arguments at a grain-size to satisfy the most skeptical and detail-oriented readers (mainly ourselves).

v-coordinate of a point in $\mathcal{O}(P)$ around the midpoint of the interval of all values it could take without the point leaving $\mathcal{O}(P)$ (while all other coordinates are considered fixed). The operation of "piecewise linear" rowmotion is still defined as the composition of these reflection maps in the same way as rowmotion is the composition of the toggles $\mathbf{t}_v$. This "piecewise linear" rowmotion extends (interpolates, even) classical rowmotion, as order ideals correspond to the vertices of the order polytope $\mathcal{O}(P)$ (see [Stan86, Corollary 1.3]). We will not study case **(a)** here, since all of the results we could find in this case can be obtained by tropicalization from similar results for case **(b)**.

In case **(b)**, instead of order ideals of P one considers maps from the poset $\widehat{P} := \{0\} \oplus P \oplus \{1\}$ (where $\oplus$ stands for the ordinal sum[2]) to a given infinite field $\mathbb{K}$ (or, more graphically, labellings of the elements of P by elements of $\mathbb{K}$, along with two additional labels "at the very bottom" and "at the very top"). The maps $\mathbf{t}_v$ are then replaced by certain birational maps which we call *birational v-toggles* (Definition 5); the resulting composition is called *birational rowmotion* and denoted by R. By a careful limiting procedure (the tropical limit), we can "degenerate" R to the "piecewise linear" rowmotion of case **(a)**, and thus it can be seen as an even higher generalization of classical rowmotion. We refer to the body of this paper for precise definitions of these maps. Note that birational v-toggles (but not birational rowmotion) in the case of a rectangle poset have also appeared in [OSZ13, (3.5)], but (apparently) have not been composed there in a way that yields birational rowmotion.

As in the case of classical rowmotion on $J(P)$, the most interesting question is the order of this map R, which in general no longer has an obvious reason to be finite (since the affine variety of $\mathbb{K}$-labellings is not a finite set like $J(P)$). Indeed, for some posets P this order is infinite. We have shown that R has finite order for a wide class of graded posets P in [GrRo14a]; this class covers (in particular) all forests which are graded as posets (i.e., have their leaves all at the same rank). In this paper we will prove the following:

- Birational rowmotion on a $p \times q$-rectangle has order $p + q$ and satisfies a further symmetry property (Theorem 32). These results have originally been conjectured by James Propp and the second author, and can be used as an alternative route to certain properties of (Schützenberger's) promotion map on semistandard Young tableaux.

- Birational rowmotion on certain triangle-shaped posets (this is made precise in Sections 9, 10, 11) also has finite order (computed explicitly below). We show this for three kinds of triangle-shaped posets (obtained by cutting the $p \times p$-square in two along either of its two diagonals) and conjecture it for a fourth (a quarter of a $p \times p$-square obtained by cutting it along both diagonals).

The proof of the most difficult and fundamental case – that of a $p \times q$-rectangle – is inspired by Volkov's proof of the "rectangular" (type-AA) Zamolodchikov conjecture

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[2]More explicitly, $\widehat{P}$ is the poset obtained by adding a new element 0 to P, which is set to be lower than every element of P, and adding a new element 1 to P, which is set to be higher than every element of P (and 0).

[Volk06], which uses a similar idea of parametrizing (generic) $\mathbb{K}$-labellings by matrices (or tuples of points in projective space). There is, of course, a striking similarity between the fact itself and the Zamolodchikov conjecture; yet, we were not able to reduce either result to the other.

Applications of the results of this paper (specifically Theorems 30 and 32) are found in [EiPr13]. Directions for further study include relations to the totally positive Grassmannian and generalizations to further classes of posets.

An extended (12-page) abstract [GrRo13] of this paper and [GrRo14a] was presented at the FPSAC 2014 conference.

0.1 Leitfaden

This paper can be read in linear order and independently of [GrRo14a] (provided the reader is willing to trust a few results quoted from [GrRo14a] or supply the rather simple proofs on their own). If the reader is not interested in proofs, it is also sufficient to cherrypick the results from Sections 1, 3, 9, 10, 11, 12 and 13 only.

0.2 Acknowledgments

1 Notation and definitions

In this section, we introduce the notions of birational toggles and birational rowmotion, and state some very basic properties. We refer to [GrRo14a] for all proofs, although those are elementary enough that an interested reader could regard them as exercises.

Throughout the paper, unless otherwise stated explicitly, $\mathbb{K}$ will denote a field, tacitly assumed to be infinite whenever necessary for certain density arguments to hold. Also,

$\mathbb{N} := \{0, 1, 2, \dots\}$ denotes the set of natural numbers, and P a *finite* poset, assumed to be *graded* starting in Section 2.

Our conventions and notations for *posets* and related notions (such as *linear extension*) closely follow that of [Stan11, Chapter 3]. In particular, we write $u \lessdot v$ to mean "u is covered by v" (i.e., $u < v$ and there is no $w \in P$ such that $u < w < v$). Our notion of "n-graded" is slightly nonstandard, as explained after Definition 14.

Definition 1. Let P be a finite poset. Then, $\widehat{P}$ will denote the poset defined as follows: As a set, let $\widehat{P}$ be the disjoint union of the set P with the two-element set $\{0, 1\}$. The smaller-or-equal relation $\leqslant$ on $\widehat{P}$ will be given by

$$(a \leqslant b) \Longleftrightarrow (\text{either } (a \in P \text{ and } b \in P \text{ and } a \leqslant b \text{ in } P) \text{ or } a = 0 \text{ or } b = 1)$$

(where "either/or" has a non-exclusive meaning). Here and in the following, we regard the canonical injection of the set P into the disjoint union $\widehat{P}$ as an inclusion; thus, P becomes a subposet of $\widehat{P}$. In the terminology of Stanley's [Stan11, section 3.2], this poset $\widehat{P}$ is the ordinal sum $\{0\} \oplus P \oplus \{1\}$.

Definition 2. Let P be a finite poset and $\mathbb{K}$ be a field (henceforth). A $\mathbb{K}$-*labelling of* P will mean a map $f : \widehat{P} \to \mathbb{K}$. Thus, $\mathbb{K}^{\widehat{P}}$ is the set of all $\mathbb{K}$-labellings of P. If f is a $\mathbb{K}$-labelling of P and v is an element of $\widehat{P}$, then $f(v)$ will be called the *label of f at v*.

Our basic object of study, birational rowmotion, will be defined as a map between $\mathbb{K}$-labelings of P. Unfortunately, it can happen that for certain choices of labels, this map will lead to division by zero and not be well-defined. To handle this we make use of some standard notions in basic algebraic geometry: algebraic varieties, the Zariski topology and dominant rational maps. However, the only algebraic varieties that we consider are products of affine spaces and their open subsets.

Definition 3. We use the punctured arrow $\dashrightarrow$ to signify **rational maps** (i.e., a rational map from a variety U to a variety V is called a rational map $U \dashrightarrow V$). A rational map $U \dashrightarrow V$ is said to be *dominant* if its image is dense in V (with respect to the Zariski topology).

Whenever we are working with a field $\mathbb{K}$, we will tacitly assume that $\mathbb{K}$ is either infinite or at least can be enlarged when necessity arises. This assumption is needed to clarify the notions of rational maps and generic elements of algebraic varieties over $\mathbb{K}$. (We will not require $\mathbb{K}$ to be algebraically closed.)

The words "generic" and "almost" will always refer to the Zariski topology. For example, if U is a finite set, then an assertion saying that some statement holds "for almost every point $p \in \mathbb{K}^U$" is supposed to mean that there is a Zariski-dense open subset D of $\mathbb{K}^U$ such that this statement holds for every point $p \in D$. A "generic" point on an algebraic variety V (for example, this can be a "generic matrix" when V is a space of matrices, or a "generic $\mathbb{K}$-labelling of a poset P" when V is the space of all $\mathbb{K}$-labellings of P) means a point lying in some fixed Zariski-dense open subset S of V; the concrete definition of S can usually be inferred from the context (often, it will be the subset of V

on which everything we want to do with our point is well-defined), but of course should never depend on the actual point. (Note that one often has to read the whole proof in order to be able to tell what this S is. This is similar to the use of the "for ϵ small enough" wording in analysis, where it is often not clear until the end of the proof how small exactly the ϵ needs to be.) We are sometimes going to abuse notation and say that an equality holds "for every point" instead of "for almost every point" when it is really clear what the S is. (For example, if we say that "the equality $\dfrac{x^3 - y^3}{x - y} = x^2 + xy + y^2$ holds for every $x \in \mathbb{K}$ and $y \in \mathbb{K}$", it is clear that S has to be the set $\mathbb{K}^2 \setminus \{(x, y) \in \mathbb{K}^2 \mid x = y\}$).

Remark 4. Most statements that we make below work not only for fields, but also more generally for semifields such as the semifield $\mathbb{Q}_+$ of positive rationals or the tropical semiring. We will not concern ourselves with stating them for semifields; a reader curious about this possibility is referred to [GrRo14a, §2] for details on how identities between subtraction-free rational functions can be transferred from fields to semifields.

We are now ready to introduce the concepts of a birational toggle and of birational rowmotion. These concepts originate in [EiPr13] (where they have been studied over $\mathbb{R}_+$ rather than over fields as we do) and are the focus of our work here and in [GrRo14a].

Definition 5. Let $v \in P$. We define a rational map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$, called the *v-toggle*, by

$$
(T_v f)(w) = \begin{cases} f(w), & \text{if } w \neq v \\[2em] \dfrac{1}{f(v)} \cdot \dfrac{\displaystyle\sum_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u)}{\displaystyle\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \dfrac{1}{f(u)}}, & \text{if } w = v \end{cases} \tag{1}
$$

for all $w \in \widehat{P}$ and $f \in \mathbb{K}^{\widehat{P}}$. Note that this rational map T_v is well-defined, because the right-hand side of (1) is well-defined on a Zariski-dense open subset of $\mathbb{K}^{\widehat{P}}$.

The following simple properties of these maps T_v are proven in [GrRo14a, §2].

Proposition 6. *Let $v \in P$. Then, the rational map T_v is an involution, i.e., the map T_v^2 is well-defined on a Zariski-dense open subset of $\mathbb{K}^{\widehat{P}}$ and satisfies $T_v^2 = \mathrm{id}$ on this subset. As a consequence, T_v is a dominant rational map.*

The reader should remember that dominant rational maps (unlike general rational maps) can be composed, and their compositions are still dominant rational maps. Of course, in using the notion of dominant maps, we are relying on our assumption that $\mathbb{K}$ is infinite.

Proposition 7. *Let $v, w \in P$. Then, $T_v \circ T_w = T_w \circ T_v$, unless we have either $v \lessdot w$ or $w \lessdot v$.*

We can now define birational rowmotion:

Definition 8. *Birational rowmotion* is defined as the dominant rational map $T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_m} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$, where $(v_1, v_2, \ldots, v_m)$ is a linear extension of P. This rational map is well-defined (in particular, it does not depend on the linear extension $(v_1, v_2, \ldots, v_m)$ chosen), as has been proven in [GrRo14a, §2]. This rational map will be denoted by R, or by R_P when we wish to make its dependence on P explicit.

Example 9. Consider the 4-element poset $P := \{1, 2\} \times \{1, 2\}$, i.e., the (Cartesian) product of two chains of length two. The Hasse diagrams of P and $\widehat{P}$ are shown below:
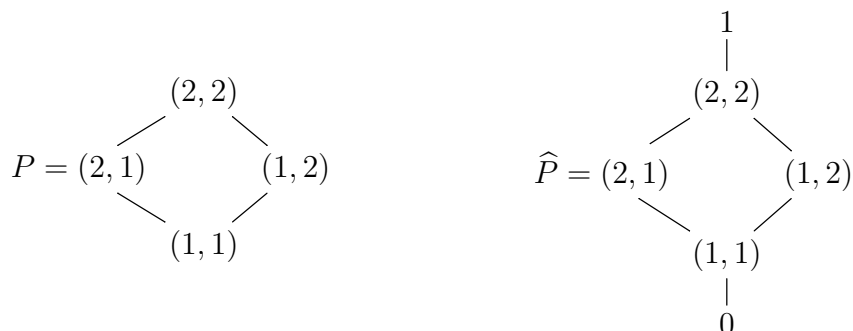
We can visualize a $\mathbb{K}$-labelling f of P by replacing, in the Hasse diagram of $\widehat{P}$, each element $v \in \widehat{P}$ by the label $f(v)$. Let f be a $\mathbb{K}$-labelling sending 0, $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, and 1 to a, w, y, x, z, and b, respectively (for some elements a, b, x, y, z, w of $\mathbb{K}$). Then f and the output of toggling f at the element $(2,2)$ are visualized as follows:
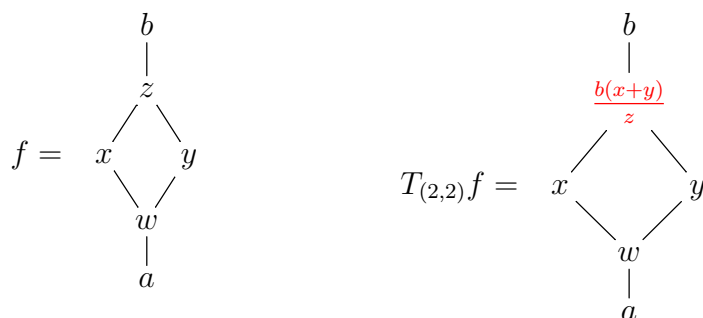
The label at $(2,2)$, which is the only one that changed, was computed via

$$
\left(T_{(2,2)}f\right)(2,2) = \frac{1}{f((2,2))} \cdot \frac{\sum\limits_{\substack{u \in \widehat{P}; \\ u \lessdot (2,2)}} f(u)}{\sum\limits_{\substack{u \in \widehat{P}; \\ u \gtrdot (2,2)}} \frac{1}{f(u)}} = \frac{1}{f((2,2))} \cdot \frac{f((1,2)) + f((2,1))}{\left(\frac{1}{f(1)}\right)}
$$

$$
= \frac{1}{z} \cdot \frac{y + x}{\left(\frac{1}{b}\right)} = \frac{b(x+y)}{z}.
$$

To compute Rf, we need to toggle at each vertex of P along a linear extension. Computing successively $T_{(2,1)}T_{(2,2)}f$, $T_{(1,2)}T_{(2,1)}T_{(2,2)}f$, and $Rf = T_{(1,1)}T_{(1,2)}T_{(2,1)}T_{(2,2)}f$ gives (respectively)

$$
\begin{array}{ccc}
\begin{array}{c}
b \\ | \\ \frac{b(x+y)}{z} \\
\diagup \quad \diagdown \\
x \qquad \frac{b(x+y)w}{yz} \\
\diagdown \quad \diagup \\
w \\ | \\ a
\end{array}
&
\begin{array}{c}
b \\ | \\ \frac{b(x+y)}{z} \\
\diagup \quad \diagdown \\
\frac{b(x+y)w}{xz} \qquad \frac{b(x+y)w}{yz} \\
\diagdown \quad \diagup \\
w \\ | \\ a
\end{array}
& \text{and} \quad
\begin{array}{c}
b \\ | \\ \frac{b(x+y)}{z} \\
\diagup \quad \diagdown \\
\frac{b(x+y)w}{xz} \qquad \frac{b(x+y)w}{yz} \\
\diagdown \quad \diagup \\
\frac{ab}{z} \\ | \\ a
\end{array}
\\
, & , &
\end{array}
$$
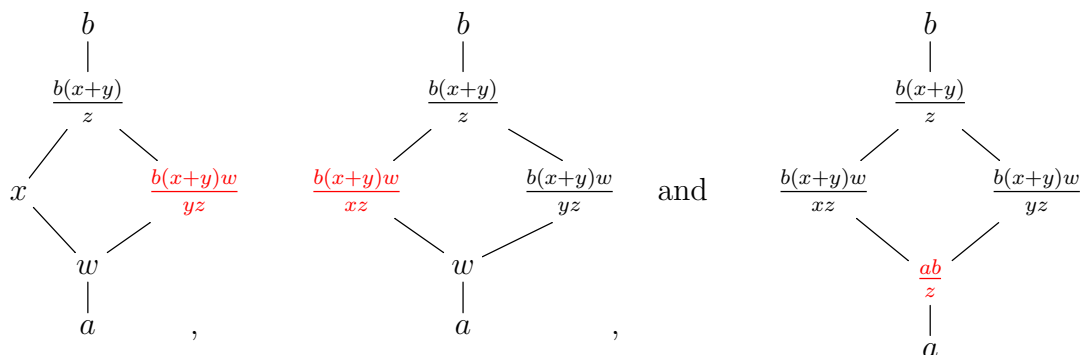
(after cancelling terms). By repeating this procedure (or just substituting the labels of Rf obtained as variables), we can compute R^2f, R^3f etc.. Specifically, we obtain:

$$
Rf = \quad
\begin{array}{c}
b \\ | \\ \frac{b(x+y)}{z} \\
\diagup \quad \diagdown \\
\frac{b(x+y)w}{xz} \qquad \frac{b(x+y)w}{yz} \\
\diagdown \quad \diagup \\
\frac{ab}{z} \\ | \\ a
\end{array}
\qquad
R^2f = \quad
\begin{array}{c}
b \\ | \\ \frac{b(x+y)w}{xy} \\
\diagup \quad \diagdown \\
\frac{ab}{y} \qquad \frac{ab}{x} \\
\diagdown \quad \diagup \\
\frac{az}{x+y} \\ | \\ a
\end{array}
$$

$$
R^3f = \quad
\begin{array}{c}
b \\ | \\ \frac{ab}{w} \\
\diagup \quad \diagdown \\
\frac{ayz}{(x+y)w} \qquad \frac{axz}{(x+y)w} \\
\diagdown \quad \diagup \\
\frac{axy}{(x+y)w} \\ | \\ a
\end{array}
\qquad
R^4f = \quad
\begin{array}{c}
b \\ | \\ z \\
\diagup \quad \diagdown \\
x \qquad y \\
\diagdown \quad \diagup \\
w \\ | \\ a
\end{array}
$$
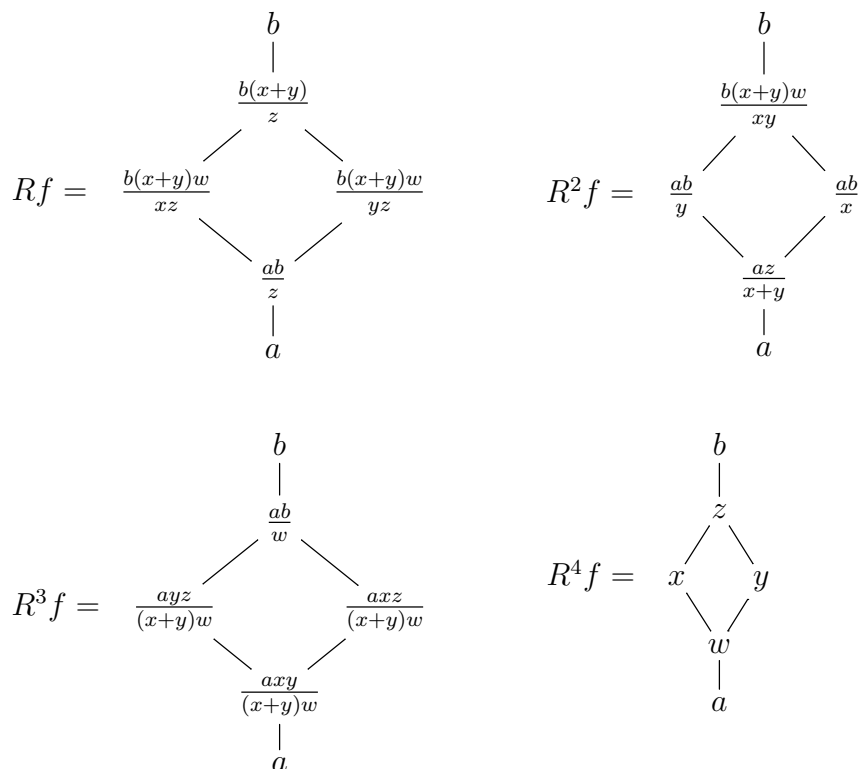
There are two surprises here. First, it turns out that $R^4f = f$. This is not obvious, but generalizes in at least two ways: On the one hand, our poset P is a particular case of what we called a "skeletal poset" in [GrRo14a, §9], a class of posets which all were shown in [GrRo14a, §9] to satisfy $R^n = \mathrm{id}$ for some sufficiently high positive integer n (which can be explicitly computed depending on P). On the other hand, our poset P is

a particular case of rectangle posets, which turn out (Theorem 30) to satisfy $R^{p+q} = \mathrm{id}$ with p and q being the side lengths of the rectangle. Second, on a more subtle level, the rational functions appearing as labels in Rf, $R^2 f$ and $R^3 f$ are not as "wild" as one might expect. The values $(Rf)((1,1))$, $(R^2 f)((1,2))$, $(R^2 f)((2,1))$ and $(R^3 f)((2,2))$ each have the form $\dfrac{ab}{f(v)}$ for some $v \in P$. This is a "reciprocity" phenomenon which turns out to generalize to arbitrary rectangles (Theorem 32).

In the above calculation, we used the linear extension $((1,1),(1,2),(2,1),(2,2))$ of P to compute R as $T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,1)} \circ T_{(2,2)}$. We could have just as well used the linear extension $((1,1),(2,1),(1,2),(2,2))$, obtaining the same result. But we could not have used the list $((1,1),(1,2),(2,2),(2,1))$ (for example), since it is not a linear extension (and indeed, the order of $T_{(1,1)} \circ T_{(1,2)} \circ T_{(2,2)} \circ T_{(2,1)}$ is infinite, as follows from the results of [EiPr13, §12.2]).

A different example for birational rowmotion is given in [GrRo14a, §2].

The next proposition merely describe the situation when one is partway through the toggling process. Here (and elsewhere), we tacitly assume that Rf is well-defined because these assumptions are satisfied when the parameters belong to some Zariski-dense open subset of their domains.

Proposition 10. *Let $v \in P$. Let $f \in \mathbb{K}^{\widehat{P}}$. Then,*

$$(Rf)(v) = \frac{1}{f(v)} \cdot \frac{\displaystyle\sum_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u)}{\displaystyle\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \frac{1}{(Rf)(u)}}. \tag{2}$$

Here (and in later statements such as Proposition 10), we take the liberty of leaving assumptions such as "Assume that Rf is well-defined" unsaid. These assumptions are satisfied when the parameters belong to some Zariski-dense open subset of their domains.

Proposition 11. *Let $f \in \mathbb{K}^{\widehat{P}}$. Then, $(Rf)(0) = f(0)$ and $(Rf)(1) = f(1)$.*

Corollary 12. *Let $f \in \mathbb{K}^{\widehat{P}}$ and $\ell \in \mathbb{N}$. Then, $(R^\ell f)(0) = f(0)$ and $(R^\ell f)(1) = f(1)$.*

Proposition 13. *Let $f \in \mathbb{K}^{\widehat{P}}$ and $g \in \mathbb{K}^{\widehat{P}}$ be such that $f(0) = g(0)$ and $f(1) = g(1)$. Assume that*

$$g(v) = \frac{1}{f(v)} \cdot \frac{\displaystyle\sum_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u)}{\displaystyle\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \frac{1}{g(u)}} \qquad \text{for every } v \in P. \tag{3}$$

(This means, in particular, that we assume that all denominators in (3) are nonzero.) Then, $g = Rf$. [3]

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[3]More precisely, Rf **is well-defined** and equals to g.

2 Auxiliary results

In this section, we collect further results from [GrRo14a] (which the interested reader may consult), needed for the proofs (but not the statements) of our results.

Definition 14. Let $n \in \mathbb{N}$. We call a finite poset P *n-graded* if there exists a surjective map $\deg : P \to \{1, 2, \ldots, n\}$ satisfying:

   Assertion 1: Any two elements u and v of P such that $u \gtrdot v$ satisfy $\deg u = \deg v + 1$.
   Assertion 2: We have $\deg u = 1$ for every minimal element u of P.
   Assertion 3: We have $\deg v = n$ for every maximal element v of P.

Throughout the rest of this paper, unless otherwise stated, P will denote an n-graded poset (for some $n \in \mathbb{N}$).

Example 15. The poset $\{1, 2\} \times \{1, 2\}$ from Example 9 is 3-graded. The empty poset is 0-graded, but not n-graded for any $n > 0$. A chain with k elements is k-graded.

Definition 16. Let P be an n-graded poset. Then, there exists a surjective map $\deg : P \to \{1, 2, \ldots, n\}$ that satisfies the Assertions 1, 2 and 3 of Definition 14. A moment of thought reveals that such a map $\deg$ is also uniquely determined by P.

Moreover, we extend this map $\deg$ to a map $\widehat{P} \to \{0, 1, \ldots, n+1\}$ by letting it map 0 to 0 and 1 to $n + 1$. This extended map will also be denoted by $\deg$. Notice that this extended map $\deg$ still satisfies Assertion 1 of Definition 14 if P is replaced by $\widehat{P}$ in that assertion.

The notion of an "n-graded poset" we just defined is identical with the notion of a "graded finite poset of rank $n - 1$" as defined in [Stan11, §3.1]. For $v \in \widehat{P}$, the integer $\deg v$ as defined in Definition 16 equals the rank of v in $\widehat{P}$ in the sense of [Stan11, §3.1], but is off by 1 from the rank of v in P in the sense of [Stan11, §3.1] (if v lies in P).

The rationale for setting $\deg 0 = 0$ and $\deg 1 = n + 1$ in Definition 16 was to make the following hold:

Proposition 17. *Let P be an n-graded poset. Let $u \in \widehat{P}$ and $v \in \widehat{P}$. Consider the map* $\deg : \widehat{P} \to \{0, 1, \ldots, n+1\}$ *defined in Definition 16.*

   (a) *If $u \lessdot v$ in $\widehat{P}$, then $\deg u = \deg v - 1$.*
   (b) *If $u < v$ in $\widehat{P}$, then $\deg u < \deg v$.*
   (c) *If $u < v$ in $\widehat{P}$ and $\deg u = \deg v - 1$, then $u \lessdot v$ in $\widehat{P}$.*

Definition 18. For every $\mathbb{K}$-labelling $f \in \mathbb{K}^{\widehat{P}}$ and any $(n+2)$-tuple $(a_0, a_1, \ldots, a_{n+1}) \in (\mathbb{K}^{\times})^{n+2}$, we define a $\mathbb{K}$-labelling $(a_0, a_1, \ldots, a_{n+1}) \flat f \in \mathbb{K}^{\widehat{P}}$ by

$$((a_0, a_1, \ldots, a_{n+1}) \flat f)(v) = a_{\deg v} \cdot f(v) \qquad \text{for every } v \in \widehat{P}.$$

The next proposition shows what happens when we rescale each rank by a constant.

Proposition 19. *For every $\mathbb{K}$-labelling $f \in \mathbb{K}^{\widehat{P}}$ and any $(n+2)$-tuple $(a_0, a_1, \ldots, a_{n+1}) \in (\mathbb{K}^{\times})^{n+2}$, we define a $\mathbb{K}$-labelling $(a_0, a_1, \ldots, a_{n+1}) \flat f \in \mathbb{K}^{\widehat{P}}$ as in Definition 18.*

Let $(a_0, a_1, \ldots, a_{n+1}) \in (\mathbb{K}^{\times})^{n+2}$. For every $\ell \in \{0, 1, \ldots, n+1\}$ and $k \in \{0, 1, \ldots, n+1\}$, define an element $\widehat{a}_k^{(\ell)} \in \mathbb{K}^{\times}$ by

$$\widehat{a}_k^{(\ell)} = \begin{cases} \dfrac{a_{n+1}a_{k-\ell}}{a_{n+1-\ell}}, & \text{if } k \geqslant \ell; \\ \dfrac{a_{n+1+k-\ell}a_0}{a_{n+1-\ell}}, & \text{if } k < \ell \end{cases}.$$

Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labelling. Then, every $\ell \in \{0, 1, \ldots, n+1\}$ satisfies

$$R^\ell \left((a_0, a_1, \ldots, a_{n+1}) \flat f \right) = \left(\widehat{a}_0^{(\ell)}, \widehat{a}_1^{(\ell)}, \ldots, \widehat{a}_{n+1}^{(\ell)} \right) \flat \left(R^\ell f \right)$$

Example 20. Let P be a 3-graded poset. Then,

$$\left(\widehat{a}_0^{(0)}, \widehat{a}_1^{(0)}, \widehat{a}_2^{(0)}, \widehat{a}_3^{(0)}, \widehat{a}_4^{(0)} \right) = (a_0, a_1, a_2, a_3, a_4) ;$$

$$\left(\widehat{a}_0^{(1)}, \widehat{a}_1^{(1)}, \widehat{a}_2^{(1)}, \widehat{a}_3^{(1)}, \widehat{a}_4^{(1)} \right) = \left(a_0, \frac{a_4 a_0}{a_3}, \frac{a_4 a_1}{a_3}, \frac{a_4 a_2}{a_3}, a_4 \right) ;$$

$$\left(\widehat{a}_0^{(2)}, \widehat{a}_1^{(2)}, \widehat{a}_2^{(2)}, \widehat{a}_3^{(2)}, \widehat{a}_4^{(2)} \right) = \left(a_0, \frac{a_3 a_0}{a_2}, \frac{a_4 a_0}{a_2}, \frac{a_4 a_1}{a_2}, a_4 \right) ;$$

$$\left(\widehat{a}_0^{(3)}, \widehat{a}_1^{(3)}, \widehat{a}_2^{(3)}, \widehat{a}_3^{(3)}, \widehat{a}_4^{(3)} \right) = \left(a_0, \frac{a_2 a_0}{a_1}, \frac{a_3 a_0}{a_1}, \frac{a_4 a_0}{a_1}, a_4 \right) ;$$

$$\left(\widehat{a}_0^{(4)}, \widehat{a}_1^{(4)}, \widehat{a}_2^{(4)}, \widehat{a}_3^{(4)}, \widehat{a}_4^{(4)} \right) = (a_0, a_1, a_2, a_3, a_4) .$$

More generally, direct computation easily shows that

$$\left(\widehat{a}_0^{(0)}, \widehat{a}_1^{(0)}, \ldots, \widehat{a}_{n+1}^{(0)} \right) = (a_0, a_1, \ldots, a_{n+1}) = \left(\widehat{a}_0^{(n+1)}, \widehat{a}_1^{(n+1)}, \ldots, \widehat{a}_{n+1}^{(n+1)} \right) .$$

Corollary 21. *In the situation of Proposition 19, we have*

$$R^{n+1} \left((a_0, a_1, \ldots, a_{n+1}) \flat f \right) = (a_0, a_1, \ldots, a_{n+1}) \flat \left(R^{n+1} f \right)$$

Definition 22. Let S be a set.

(a) If α and β are two partial maps from the set S, then we write "$\alpha = \beta$" to mean: every $s \in S$ for which both $\alpha(s)$ and $\beta(s)$ are well-defined satisfies $\alpha(s) = \beta(s)$. This is, per se, not a well-behaved notation (e.g., it is possible that three partial maps α, β and γ satisfy $\alpha = \beta$ and $\beta = \gamma$ but not $\alpha = \gamma$). However, we are going to use this notation for rational maps and their quotients (and, of course, total maps) only; in all of these cases, the notation **is** well-behaved (e.g., if α, β and γ are three rational maps satisfying $\alpha = \beta$ and $\beta = \gamma$, then $\alpha = \gamma$, because the intersection of two Zariski-dense open subsets is Zariski-dense and open).

(b) The *order* of a partial map $\varphi : S \dashrightarrow S$ is defined to be the smallest positive integer k satisfying $\varphi^k = \mathrm{id}_S$, if such a positive integer k exists, and ∞ otherwise. Here,

we are disregarding the fact that φ is only a partial map; we will be working only with dominant rational maps and their quotients (and total maps), so nothing will go wrong.

We denote the order of a partial map $\varphi : S \dashrightarrow S$ as $\operatorname{ord} \varphi$.

Definition 23. Let P be a poset. Then, P^{op} will denote the poset defined on the same ground set as P but with the order relation defined by

$$((a <_{P^{\mathrm{op}}} b \text{ if and only if } b <_P a) \text{ for all } a \in P \text{ and } b \in P)$$

(where $<_P$ denotes the smaller-than relation of the poset P, and where $<_{P^{\mathrm{op}}}$ denotes the smaller-than relation of the poset P^{op} which we are defining). The poset P^{op} is called the *opposite poset* of P.

Note that P^{op} is called the *dual* of the poset P in [Stan11].

Remark 24. It is clear that $(P^{\mathrm{op}})^{\mathrm{op}} = P$ for any poset P. Also, if $n \in \mathbb{N}$, and if P is an n-graded poset, then P^{op} is an n-graded poset.

Proposition 25. *Let P be a finite poset. Let $\mathbb{K}$ be a field. Then,* $\operatorname{ord}(R_{P^{\mathrm{op}}}) = \operatorname{ord}(R_P)$.

We notice one further result, which was never explicitly stated in [GrRo14a] but follows from [GrRo14a, Proposition 62]. This lemma will only be used to show that $\operatorname{ord} R$ is equal to (rather than only a divisor of) a certain value.

Lemma 26. *Let $n \in \mathbb{N}$. Let $\mathbb{K}$ be a field. Let P be an n-graded poset. Then, $n+1 \mid \operatorname{ord} R$.* *(We understand that $m \mid \infty$ for any positive integer m.)*

3 The rectangle: statements of the results

We now are ready to state our main results.

Definition 27. Let p and q be two positive integers. The $p \times q$-*rectangle*, $\operatorname{Rect}(p, q)$, will denote the poset $\{1, 2, \ldots, p\} \times \{1, 2, \ldots, q\}$ with order defined as follows: For two elements (i, k) and (i', k') of $\{1, 2, \ldots, p\} \times \{1, 2, \ldots, q\}$, we set $(i, k) \leqslant (i', k')$ if and only if $(i \leqslant i' \text{ and } k \leqslant k')$.

Example 28. Here is the Hasse diagram of the 2×3-rectangle:
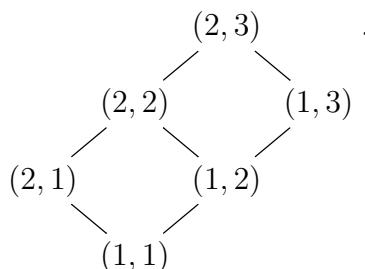
Remark 29. The $p \times q$-rectangle is denoted by $[p] \times [q]$ in [StWi11, PrRo13, EiPr13]. It is clear that $\mathrm{Rect}\,(p,q)$ is a $(p+q-1)$-graded poset, with $\deg\,((i,k)) = i+k-1$ for all $(i,k) \in \mathrm{Rect}\,(p,q)$. Also the covering relations are given by $(i,k) \lessdot (i',k')$ in $\mathrm{Rect}\,(p,q)$ if and only if either ($i' = i$ and $k' = k+1$) or ($k' = k$ and $i' = i+1$).

The following periodicity theorem was conjectured by James Propp and the second author:

Theorem 30. *The order of birational rowmotion on the $\mathbb{K}$-labelings of a $p \times q$-rectangle is $p+q$, i.e.,* $\mathrm{ord}\left(R_{\mathrm{Rect}(p,q)}\right) = p+q$.

This is a birational analogue (and, using the reasoning of [EiPr13], generalization) of the classical fact (appearing in [StWi11, Theorem 3.1] and [Flaa93, Theorem 2]) that $\mathrm{ord}\left(\mathbf{r}_{\mathrm{Rect}(p,q)}\right) = p+q$ (where $\mathbf{r}_P$ denotes the classical rowmotion map on the order ideals of a poset P). When $p \leqslant 2$ and $q \leqslant 2$, Theorem 30 follows rather easily from results in [GrRo14a, §9] (because $\mathrm{Rect}\,(p,q)$ is a skeletal poset in this case).

Remark 31. Theorem 30 generalizes a well-known property of promotion on semistandard Young tableaux of rectangular shape, albeit not in an obvious way. Let N be a nonnegative integer, and let λ be a partition. Let $\mathrm{SSYT}_N\,\lambda$ denote the set of all semistandard Young tableaux of shape λ whose entries are all $\leqslant N$. One can define a map $\mathrm{Pro} : \mathrm{SSYT}_N\,\lambda \to \mathrm{SSYT}_N\,\lambda$ called *jeu-de-taquin promotion* (or *Schützenberger promotion*, or simply *promotion* when no ambiguities can arise); see [Russ13, §5.1] for a precise definition. This map has some interesting properties already for arbitrary λ, but the most interesting situation is that of λ being a rectangular partition (i.e., a partition all of whose nonzero parts are equal). In this situation, a folklore theorem states that $\mathrm{Pro}^N = \mathrm{id}$. (The particular case of this theorem when Pro is applied only to **standard** Young tableaux is well-known (see, e.g., [Haiman92, Theorem 4.4]), but the only proof of the general theorem that we were able to find in the literature is Rhoades's [Rhoa10, Corollary 5.6], which makes use of Kazhdan-Lusztig theory.)

Theorem 30 can be used to give an alternative proof of this $\mathrm{Pro}^N = \mathrm{id}$ theorem. See a future version of [EiPr13] (or, for the time being, [EiPr14, §2, pp. 4–5]) for how this works.

Note that [Russ13, §5.1], [Rhoa10, §2] and [EiPr13] give three different definitions of promotion. The definitions in [Russ13, §5.1] and in [EiPr13] are equivalent, while the definition in [Rhoa10, §2] defines the inverse of the map defined in the other two sources. Unfortunately, we were unable to find the proofs of these facts in existing literature; they are claimed in [KiBe95, Propositions 2.5 and 2.6], and can be proven using the concept of tableau switching [Leeu01, Definition 2.2.1].

Besides Theorem 30, our other main theorem states a symmetry property of birational rowmotion on the $p \times q$-rectangle (referred to as the "pairing property" in [EiPr13]), which was also conjectured by Propp and the second author. It generalizes the "reciprocity phenomenon" observed on the 2×2-rectangle in Example 9.

Theorem 32. *Let $f \in \mathbb{K}^{\widehat{\mathrm{Rect}(p,q)}}$. Assume that $R_{\mathrm{Rect}(p,q)}^{\ell} f$ is well-defined for every $\ell \in \{0, 1, \ldots, i+k-1\}$. Then for any $(i,k) \in \mathrm{Rect}(p,q)$ we have*

$$f\left((p+1-i, q+1-k)\right) = \frac{f(0) f(1)}{\left(R_{\mathrm{Rect}(p,q)}^{i+k-1} f\right)((i,k))}.$$

Remark 33. While Theorem 30 only makes a statement about $R_{\mathrm{Rect}(p,q)}$, it can be used (in combination with results from [GrRo14a]) to derive upper bounds on the order of R_P for some other posets P. For example, let N denote the (eponymously named) poset $\mathrm{Rect}(2,3) \setminus \{(1,1), (2,3)\}$, with Hasse diagram

$$\begin{array}{cc} (2,2) & (1,3) \\ \diagup \diagdown & \diagup \\ (2,1) \qquad (1,2) \end{array}. \text{ Then}$$
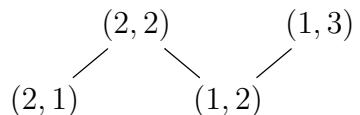
$\mathrm{ord}(R_N) \mid 15$. (See [GrRo14b] for details). It can actually be shown that $\mathrm{ord}(R_N) = 15$ by direct computation.

In the same vein it can be shown that $\mathrm{ord}\left(R_{\mathrm{Rect}(p,q)\setminus\{(1,1),(p,q)\}}\right) \mid \mathrm{lcm}(p+q-2, p+q)$ for any integers $p > 1$ and $q > 1$. This doesn't, however, generalize to arbitrary posets obtained by removing some ranks from $\mathrm{Rect}(p,q)$ (indeed $\mathrm{ord}\, R_P$ is infinite for some posets of this type, cf. Section 12).

4 Reduced labellings

The proof that we give for Theorem 30 and Theorem 32 is largely inspired by the proof of Zamolodchikov's conjecture in case AA given by Volkov in [Volk06][4]. This is not very surprising because the orbit of a $\mathbb{K}$-labelling under birational rowmotion appears superficially similar to a solution of a Y-system of type AA. Yet we do not see a way to derive Theorem 30 from Zamolodchikov's conjecture or vice versa. (Here the Y-system has an obvious "reducibility property", consisting of two decoupled subsystems – a property not obviously satisfied in the case of birational rowmotion.)

The first step towards our proof of Theorem 30 is to restrict attention to so-called *reduced labellings*, which are not much less general than arbitrary labellings: Many results can be proven for all labellings by means of proving them for reduced labellings first, and then extending them to general labellings by fairly simple arguments. We will use this tactic in our proof of Theorem 30. A slightly different way to reduce the case of a general labelling to that of a reduced one is taken in [EiPr13, §4].

Definition 34. A labelling $f \in \mathbb{K}^{\widehat{\mathrm{Rect}(p,q)}}$ is said to be *reduced* if $f(0) = f(1) = 1$. The set of all reduced labellings $f \in \mathbb{K}^{\widehat{\mathrm{Rect}(p,q)}}$ will be identified with $\mathbb{K}^{\mathrm{Rect}(p,q)}$ in the obvious way.

Note that fixing the values of $f(0)$ and $f(1)$ like this makes f "less generic", but still the operator $R_{\mathrm{Rect}(p,q)}$ restricts to a rational map from the variety of all reduced labellings

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[4] "Case AA" refers to the Cartesian product of the Dynkin diagrams of two type-A root systems. This, of course, is a rectangle, just as in our Theorem 30.

$f \in \mathbb{K}^{\widehat{\mathrm{Rect}(p,q)}}$ to itself. (This is because the operator $R_{\mathrm{Rect}(p,q)}$ does not change the values at 0 and 1, and does not degenerate from setting $f(0) = f(1) = 1$.)

Proposition 35. *Assume that almost every (in the Zariski sense) reduced labelling $f \in \mathbb{K}^{\widehat{\mathrm{Rect}(p,q)}}$ satisfies $R^{p+q}_{\mathrm{Rect}(p,q)}f = f$. Then, $\mathrm{ord}\left(R_{\mathrm{Rect}(p,q)}\right) = p+q$.*

Proof. Let $g \in \mathbb{K}^{\widehat{\mathrm{Rect}(p,q)}}$ be any $\mathbb{K}$-labelling of $\mathrm{Rect}(p,q)$ which is sufficiently generic for $R^{p+q}_{\mathrm{Rect}(p,q)}g$ to be well-defined.

We can easily find a $(p+q+1)$-tuple $(a_0, a_1, \ldots, a_{p+q}) \in (\mathbb{K}^{\times})^{p+q+1}$ such that $(a_0, a_1, \ldots, a_{p+q}) \flat g$ is a reduced $\mathbb{K}$-labelling (in fact, set $a_0 = \dfrac{1}{g(0)}$ and $a_{p+q} = \dfrac{1}{g(1)}$, and choose all other a_i arbitrarily). Corollary 21 then yields

$$R^{p+q}_{\mathrm{Rect}(p,q)}\left((a_0, a_1, \ldots, a_{p+q}) \flat g\right) = (a_0, a_1, \ldots, a_{p+q}) \flat \left(R^{p+q}_{\mathrm{Rect}(p,q)}g\right). \tag{4}$$

We have assumed that almost every (in the Zariski sense) reduced labelling $f \in \mathbb{K}^{\widehat{\mathrm{Rect}(p,q)}}$ satisfies $R^{p+q}_{\mathrm{Rect}(p,q)}f = f$. Thus, every reduced labelling $f \in \mathbb{K}^{\widehat{\mathrm{Rect}(p,q)}}$ for which $R^{p+q}_{\mathrm{Rect}(p,q)}f$ is well-defined satisfies $R^{p+q}_{\mathrm{Rect}(p,q)}f = f$ (because $R^{p+q}_{\mathrm{Rect}(p,q)}f = f$ can be written as an equality between rational functions in the labels of f, and thus it must hold everywhere if it holds on a Zariski-dense open subset). Applying this to $f = (a_0, a_1, \ldots, a_{p+q}) \flat g$, we obtain that $R^{p+q}_{\mathrm{Rect}(p,q)}\left((a_0, a_1, \ldots, a_{p+q}) \flat g\right) = (a_0, a_1, \ldots, a_{p+q}) \flat g$. Thus,

$$(a_0, a_1, \ldots, a_{p+q}) \flat g = R^{p+q}_{\mathrm{Rect}(p,q)}\left((a_0, a_1, \ldots, a_{p+q}) \flat g\right) = (a_0, a_1, \ldots, a_{p+q}) \flat \left(R^{p+q}_{\mathrm{Rect}(p,q)}g\right)$$

(by (4)). We can cancel the "$(a_0, a_1, \ldots, a_{p+q}) \flat$" from both sides of this equality (because all the a_i are nonzero), and thus obtain $g = R^{p+q}_{\mathrm{Rect}(p,q)}g$.

Now, forget that we fixed g. We thus have proven that $g = R^{p+q}_{\mathrm{Rect}(p,q)}g$ holds for every $\mathbb{K}$-labelling $g \in \mathbb{K}^{\widehat{\mathrm{Rect}(p,q)}}$ of $\mathrm{Rect}(p,q)$ which is sufficiently generic for $R^{p+q}_{\mathrm{Rect}(p,q)}g$ to be well-defined. In other words, $R^{p+q}_{\mathrm{Rect}(p,q)} = \mathrm{id}$ as partial maps. Hence, $\mathrm{ord}\left(R_{\mathrm{Rect}(p,q)}\right) \mid p+q$.

On the other hand, Lemma 26 yields that $\mathrm{ord}\left(R_{\mathrm{Rect}(p,q)}\right)$ is divisible by $(p+q-1)+1 = p+q$. Combined with $\mathrm{ord}\left(R_{\mathrm{Rect}(p,q)}\right) \mid p+q$, this yields $\mathrm{ord}\left(R_{\mathrm{Rect}(p,q)}\right) = p+q$. $\square$

Let us also formulate the particular case of Theorem 32 for reduced labellings, which we will use a stepping stone to the more general theorem.

Theorem 36. *Let $f \in \mathbb{K}^{\widehat{\mathrm{Rect}(p,q)}}$ be reduced. Assume that $R^{\ell}_{\mathrm{Rect}(p,q)}f$ is well-defined for every $\ell \in \{0, 1, \ldots, i+k-1\}$. Then for any $(i,k) \in \mathrm{Rect}(p,q)$ we have*

$$f((p+1-i, q+1-k)) = \frac{1}{\left(R^{i+k-1}_{\mathrm{Rect}(p,q)}f\right)((i,k))}.$$

5　The Grassmannian parametrization: statements

In this section, we introduce the main actor in our proof of Theorem 30: an assignment of a reduced $\mathbb{K}$-labelling of $\mathrm{Rect}\,(p,q)$, denoted $\mathrm{Grasp}_j\,A$, to any integer j and almost any matrix $A \in \mathbb{K}^{p \times (p+q)}$ (Definition 44). This assignment will give us a family of $\mathbb{K}$-labellings of $\mathrm{Rect}\,(p,q)$ which is large enough to cover almost all reduced $\mathbb{K}$-labellings of $\mathrm{Rect}\,(p,q)$ (Proposition 49), while at the same time the construction of this assignment makes it easy to track the behavior of the $\mathbb{K}$-labellings in this family through multiple iterations of birational rowmotion. Indeed, we will see that birational rowmotion has a very simple effect on the reduced $\mathbb{K}$-labelling $\mathrm{Grasp}_j\,A$ (Proposition 48).

Definition 37. Let $A \in \mathbb{K}^{u \times v}$ be a $u \times v$-matrix for some nonnegative integers u and v.
  (a) For every $i \in \{1, 2, \ldots, v\}$, let A_i denote the i-th column of A.
  (b) Moreover, we extend this definition to all $i \in \mathbb{Z}$ as follows: For every $i \in \mathbb{Z}$, let

$$A_i = (-1)^{(u-1)(i-i')/v} \cdot A_{i'},$$

where i' is the element of $\{1, 2, \ldots, v\}$ which is congruent to i modulo v. (Thus, $A_{v+i} = (-1)^{u-1} A_i$ for every $i \in \mathbb{Z}$. Consequently, the sequence $(A_i)_{i \in \mathbb{Z}}$ is periodic with period dividing $2v$, and if u is odd, the period also divides v.)
  (c) For any two integers a and b satisfying $a \leqslant b$, we let $A\,[a:b]$ be the matrix whose columns (from left to right) are $A_a, A_{a+1}, \ldots, A_{b-1}$.
  (d) For any four integers a, b, c and d satisfying $a \leqslant b$ and $c \leqslant d$, we let $A\,[a:b \mid c:d]$ be the matrix whose columns (from left to right) are $A_a, A_{a+1}, \ldots, A_{b-1}, A_c, A_{c+1}, \ldots, A_{d-1}$. (This matrix has $b - a + d - c$ columns.[5])[6] When $b - a + d - c = u$, this matrix $A\,[a:b \mid c:d]$ is a square matrix, and thus has a determinant $\det\,(A\,[a:b \mid c:d])$.
  (e) We extend the definition of $\det\,(A\,[a:b \mid c:d])$ to encompass the case when $b = a - 1$ or $d = c - 1$, by setting $\det\,(A\,[a:b \mid c:d]) = 0$ in this case (although the matrix $A\,[a:b \mid c:d]$ itself is not defined in this case).

Example 38. If $A = \begin{pmatrix} 3 & 5 & 7 \\ 4 & 1 & 9 \end{pmatrix}$, then $A_5 = (-1)^{(2-1)(5-2)/3} \cdot A_2 = -A_2 = -\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$ and $A_{-4} = (-1)^{(2-1)((-4)-2)/3} \cdot A_2 = A_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

If $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ -5 & 4 \end{pmatrix}$, then $A_0 = (-1)^{(3-1)(0-2)/2} \cdot A_2 = A_2 = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$.

Remark 39. Some parts of Definition 37 might look accidental and haphazard; here are some motivations and aide-memoires:

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[5]It is not always a submatrix of A. Its columns are columns of A multiplied with 1 or -1; they can appear several times and need not appear in the same order as they appear in A.

[6]We notice that we allow the case $a = b$. In this case, obviously, the columns of the matrix $A\,[a:b \mid c:d]$ are $A_c, A_{c+1}, \ldots, A_{d-1}$, so we have $A\,[a:b \mid c:d] = A\,[c:d]$. Similarly, the case $c = d$ is allowed.

The choice of sign in Definition 37 **(b)** is not only the "right" one for what we are going to do below, but also naturally appears in [Post06, Remark 3.3]. It guarantees, among other things, that if $A \in \mathbb{R}^{u \times v}$ is totally nonnegative, then the matrix having columns $A_{1+i}, A_{2+i}, \ldots, A_{v+i}$ is totally nonnegative for every $i \in \mathbb{Z}$.

The notation $A[a : b \mid c : d]$ in Definition 37 **(d)** borrows from Python's notation $[x : y]$ for taking indices from the interval $\{x, x+1, \ldots, y-1\}$.

The convention to define $\det(A[a : b \mid c : d])$ as 0 in Definition 37 **(e)** can be motivated using exterior algebra as follows: If we identify $\wedge^u(\mathbb{K}^u)$ with $\mathbb{K}$ by equating with $1 \in \mathbb{K}$ the wedge product $e_1 \wedge e_2 \wedge \ldots \wedge e_u$ of the standard basis vectors, then $\det(A[a : b \mid c : d]) = A_a \wedge A_{a+1} \wedge \ldots \wedge A_{b-1} \wedge A_c \wedge A_{c+1} \wedge \ldots \wedge A_{d-1}$; this belongs to the product of $\wedge^{b-a}(\mathbb{K}^u)$ with $\wedge^{d-c}(\mathbb{K}^u)$ in $\wedge^u(\mathbb{K}^u)$. If $b = a-1$, then this product is 0 (since $\wedge^{b-a}(\mathbb{K}^u) = \wedge^{-1}(\mathbb{K}^u) = 0$), so $\det(A[a : b \mid c : d])$ has to be 0 in this case.

The following four propositions are all straightforward observations.

Proposition 40. *Let $A \in \mathbb{K}^{u \times v}$. Let $a \leqslant b$ and $c \leqslant d$ be four integers satisfying $b-a+d-c = u$. Assume that some element of the interval $\{a, a+1, \ldots, b-1\}$ is congruent to some element of the interval $\{c, c+1, \ldots, d-1\}$ modulo v. Then, $\det(A[a : b \mid c : d]) = 0$.*

Proof. The assumption yields that the matrix $A[a : b \mid c : d]$ has two columns which are proportional to each other by a factor of ± 1. Hence, this matrix has determinant 0. □

Proposition 41. *Let $A \in \mathbb{K}^{u \times v}$. Let $a \leqslant b$ and $c \leqslant d$ be four integers satisfying $b - a + d - c = u$. Then,*

$$\det(A[a : b \mid c : d]) = (-1)^{(b-a)(d-c)} \det(A[c : d \mid a : b]).$$

Proof. This follows from the fact that permuting the columns of a matrix multiplies its determinant by the sign of the corresponding permutation. □

Proposition 42. *Let $A \in \mathbb{K}^{u \times v}$. Let a, b_1, b_2 and c be four integers satisfying $a \leqslant b_1 \leqslant c$ and $a \leqslant b_2 \leqslant c$. Then,*
$$A[a : b_1 \mid b_1 : c] = A[a : b_2 \mid b_2 : c].$$

Proof. Both matrices $A[a : b_1 \mid b_1 : c]$ and $A[a : b_2 \mid b_2 : c]$ are simply the matrix with columns $A_a, A_{a+1}, \ldots, A_{c-1}$. □

Proposition 43. *Let $A \in \mathbb{K}^{u \times v}$. Let $a \leqslant b$ and $c \leqslant d$ be four integers satisfying $b - a + d - c = u$. Then*
   (a) $\det(A[v+a : v+b \mid v+c : v+d]) = \det(A[a : b \mid c : d])$.
   (b) $\det(A[a : b \mid v+c : v+d]) = (-1)^{(u-1)(d-c)} \det(A[a : b \mid c : d])$.
   (c) $\det(A[a : b \mid v+c : v+d]) = \det(A[c : d \mid a : b])$.

Proof. Straightforward from the definition and basic properties of the determinant. □

Definition 44. Let p and q be two positive integers. Let $A \in \mathbb{K}^{p \times (p+q)}$. Let $j \in \mathbb{Z}$.

(a) We define a map $\mathrm{Grasp}_j A \in \mathbb{K}^{\mathrm{Rect}(p,q)}$ by

$$\left(\mathrm{Grasp}_j A\right)((i,k)) = \frac{\det\left(A\left[j+1 : j+i \mid j+i+k-1 : j+p+k\right]\right)}{\det\left(A\left[j : j+i \mid j+i+k : j+p+k\right]\right)} \tag{5}$$

This is well-defined when the matrix A is sufficiently generic (in the sense of Zariski topology), since the matrix $A\left[j : j+i \mid j+i+k : j+p+k\right]$ is obtained by picking p distinct columns out of A, some possibly multiplied with $(-1)^{u-1}$. This map $\mathrm{Grasp}_j A$ will be considered as a reduced $\mathbb{K}$-labelling of $\mathrm{Rect}(p,q)$ (since we are identifying the set of all reduced labellings $f \in \mathbb{K}^{\widehat{\mathrm{Rect}(p,q)}}$ with $\mathbb{K}^{\mathrm{Rect}(p,q)}$).

(b) It will be handy to extend the map $\mathrm{Grasp}_j A$ to a slightly larger domain by blindly following (5) (and using Definition 37 **(e)**), accepting the fact that outside $\{1, 2, \ldots, p\} \times \{1, 2, \ldots, q\}$ its values can be "infinity" (whatever this means):

$$\begin{aligned}
\left(\mathrm{Grasp}_j A\right)((0,k)) &= 0 &\quad \text{for all } k \in \{1, 2, \ldots, q\}; \\
\left(\mathrm{Grasp}_j A\right)((p+1,k)) &= \infty &\quad \text{for all } k \in \{1, 2, \ldots, q\}; \\
\left(\mathrm{Grasp}_j A\right)((i,0)) &= 0 &\quad \text{for all } i \in \{1, 2, \ldots, p\}; \\
\left(\mathrm{Grasp}_j A\right)((i,q+1)) &= \infty &\quad \text{for all } i \in \{1, 2, \ldots, p\}.
\end{aligned}$$

The term "Grasp" is meant to suggest "Grassmannian parametrization", as we will later parametrize (generic) reduced labellings on $\mathrm{Rect}(p,q)$ by matrices via this map Grasp_0. The reason for the word "Grassmannian" is that, while we have defined Grasp_j as a rational map from the matrix space $\mathbb{K}^{p \times (p+q)}$, it actually is not defined outside of the Zariski-dense open subset $\mathbb{K}^{p \times (p+q)}_{\mathrm{rk}=p}$ of $\mathbb{K}^{p \times (p+q)}$ formed by all matrices whose rank is p, on which it factors through the quotient of $\mathbb{K}^{p \times (p+q)}_{\mathrm{rk}=p}$ by the left multiplication action of $\mathrm{GL}_p \mathbb{K}$ (because it is easy to see that $\mathrm{Grasp}_j A$ is invariant under row transformations of A); this quotient is a well-known avatar of the Grassmannian.

The formula (5) is inspired by the Y_{ijk} of Volkov's [Volk06]; similar expressions (in a different context) also appear in [Kiri00, Theorem 4.21].

Example 45. If $p = 2$, $q = 2$ and $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$, then

$$\left(\mathrm{Grasp}_0 A\right)((1,1)) = \frac{\det\left(A\left[1 : 1 \mid 1 : 3\right]\right)}{\det\left(A\left[0 : 1 \mid 2 : 3\right]\right)} = \frac{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}{\det\begin{pmatrix} -a_{14} & a_{12} \\ -a_{24} & a_{22} \end{pmatrix}} = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{12}a_{24} - a_{14}a_{22}}$$

and

$$\left(\mathrm{Grasp}_1 A\right)((1,2)) = \frac{\det\left(A\left[2 : 2 \mid 3 : 5\right]\right)}{\det\left(A\left[1 : 2 \mid 4 : 5\right]\right)} = \frac{\det\begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{pmatrix}} = \frac{a_{13}a_{24} - a_{14}a_{23}}{a_{11}a_{24} - a_{14}a_{21}}.$$

We will see more examples of values of $\mathrm{Grasp}_0 A$ in Example 52.

The next two propositions follow easily from the definition and elementary properties listed above.

Proposition 46. *Let p and q be two positive integers. Let $A \in \mathbb{K}^{p \times (p+q)}$ be a matrix. Then, $\mathrm{Grasp}_j A = \mathrm{Grasp}_{p+q+j} A$ for every $j \in \mathbb{Z}$ (provided that A is sufficiently generic in the sense of Zariski topology for $\mathrm{Grasp}_j A$ to be well-defined).*

Proposition 47. *Let $A \in \mathbb{K}^{p \times (p+q)}$. Let $(i,k) \in \mathrm{Rect}\,(p,q)$ and $j \in \mathbb{Z}$. Then*

$$\left(\mathrm{Grasp}_j A\right)((i,k)) = \frac{1}{\left(\mathrm{Grasp}_{j+i+k-1} A\right)((p+1-i, q+1-k))}$$

(provided that A is sufficiently generic in the sense of Zariski topology for $\left(\mathrm{Grasp}_j A\right)((i,k))$ and $\left(\mathrm{Grasp}_{j+i+k-1} A\right)((p+1-i, q+1-k))$ to be well-defined).

Proof. Expand the definitions of $\left(\mathrm{Grasp}_j A\right)((i,k))$ and $\left(\mathrm{Grasp}_{j+i+k-1} A\right)((p+1-i, q+1-k))$ and apply Proposition 43 **(c)** twice. $\square$

Each of the next two propositions has one of the following sections devoted to its proof. These are the key lemmas that will allow us fairly easily to prove our main Theorems 30, 36 and 32 in Section 8.

Proposition 48. *Let $A \in \mathbb{K}^{p \times (p+q)}$. Let $j \in \mathbb{Z}$. Then*

$$\mathrm{Grasp}_j A = R_{\mathrm{Rect}(p,q)} \left(\mathrm{Grasp}_{j+1} A\right)$$

(provided that A is sufficiently generic in the sense of Zariski topology for the two sides of this equality to be well-defined).

Proposition 49. *For almost every (in the Zariski sense) $f \in \mathbb{K}^{\mathrm{Rect}(p,q)}$, there exists a matrix $A \in \mathbb{K}^{p \times (p+q)}$ satisfying $f = \mathrm{Grasp}_0 A$.*

6   The Plücker-Ptolemy relation

This section is devoted to proving Proposition 48. Our main tool is a fundamental determinantal identity, which we call the *Plücker-Ptolemy relation*:

Theorem 50. *Let $A \in \mathbb{K}^{u \times v}$ be a $u \times v$-matrix for some nonnegative integers u and v. Let a, b, c and d be four integers satisfying $a \leqslant b+1$ and $c \leqslant d+1$ and $b-a+d-c = u-2$. Then,*

$$\begin{aligned}
&\det\left(A\,[a-1:b \mid c:d+1]\right) \cdot \det\left(A\,[a:b+1 \mid c-1:d]\right) \\
&+ \det\left(A\,[a:b \mid c-1:d+1]\right) \cdot \det\left(A\,[a-1:b+1 \mid c:d]\right) \\
&= \det\left(A\,[a-1:b \mid c-1:d]\right) \cdot \det\left(A\,[a:b+1 \mid c:d+1]\right).
\end{aligned}$$

Notice that the special case of this theorem for $v = u + 2$, $a = 2$, $b = p$, $c = p + 2$ and $d = p + q$ is the following lemma:

Lemma 51. *Let $u \in \mathbb{N}$. Let $B \in \mathbb{K}^{u \times (u+2)}$ be a $u \times (u + 2)$-matrix. Let p and q be two integers $\geqslant 2$ satisfying $p + q = u + 2$. Then,*

$$
\det\left(B\left[1:p \mid p+2:p+q+1\right]\right) \cdot \det\left(B\left[2:p+1 \mid p+1:p+q\right]\right)
$$
$$
+ \det\left(B\left[2:p \mid p+1:p+q+1\right]\right) \cdot \det\left(B\left[1:p+1 \mid p+2:p+q\right]\right)
$$
$$
= \det\left(B\left[1:p \mid p+1:p+q\right]\right) \cdot \det\left(B\left[2:p+1 \mid p+2:p+q+1\right]\right). \tag{6}
$$

Proof of Theorem 50. Theorem 50 follows from the well-known Plücker relations (see, e.g., [KlLa72, (QR)]) applied to the $u \times (u + 2)$-matrix $A\left[a - 1 : b + 1 \mid c - 1 : d + 1\right]$. The extended versions [GrRo14b] of this paper have a self-contained proof, which we briefly outline here. First we reduce Theorem 50 to its special case, Lemma 51, by shifting columns. The latter can now be derived by (a) using row-reduction to transform as many columns as possible into standard basis vectors; (b) permuting columns to bring the matrices in (6) into block triangular form; and (c) using that the determinant of such a matrix is the product of the determinants of its blocks. □

We are now ready to prove the key lemma that birational rowmotion acts by a cyclic shifted on Grasp-labelings.

Proof of Proposition 48. Let $f = \mathrm{Grasp}_{j+1} A$ and $g = \mathrm{Grasp}_j A$. We want to show that $g = R_{\mathrm{Rect}(p,q)}(f)$. By Proposition 13 this will follow once we can show that

$$
g(v) = \frac{1}{f(v)} \cdot \frac{\displaystyle\sum_{\substack{u \in \widehat{\mathrm{Rect}(p,q)}; \\ u \lessdot v}} f(u)}{\displaystyle\sum_{\substack{u \in \widehat{\mathrm{Rect}(p,q)}; \\ u \gtrdot v}} \frac{1}{g(u)}} \qquad \text{for every } v \in \mathrm{Rect}(p,q). \tag{7}
$$

Let $v = (i, j) \in \mathrm{Rect}(p, q)$. We are clearly in one of the following four cases:
Case 1: We have $v \neq (1, 1)$ and $v \neq (p, q)$.
Case 2: We have $v = (1, 1)$ and $v \neq (p, q)$.
Case 3: We have $v \neq (1, 1)$ and $v = (p, q)$.
Case 4: We have $v = (1, 1)$ and $v = (p, q)$.

For Case 1 all elements $u \in \widehat{\mathrm{Rect}(p,q)}$ satisfying $u \lessdot v$ belong to $\mathrm{Rect}(p, q)$, and the same holds for all $u \in \widehat{\mathrm{Rect}(p,q)}$ satisfying $u \gtrdot v$.

Now in $\mathrm{Rect}(p, q)$ there are at most two elements u of $\widehat{\mathrm{Rect}(p,q)}$ satisfying $u \lessdot v$, namely $(i, k - 1)$ and $(i - 1, k)$. Hence, the sum $\displaystyle\sum_{\substack{u \in \widehat{\mathrm{Rect}(p,q)}; \\ u \lessdot v}} f(u)$ takes one of the three forms $f((i, k - 1)) + f((i - 1, k))$, $f((i, k - 1))$ and $f((i - 1, k))$. By the convention of Definition 44 **(b)**, all of these three forms can be rewritten uniformly as $f((i, k - 1)) + f((i - 1, k))$.

So we have

$$\sum_{\substack{u\in\widehat{\mathrm{Rect}(p,q)};\\ u\lessdot v}} f\left(u\right) = f\left(\left(i,k-1\right)\right) + f\left(\left(i-1,k\right)\right). \tag{8}$$

Similarly,

$$\sum_{\substack{u\in\widehat{\mathrm{Rect}(p,q)};\\ u\gtrdot v}} \frac{1}{g\left(u\right)} = \frac{1}{g\left(\left(i,k+1\right)\right)} + \frac{1}{g\left(\left(i+1,k\right)\right)}, \tag{9}$$

where we set $\dfrac{1}{\infty} = 0$ as usual.

But $f = \mathrm{Grasp}_{j+1} A$. Hence,

$$
\begin{aligned}
&f\left(\left(i,k-1\right)\right)\\
&= \left(\mathrm{Grasp}_{j+1} A\right)\left(\left(i,k-1\right)\right)\\
&= \frac{\det\left(A\left[j+2:j+i+1 \mid j+i+k-1:j+p+k\right]\right)}{\det\left(A\left[j+1:j+i+1 \mid j+i+k:j+p+k\right]\right)}
\end{aligned}
$$

and

$$
\begin{aligned}
&f\left(\left(i-1,k\right)\right)\\
&= \left(\mathrm{Grasp}_{j+1} A\right)\left(\left(i-1,k\right)\right)\\
&= \frac{\det\left(A\left[j+2:j+i \mid j+i+k-1:j+p+k+1\right]\right)}{\det\left(A\left[j+1:j+i \mid j+i+k:j+p+k+1\right]\right)}.
\end{aligned}
$$

Due to these two equalities, (8) becomes

$$
\begin{aligned}
\sum_{\substack{u\in\widehat{\mathrm{Rect}(p,q)};\\ u\lessdot v}} f\left(u\right) &= \frac{\det\left(A\left[j+2:j+i+1 \mid j+i+k-1:j+p+k\right]\right)}{\det\left(A\left[j+1:j+i+1 \mid j+i+k:j+p+k\right]\right)}\\[2mm]
&\quad + \frac{\det\left(A\left[j+2:j+i \mid j+i+k-1:j+p+k+1\right]\right)}{\det\left(A\left[j+1:j+i \mid j+i+k:j+p+k+1\right]\right)}\\
&= \left(\det\left(A\left[j+1:j+i+1 \mid j+i+k:j+p+k\right]\right)\right)^{-1}\\
&\quad\cdot \left(\det\left(A\left[j+1:j+i \mid j+i+k:j+p+k+1\right]\right)\right)^{-1}\\
&\quad\cdot \left(\det\left(A\left[j+1:j+i \mid j+i+k:j+p+k+1\right]\right)\right.\\
&\qquad\quad \cdot \det\left(A\left[j+2:j+i+1 \mid j+i+k-1:j+p+k\right]\right)\\
&\qquad + \det\left(A\left[j+2:j+i \mid j+i+k-1:j+p+k+1\right]\right)\\
&\qquad\quad \left.\cdot \det\left(A\left[j+1:j+i+1 \mid j+i+k:j+p+k\right]\right)\right)\\
&= \left(\det\left(A\left[j+1:j+i+1 \mid j+i+k:j+p+k\right]\right)\right)^{-1}\\
&\quad\cdot \left(\det\left(A\left[j+1:j+i \mid j+i+k:j+p+k+1\right]\right)\right)^{-1}\\
&\quad\cdot \det\left(A\left[j+1:j+i \mid j+i+k-1:j+p+k\right]\right)\\
&\quad\cdot \det\left(A\left[j+2:j+i+1 \mid j+i+k:j+p+k+1\right]\right) \tag{10}
\end{aligned}
$$

(by Theorem 50, applied to $a = j + 2$, $b = j + i$, $c = j + i + k$ and $d = j + p + k$).

On the other hand, $g = \mathrm{Grasp}_j A$, so a similar series of computations gives

$$\sum_{\substack{u \in \widehat{\mathrm{Rect}(p,q)}; \\ u \gtrdot v}} \frac{1}{g(u)} = \frac{\det\left(A\left[j : j+i \mid j+i+k+1 : j+p+k+1\right]\right)}{\det\left(A\left[j+1 : j+i \mid j+i+k : j+p+k+1\right]\right)}$$

$$+ \frac{\det\left(A\left[j : j+i+1 \mid j+i+k+1 : j+p+k\right]\right)}{\det\left(A\left[j+1 : j+i+1 \mid j+i+k : j+p+k\right]\right)}$$

$$= \left(\det\left(A\left[j+1 : j+i \mid j+i+k : j+p+k+1\right]\right)\right)^{-1}$$
$$\cdot \left(\det\left(A\left[j+1 : j+i+1 \mid j+i+k : j+p+k\right]\right)\right)^{-1}$$
$$\cdot \left(\det\left(A\left[j : j+i \mid j+i+k+1 : j+p+k+1\right]\right)\right.$$
$$\cdot \det\left(A\left[j+1 : j+i+1 \mid j+i+k : j+p+k\right]\right)$$
$$+ \det\left(A\left[j+1 : j+i \mid j+i+k : j+p+k+1\right]\right)$$
$$\left. \cdot \det\left(A\left[j : j+i+1 \mid j+i+k+1 : j+p+k\right]\right)\right)$$

$$= \left(\det\left(A\left[j+1 : j+i \mid j+i+k : j+p+k+1\right]\right)\right)^{-1}$$
$$\cdot \left(\det\left(A\left[j+1 : j+i+1 \mid j+i+k : j+p+k\right]\right)\right)^{-1}$$
$$\cdot \det\left(A\left[j : j+i \mid j+i+k : j+p+k\right]\right)$$
$$\cdot \det\left(A\left[j+1 : j+i+1 \mid j+i+k+1 : j+p+k+1\right]\right) \quad (11)$$

(by Theorem 50, applied to $a = j + 1$, $b = j + i$, $c = j + i + k + 1$ and $d = j + p + k$).

Now by definition we get:

$$g(v) = \left(\mathrm{Grasp}_j A\right)((i,k)) = \frac{\det\left(A\left[j+1 : j+i \mid j+i+k-1 : j+p+k\right]\right)}{\det\left(A\left[j : j+i \mid j+i+k : j+p+k\right]\right)} \quad (12)$$

while $f(v) = \left(\mathrm{Grasp}_{j+1} A\right)((i,k))$

$$= \frac{\det\left(A\left[j+2 : j+i+1 \mid j+i+k : j+p+k+1\right]\right)}{\det\left(A\left[j+1 : j+i+1 \mid j+i+k+1 : j+p+k+1\right]\right)}. \quad (13)$$

So we can rewrite the terms $\displaystyle\sum_{\substack{u \in \widehat{\mathrm{Rect}(p,q)}; \\ u \lessdot v}} f(u)$, $\displaystyle\sum_{\substack{u \in \widehat{\mathrm{Rect}(p,q)}; \\ u \gtrdot v}} \frac{1}{g(u)}$, $g(v)$ and $f(v)$ in (7) using

the equalities (10), (11), (12) and (13), respectively. The resulting equation is a tautology because all determinants cancel out. This proves (7) in Case 1.

Proofs of the other three cases follow the same lines of argument, but are simpler. Note, however, that it is only in Cases 3 and 4 that we use the fact that the sequence $(A_n)_{n \in \mathbb{Z}}$ is "$(p+q)$-periodic up to sign" as opposed to an arbitrary sequence of length-p column vectors. $\qquad\square$

7 Dominance of the Grassmannian parametrization

In this section we prove Proposition 49 that the space of $\mathbb{K}$-labelings that we can obtain in the form $\mathrm{Grasp}_0 A$ is sufficiently diverse to cover everything we need. Before plunging

into the details of the general case, we illustrate the approach we take with an example.

Example 52. Let $p = q = 2$ and $f \in \mathbb{K}^{\widehat{\text{Rect}(2,2)}}$ be a generic reduced labelling. We want to construct a matrix $A \in \mathbb{K}^{2 \times (2+2)}$ satisfying $f = \text{Grasp}_0 \, A$.

Clearly the condition $f = \text{Grasp}_0 \, A$ imposes 4 equations on the eight entries of A; thus, we are trying to solve an underdetermined system. However, we can get rid of the superfluous freedom if we additionally try to ensure that our matrix A has the form

$$A = (I_p \mid B) = \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & z & w \end{pmatrix} \text{ for some } B = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbb{K}^{2 \times 2}.$$

Now,

$$(\text{Grasp}_0 \, (I_p \mid B)) \, ((1,1)) = \frac{\det \left((I_p \mid B) \, [1:1 \mid 1:3] \right)}{\det \left((I_p \mid B) \, [0:1 \mid 2:3] \right)} = \frac{\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{\det \begin{pmatrix} -y & 0 \\ -w & 1 \end{pmatrix}} = \frac{-1}{y};$$

$$(\text{Grasp}_0 \, (I_p \mid B)) \, ((1,2)) = \frac{\det \left((I_p \mid B) \, [1:1 \mid 2:4] \right)}{\det \left((I_p \mid B) \, [0:1 \mid 3:4] \right)} = \frac{\det \begin{pmatrix} 0 & x \\ 1 & z \end{pmatrix}}{\det \begin{pmatrix} -y & x \\ -w & z \end{pmatrix}} = \frac{-x}{wx - yz};$$

$$(\text{Grasp}_0 \, (I_p \mid B)) \, ((2,1)) = \frac{\det \left((I_p \mid B) \, [1:2 \mid 2:3] \right)}{\det \left((I_p \mid B) \, [0:2 \mid 3:3] \right)} = \frac{\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{\det \begin{pmatrix} -y & 1 \\ -w & 0 \end{pmatrix}} = \frac{1}{w};$$

$$(\text{Grasp}_0 \, (I_p \mid B)) \, ((2,2)) = \frac{\det \left((I_p \mid B) \, [1:2 \mid 3:4] \right)}{\det \left((I_p \mid B) \, [0:2 \mid 4:4] \right)} = \frac{\det \begin{pmatrix} 1 & x \\ 0 & z \end{pmatrix}}{\det \begin{pmatrix} -y & 1 \\ -w & 0 \end{pmatrix}} = \frac{z}{w}.$$

The requirement $f = \text{Grasp}_0 \, (I_p \mid B)$ therefore translates into the following system, which is solved by elimination (in order w, y, z, x) as shown:

$$\begin{cases} f((1,1)) &= \dfrac{-1}{y}; \\ f((1,2)) &= \dfrac{-x}{wx - yz}; \\ f((2,1)) &= \dfrac{1}{w}; \\ f((2,2)) &= \dfrac{z}{w} \end{cases} \implies \begin{cases} w &= \dfrac{1}{f((2,1))}; \\ x &= \dfrac{-f((1,2))f((2,2))}{[f((1,2)) + f((2,1))]f((1,1))}; \\ y &= \dfrac{-1}{f((1,1))}; \\ z &= \dfrac{f((2,2))}{f((2,1))} \end{cases}$$

While the denominators in these fractions can vanish, leading to underdetermination or unsolvability, this will not happen for **generic** f.

We apply this same technique to the general proof of Proposition 49. For any fixed $f \in \mathbb{K}^{\text{Rect}(p,q)}$, solving the equation $f = \text{Grasp}_0 A$ for $A \in \mathbb{K}^{p \times (p+q)}$ can be considered as a system of pq equations on $p(p+q)$ unknowns. While this (nonlinear) system is usually underdetermined, we can restrict the entries of A by requiring that the leftmost p columns of A form the $p \times p$ identity matrix, leaving us with only pq unknowns only; for f sufficiently generic, the resulting system will be uniquely solvable by "triangular elimination" (i.e., there is an equation containing only one unknown; then, when this unknown is eliminated, the resulting system again contains an equation with only one unknown, and once this one is eliminated, one gets a further system containing an equation with only one unknown, and so forth).

We will sketch the ideas of this proof, leaving all straightforward details to the reader. We word the argument using algebraic properties of families of rational functions instead of using the algorithmic nature of "triangular elimination" (similarly to how most applications of linear algebra use the language of bases of vector spaces rather than talk about the process of solving systems by Gaussian elimination). While this clarity comes at the cost of a slight disconnect from the motivation of the proof, we hope that the reader will still see how the wind blows. We first introduce some notation to capture the essence of "triangular elimination" without having to talk about actually moving around variables in equations.

Definition 53. Let $\mathbb{F}$ be a field. Let $\mathbf{P}$ be a finite set.

(a) Let $x_{\mathbf{p}}$ be a new symbol for every $\mathbf{p} \in \mathbf{P}$. We will denote by $\mathbb{F}(x_{\mathbf{P}})$ the field of rational functions over $\mathbb{F}$ in the indeterminates $x_{\mathbf{p}}$ with $\mathbf{p}$ ranging over all elements of $\mathbf{P}$ (hence altogether $|\mathbf{P}|$ indeterminates). We also will denote by $\mathbb{F}[x_{\mathbf{P}}]$ the ring of polynomials over $\mathbb{F}$ in the indeterminates $x_{\mathbf{p}}$ with $\mathbf{p}$ ranging over all elements of $\mathbf{P}$. (Thus, $\mathbb{F}(x_{\mathbf{P}}) = \mathbb{F}(x_{\mathbf{p}_1}, x_{\mathbf{p}_2}, \ldots, x_{\mathbf{p}_n})$ and $\mathbb{F}[x_{\mathbf{P}}] = \mathbb{F}[x_{\mathbf{p}_1}, x_{\mathbf{p}_2}, \ldots, x_{\mathbf{p}_n}]$ if $\mathbf{P}$ is written in the form $\mathbf{P} = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$.) The symbols $x_{\mathbf{p}}$ are understood to be distinct, and are used as commuting indeterminates. We regard $\mathbb{F}[x_{\mathbf{P}}]$ as a subring of $\mathbb{F}(x_{\mathbf{P}})$, and $\mathbb{F}(x_{\mathbf{P}})$ as the field of quotients of $\mathbb{F}[x_{\mathbf{P}}]$.

(b) If $\mathbf{Q}$ is a subset of $\mathbf{P}$, then $\mathbb{F}(x_{\mathbf{Q}})$ can be canonically embedded into $\mathbb{F}(x_{\mathbf{P}})$, and $\mathbb{F}[x_{\mathbf{Q}}]$ can be canonically embedded into $\mathbb{F}[x_{\mathbf{P}}]$. We regard these embeddings as inclusions.

(c) Let $\mathbb{K}$ be a field extension of $\mathbb{F}$. Let f be an element of $\mathbb{F}(x_{\mathbf{P}})$. If $(a_{\mathbf{p}})_{\mathbf{p} \in \mathbf{P}} \in \mathbb{K}^{\mathbf{P}}$ is a family of elements of $\mathbb{K}$ indexed by elements of $\mathbf{P}$, then we let $f\left((a_{\mathbf{p}})_{\mathbf{p} \in \mathbf{P}}\right)$ denote the element of $\mathbb{K}$ obtained by substituting $a_{\mathbf{p}}$ for $x_{\mathbf{p}}$ for each $\mathbf{p} \in \mathbf{P}$ in the rational function f. This $f\left((a_{\mathbf{p}})_{\mathbf{p} \in \mathbf{P}}\right)$ is defined only if the substitution does not render the denominator equal to 0. If $\mathbb{K}$ is infinite, this shows that $f\left((a_{\mathbf{p}})_{\mathbf{p} \in \mathbf{P}}\right)$ is defined for almost all $(a_{\mathbf{p}})_{\mathbf{p} \in \mathbf{P}} \in \mathbb{K}^{\mathbf{P}}$ (with respect to the Zariski topology).

(d) Let $\mathbf{P}$ now be a finite totally ordered set, and let $\lhd$ be the smaller-than relation of $\mathbf{P}$. For every $\mathbf{p} \in \mathbf{P}$, let $\mathbf{p} \Downarrow$ denote the subset $\{\mathbf{v} \in \mathbf{P} \mid \mathbf{v} \lhd \mathbf{p}\}$ of $\mathbf{P}$. For every $\mathbf{p} \in \mathbf{P}$, let $Q_{\mathbf{p}}$ be an element of $\mathbb{F}(x_{\mathbf{P}})$.

We say that the family $(Q_{\mathbf{p}})_{\mathbf{p} \in \mathbf{P}}$ is $\mathbf{P}$-*triangular* if and only if the following condition holds:

Algebraic triangularity condition: For every $\mathbf{p} \in \mathbf{P}$, there exist elements $\alpha_{\mathbf{p}}$, $\beta_{\mathbf{p}}$, $\gamma_{\mathbf{p}}$, $\delta_{\mathbf{p}}$ of $\mathbb{F}\left(x_{\mathbf{p}\Downarrow}\right)$ such that $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$ and $Q_{\mathbf{p}} = \dfrac{\alpha_{\mathbf{p}}x_{\mathbf{p}} + \beta_{\mathbf{p}}}{\gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}}}.$

We will use $\mathbf{P}$-triangularity via the following fact:

Lemma 54. *Let $\mathbb{F}$ be a field. Let $\mathbf{P}$ be a finite totally ordered set. For every $\mathbf{p} \in \mathbf{P}$, let $Q_{\mathbf{p}}$ be an element of $\mathbb{F}\left(x_{\mathbf{P}}\right)$. Assume that $\left(Q_{\mathbf{p}}\right)_{\mathbf{p}\in\mathbf{P}}$ is a $\mathbf{P}$-triangular family. Then:*

(a) The family $\left(Q_{\mathbf{p}}\right)_{\mathbf{p}\in\mathbf{P}} \in \left(\mathbb{F}\left(x_{\mathbf{P}}\right)\right)^{\mathbf{P}}$ is algebraically independent (over $\mathbb{F}$).

(b) There exists a $\mathbf{P}$-triangular family $\left(R_{\mathbf{p}}\right)_{\mathbf{p}\in\mathbf{P}} \in \left(\mathbb{F}\left(x_{\mathbf{P}}\right)\right)^{\mathbf{P}}$ such that every $\mathbf{q} \in \mathbf{P}$ satisfies $Q_{\mathbf{q}}\left(\left(R_{\mathbf{p}}\right)_{\mathbf{p}\in\mathbf{P}}\right) = x_{\mathbf{q}}.$

Proof. The proof of this lemma – an exercise in elementary algebra and induction – is omitted; it can be found in [GrRo14b, Lemma 15.3]. $\qquad\square$

Armed with this definition, we are ready to tackle the proof of Proposition 49 that $\mathbb{K}$-labelings can be generically parametrized by $\mathrm{Grasp}_0 A$.

Proof of Proposition 49. Let $\mathbb{F}$ be the prime field of $\mathbb{K}$. (This means either $\mathbb{Q}$ or $\mathbb{F}_p$ depending on the characteristic of $\mathbb{K}$.) In the following, the word "algebraically independent" will always mean "algebraically independent over $\mathbb{F}$" (rather than over $\mathbb{K}$ or over $\mathbb{Z}$).

Let $\mathbf{P}$ be a totally ordered set such that $\mathbf{P} = \{1, 2, \ldots, p\} \times \{1, 2, \ldots, q\}$ as sets, and such that

$$(i, k) \trianglelefteq (i', k') \text{ for all } (i, k) \in \mathbf{P} \text{ and } (i', k') \in \mathbf{P} \text{ satisfying } (i \geqslant i' \text{ and } k \leqslant k'),$$

where $\trianglelefteq$ denotes the smaller-or-equal relation of $\mathbf{P}$. Such a $\mathbf{P}$ clearly exists (in fact, there usually exist several such $\mathbf{P}$, and it doesn't matter which of them we choose). We denote the smaller-than relation of $\mathbf{P}$ by $\triangleleft$. We will later see what this total order is good for (intuitively, it is an order in which the variables can be eliminated; in other words, it makes our system behave like a triangular matrix rather than like a triangular matrix with permuted columns), but for now let us notice that it is generally not compatible with $\mathrm{Rect}\,(p, q)$.

Let $Z : \{1, 2, \ldots, q\} \to \{1, 2, \ldots, q\}$ denote the map which sends every $k \in \{1, 2, \ldots, q-1\}$ to $k+1$ and sends q to 1. Thus, Z is a permutation in the symmetric group S_q, and can be written in cycle notation as $(1, 2, \ldots, q)$.

Consider the field $\mathbb{F}\left(x_{\mathbf{P}}\right)$ and the ring $\mathbb{F}\left[x_{\mathbf{P}}\right]$ defined as in Definition 53. In order to prove Proposition 49, it is enough to show that there exists a matrix $\widetilde{D} \in \left(\mathbb{F}\left(x_{\mathbf{P}}\right)\right)^{p \times (p+q)}$ satisfying

$$x_{\mathbf{p}} = \left(\mathrm{Grasp}_0 \widetilde{D}\right)(\mathbf{p}) \qquad \text{for every } \mathbf{p} \in \mathbf{P}. \tag{14}$$

---

[6]Notice that the fraction $\dfrac{\alpha_{\mathbf{p}}x_{\mathbf{p}} + \beta_{\mathbf{p}}}{\gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}}}$ is well-defined for any four elements $\alpha_{\mathbf{p}}, \beta_{\mathbf{p}}, \gamma_{\mathbf{p}}, \delta_{\mathbf{p}}$ of $\mathbb{F}\left(x_{\mathbf{p}\Downarrow}\right)$ such that $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$. (Indeed, $\gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}} \neq 0$ in this case, as can easily be checked.)

For then we can obtain a matrix $A \in \mathbb{K}^{p \times (p+q)}$ satisfying $f = \mathrm{Grasp}_0 A$ for almost every $f \in \mathbb{K}^{\mathrm{Rect}(p,q)}$ simply by substituting $f(\mathbf{p})$ for every $x_{\mathbf{p}}$ in all entries of the matrix $\widetilde{D}$

Now define a matrix $C \in (\mathbb{F}[x_{\mathbf{P}}])^{p \times q}$ by

$$C = \left(x_{(i, Z(k))} \right)_{1 \leqslant i \leqslant p, \ 1 \leqslant k \leqslant q}.$$

This is simply a matrix whose entries are all the indeterminates $x_{\mathbf{p}}$ of the polynomial ring $\mathbb{F}[x_{\mathbf{P}}]$, albeit in a strange order (tailored to make the "triangularity" argument work nicely). This matrix C is not directly related to the $\widetilde{D}$ we will construct, but will be used in its construction.

For every $(i, k) \in \mathbf{P}$, define element $\mathfrak{N}_{(i,k)}, \mathfrak{D}_{(i,k)} \in \mathbb{F}[x_{\mathbf{P}}]$ by

$$\mathfrak{N}_{(i,k)} = \det \left((I_p \mid C) [1 : i \mid i + k - 1 : p + k] \right). \tag{15}$$
$$\mathfrak{D}_{(i,k)} = \det \left((I_p \mid C) [0 : i \mid i + k : p + k] \right). \tag{16}$$

Our plan from here is the following:

Step 1: We will find alternate expressions for the polynomials $\mathfrak{N}_{(i,k)}$ and $\mathfrak{D}_{(i,k)}$ which will give us a better idea of what variables occur in these polynomials.

Step 2: We will show that $\mathfrak{N}_{(i,k)}$ and $\mathfrak{D}_{(i,k)}$ are nonzero for all $(i, k) \in \mathbf{P}$.

Step 3: We will define a $Q_{\mathbf{p}} \in \mathbb{F}(x_{\mathbf{P}})$ for every $\mathbf{p} \in \mathbf{P}$ by $Q_{\mathbf{p}} = \dfrac{\mathfrak{N}_{\mathbf{p}}}{\mathfrak{D}_{\mathbf{p}}}$, and we will show that $Q_{\mathbf{p}} = (\mathrm{Grasp}_0 (I_p \mid C))(\mathbf{p})$.

Step 4: We will prove that the family $(Q_{\mathbf{p}})_{\mathbf{p} \in \mathbf{P}} \in (\mathbb{F}(x_{\mathbf{P}}))^{\mathbf{P}}$ is $\mathbf{P}$-triangular.

Step 5: We will use Lemma 54 **(b)** and the result of Step 4 to find a matrix $\widetilde{D} \in (\mathbb{F}(x_{\mathbf{P}}))^{p \times (p+q)}$ satisfying (14).

We now fill in a few details for each step.

Details of Step 1: We introduce two more pieces of notation pertaining to matrices:

- If $\ell \in \mathbb{N}$, and if A_1, A_2, …, A_k are several matrices with ℓ rows each, then $(A_1 \mid A_2 \mid \ldots \mid A_k)$ will denote the matrix obtained by starting with an (empty) $\ell \times 0$-matrix, then attaching the matrix A_1 to it on the right, then attaching the matrix A_2 to the result on the right, etc., and finally attaching the matrix A_k to the result on the right. For example, $\left(I_2 \mid \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 3 & 0 \end{pmatrix}$.

- If $\ell \in \mathbb{N}$, if B is a matrix with ℓ rows, and if i_1, i_2, …, i_k are some elements of $\{1, 2, \ldots, \ell\}$, then $\mathrm{rows}_{i_1, i_2, \ldots, i_k} B$ will denote the matrix whose rows (from top to bottom) are the rows labelled i_1, i_2, …, i_k of the matrix B.

We will use without proof a standard fact about determinants of block matrices:

- Given a commutative ring $\mathbb{L}$, two nonnegative integers a and b satisfying $a \geqslant b$, and a matrix $U \in \mathbb{L}^{a \times b}$, we have

$$\det \left(\begin{pmatrix} I_{a-b} \\ 0_{b \times (a-b)} \end{pmatrix} \mid U \right) = \det \left(\mathrm{rows}_{a-b+1, a-b+2, \ldots, a} U \right) \tag{17}$$

and

$$\det\left(\left(\begin{array}{c} 0_{b\times(a-b)} \\ I_{a-b} \end{array}\right) \mid U\right) = (-1)^{b(a-b)} \det\left(\mathrm{rows}_{1,2,\ldots,b}\, U\right). \tag{18}$$

Using this we can rewrite

$$\begin{aligned}
\mathfrak{N}_{(i,k)} &= \det\left((I_p \mid C)\,[1 : i \mid i + k - 1 : p + k]\right) \\
&= \det\left(\left(\begin{array}{c} I_{i-1} \\ 0_{(p-(i-1))\times(i-1)} \end{array}\right) \mid (I_p \mid C)\,[i + k - 1 : p + k]\right) \tag{19} \\
&= \det\left(\mathrm{rows}_{i,i+1,\ldots,p}\left((I_p \mid C)\,[i + k - 1 : p + k]\right)\right). \tag{20}
\end{aligned}$$

Also,

$$(I_p \mid C)\,[0 : i \mid i + k : p + k]$$

$$= \left(\begin{array}{c|c|c} \underbrace{(I_p \mid C)_0}_{\substack{=(-1)^{p-1}C_q \\ \text{(due to Definition 37 (b))}}} & \left(\begin{array}{c} I_{i-1} \\ 0_{(p-(i-1))\times(i-1)} \end{array}\right) & (I_p \mid C)\,[i + k : p + k] \end{array}\right)$$

$$= \left((-1)^{p-1}C_q \;\middle|\; \left(\begin{array}{c} I_{i-1} \\ 0_{(p-(i-1))\times(i-1)} \end{array}\right) \;\middle|\; (I_p \mid C)\,[i + k : p + k]\right),$$

whence

$$\begin{aligned}
\mathfrak{D}_{(i,k)} &= \det\left((I_p \mid C)\,[0 : i \mid i + k : p + k]\right) \\
&= \det\left((-1)^{p-1}C_q \;\middle|\; \left(\begin{array}{c} I_{i-1} \\ 0_{(p-(i-1))\times(i-1)} \end{array}\right) \;\middle|\; (I_p \mid C)\,[i + k : p + k]\right) \\
&= (-1)^{p-1}\det\left(C_q \;\middle|\; \left(\begin{array}{c} I_{i-1} \\ 0_{(p-(i-1))\times(i-1)} \end{array}\right) \;\middle|\; (I_p \mid C)\,[i + k : p + k]\right) \\
&= \underbrace{(-1)^{p-1}(-1)^{i-1}}_{=(-1)^{p-i}}\det\left(\left(\begin{array}{c} I_{i-1} \\ 0_{(p-(i-1))\times(i-1)} \end{array}\right) \;\middle|\; C_q \;\middle|\; (I_p \mid C)\,[i + k : p + k]\right) \\
&= (-1)^{p-i}\det\left(\left(\begin{array}{c} I_{i-1} \\ 0_{(p-(i-1))\times(i-1)} \end{array}\right) \;\middle|\; C_q \;\middle|\; (I_p \mid C)\,[i + k : p + k]\right) \\
&= (-1)^{p-i}\det\left(\mathrm{rows}_{i,i+1,\ldots,p}\left(C_q \;\middle|\; (I_p \mid C)\,[i + k : p + k]\right)\right). \tag{21}
\end{aligned}$$

Although these alternative formulas (20) and (21) for $\mathfrak{N}_{(i,k)}$ and $\mathfrak{D}_{(i,k)}$ are not shorter than the definitions, they involve smaller matrices (unless $i = 1$) and are more useful in understanding the monomials appearing in $\mathfrak{N}_{(i,k)}$ and $\mathfrak{D}_{(i,k)}$.

Details of Step 2: We claim that $\mathfrak{N}_{(i,k)}$ and $\mathfrak{D}_{(i,k)}$ are nonzero for all $(i, k) \in \mathbf{P}$.

Proof. Let $(i, k) \in \mathbf{P}$. Let us first check that $\mathfrak{N}_{(i,k)}$ is nonzero. This follows from observing that, if 0's and 1's are substituted for the indeterminates $x_{(i,k)}$ in an appropriate way, then the columns of the matrix $(I_p \mid C)\,[1 : i \mid i + k - 1 : p + k]$ become the standard

basis vectors of $\mathbb{K}^p$ (in some order), and so the determinant $\mathfrak{N}_{(i,k)}$ of this matrix becomes ± 1, which is nonzero.

Similarly, $\mathfrak{D}_{(i,k)}$ is nonzero.

Details of Step 3: Define $Q_{\mathbf{p}} \in \mathbb{F}(x_{\mathbf{P}})$ for every $\mathbf{p} \in \mathbf{P}$ by $Q_{\mathbf{p}} = \dfrac{\mathfrak{N}_{\mathbf{p}}}{\mathfrak{D}_{\mathbf{p}}}$. This is well-defined because Step 2 has shown that $\mathfrak{D}_{\mathbf{p}}$ is nonzero. Moreover, it is easy to see that every $\mathbf{p} = (i,k) \in \mathbf{P}$ satisfies

$$Q_{(i,k)} = (\mathrm{Grasp}_0 (I_p \mid C)) ((i,k)) , \text{ i.e., } Q_{\mathbf{p}} = (\mathrm{Grasp}_0 (I_p \mid C)) (\mathbf{p}) . \tag{22}$$

Details of Step 4: To prove the family $(Q_{\mathbf{p}})_{\mathbf{p} \in \mathbf{P}} \in (\mathbb{F}(x_{\mathbf{P}}))^{\mathbf{P}}$ is $\mathbf{P}$-triangular, we need that for every $\mathbf{p} \in \mathbf{P}$, there exist elements $\alpha_{\mathbf{p}}$, $\beta_{\mathbf{p}}$, $\gamma_{\mathbf{p}}$, $\delta_{\mathbf{p}}$ of $\mathbb{F}(x_{\mathbf{p}\Downarrow})$ such that $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$ and $Q_{\mathbf{p}} = \dfrac{\alpha_{\mathbf{p}}x_{\mathbf{p}} + \beta_{\mathbf{p}}}{\gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}}}$ (where $\mathbf{p} \Downarrow$ is defined as in Definition 53 **(d)**). So fix $\mathbf{p} = (i,k) \in \mathbf{P}$.

We will actually do something slightly better than we need. We will find elements $\alpha_{\mathbf{p}}$, $\beta_{\mathbf{p}}$, $\gamma_{\mathbf{p}}$, $\delta_{\mathbf{p}}$ of $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$ (not just of $\mathbb{F}(x_{\mathbf{p}\Downarrow})$) such that $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$ and $\mathfrak{N}_{\mathbf{p}} = \alpha_{\mathbf{p}}x_{\mathbf{p}} + \beta_{\mathbf{p}}$ and $\mathfrak{D}_{\mathbf{p}} = \gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}}$. (Of course, the conditions $\mathfrak{N}_{\mathbf{p}} = \alpha_{\mathbf{p}}x_{\mathbf{p}} + \beta_{\mathbf{p}}$ and $\mathfrak{D}_{\mathbf{p}} = \gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}}$ combined imply $Q_{\mathbf{p}} = \dfrac{\alpha_{\mathbf{p}}x_{\mathbf{p}} + \beta_{\mathbf{p}}}{\gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}}}$, hence the yearned-for $\mathbf{P}$-triangularity.)

We first handle two "boundary" cases: (a) $k = 1$, and (b) $k \neq 1$ but $i = p$.

The case when $k = 1$ is very easy: we get that $\mathfrak{N}_{\mathbf{p}} = 1$ (using (20)) and that $\mathfrak{D}_{\mathbf{p}} = (-1)^{i+p} x_{\mathbf{p}}$ (using (21)). Consequently, we can take $\alpha_{\mathbf{p}} = 0$, $\beta_{\mathbf{p}} = 1$, $\gamma_{\mathbf{p}} = (-1)^{i+p}$ and $\delta_{\mathbf{p}} = 0$, and it is clear that all three requirements $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$ and $\mathfrak{N}_{\mathbf{p}} = \alpha_{\mathbf{p}}x_{\mathbf{p}} + \beta_{\mathbf{p}}$ and $\mathfrak{D}_{\mathbf{p}} = \gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}}$ are satisfied.

The case when $k \neq 1$ but $i = p$ is not much harder. In this case, (20) simplifies to $\mathfrak{N}_{\mathbf{p}} = x_{\mathbf{p}}$, and (21) simplifies to $\mathfrak{D}_{\mathbf{p}} = x_{(p,1)}$. Hence, we can take $\alpha_{\mathbf{p}} = 1$, $\beta_{\mathbf{p}} = 0$, $\gamma_{\mathbf{p}} = 0$ and $\delta_{\mathbf{p}} = x_{(p,1)}$ to achieve $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$ and $\mathfrak{N}_{\mathbf{p}} = \alpha_{\mathbf{p}}x_{\mathbf{p}} + \beta_{\mathbf{p}}$ and $\mathfrak{D}_{\mathbf{p}} = \gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}}$. Note that this choice of $\delta_{\mathbf{p}}$ is legitimate because $x_{(p,1)}$ does lie in $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$ (since $(p,1) \in \mathbf{p} \Downarrow$).

The remaining case, where neither $k = 1$ nor $i = p$ takes a bit more work. Consider the matrix $\mathrm{rows}_{i,i+1,\ldots,p}((I_p \mid C)[i+k-1 : p+k])$ (this matrix appears on the right hand side of (20)). Each entry of this matrix comes either from the matrix I_p or from the matrix C. If it comes from I_p, it clearly lies in $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$. If it comes from C, it has the form $x_{\mathbf{q}}$ for some $\mathbf{q} \in \mathbf{P}$, and this $\mathbf{q}$ belongs to $\mathbf{p} \Downarrow$ unless the entry is the $(1, p - i + 1)$-th entry. Therefore, each entry of the matrix $(I_p \mid C)[i+k-1 : p+k]$ apart from the $(1, p - i + 1)$-th entry lies in $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$, whereas the $(1, p - i + 1)$-th entry is $x_{\mathbf{p}}$. Hence, if we use the Laplace expansion with respect to the first row to compute the determinant of this matrix, we obtain a formula of the form

$$\det (\mathrm{rows}_{i,i+1,\ldots,p}((I_p \mid C)[i+k-1 : p+k]))$$
$$= x_{\mathbf{p}} \cdot (\text{some polynomial in entries lying in } \mathbb{F}[x_{\mathbf{p}\Downarrow}])$$
$$+ (\text{more polynomials in entries lying in } \mathbb{F}[x_{\mathbf{p}\Downarrow}])$$
$$\in \mathbb{F}[x_{\mathbf{p}\Downarrow}] \cdot x_{\mathbf{p}} + \mathbb{F}[x_{\mathbf{p}\Downarrow}] .$$

In other words, there exist elements $\alpha_{\mathbf{p}}$ and $\beta_{\mathbf{p}}$ of $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$ such that $\det(\mathrm{rows}_{i,i+1,\dots,p}((I_p \mid C)[i+k-1:p+k])) = \alpha_{\mathbf{p}}x_{\mathbf{p}}+\beta_{\mathbf{p}}$. Consider these $\alpha_{\mathbf{p}}$ and $\beta_{\mathbf{p}}$. We have

$$\mathfrak{N}_{\mathbf{p}} = \mathfrak{N}_{(i,k)} = \det(\mathrm{rows}_{i,i+1,\dots,p}((I_p \mid C)[i+k-1:p+k])) \qquad \text{(by (20))} \qquad (23)$$
$$= \alpha_{\mathbf{p}}x_{\mathbf{p}} + \beta_{\mathbf{p}}. \qquad (24)$$

We can similarly deal with the matrix $\mathrm{rows}_{i,i+1,\dots,p}(C_q \mid (I_p \mid C)[i+k:p+k])$ which appears on the right hand side of (21). Again, each entry of this matrix apart from the $(1, p-i+1)$-th entry lies in $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$, whereas the $(1, p-i+1)$-th entry is $x_{\mathbf{p}}$. Using the Laplace expansion again, we thus see that

$$\det(\mathrm{rows}_{i,i+1,\dots,p}(C_q \mid (I_p \mid C)[i+k:p+k])) \in \mathbb{F}[x_{\mathbf{p}\Downarrow}] \cdot x_{\mathbf{p}} + \mathbb{F}[x_{\mathbf{p}\Downarrow}],$$

so that

$$(-1)^{p-i}\det(\mathrm{rows}_{i,i+1,\dots,p}(C_q \mid (I_p \mid C)[i+k:p+k])) \in \mathbb{F}[x_{\mathbf{p}\Downarrow}] \cdot x_{\mathbf{p}} + \mathbb{F}[x_{\mathbf{p}\Downarrow}].$$

Hence, there exist elements $\gamma_{\mathbf{p}}$ and $\delta_{\mathbf{p}}$ of $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$ such that $(-1)^{p-i}\det(\mathrm{rows}_{i,i+1,\dots,p}(C_q \mid (I_p \mid C)[i+k:p+k])) = \gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}}$. Consider these $\gamma_{\mathbf{p}}$ and $\delta_{\mathbf{p}}$. We have

$$\mathfrak{D}_{\mathbf{p}} = \mathfrak{D}_{(i,k)} = (-1)^{p-i}\det(\mathrm{rows}_{i,i+1,\dots,p}(C_q \mid (I_p \mid C)[i+k:p+k])) \qquad \text{(by (21))}$$
$$(25)$$
$$= \gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}}.$$

We thus have found elements $\alpha_{\mathbf{p}}$, $\beta_{\mathbf{p}}$, $\gamma_{\mathbf{p}}$, $\delta_{\mathbf{p}}$ of $\mathbb{F}[x_{\mathbf{p}\Downarrow}]$ satisfying $\mathfrak{N}_{\mathbf{p}} = \alpha_{\mathbf{p}}x_{\mathbf{p}} + \beta_{\mathbf{p}}$ and $\mathfrak{D}_{\mathbf{p}} = \gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}}$. In order to finish the proof of $\mathbf{P}$-triangularity, we only need to show that $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$.

In order to achieve this goal, we notice that

$$\alpha_{\mathbf{p}}\underbrace{\mathfrak{D}_{\mathbf{p}}}_{=\gamma_{\mathbf{p}}x_{\mathbf{p}}+\delta_{\mathbf{p}}} - \underbrace{\mathfrak{N}_{\mathbf{p}}}_{=\alpha_{\mathbf{p}}x_{\mathbf{p}}+\beta_{\mathbf{p}}}\gamma_{\mathbf{p}} = \alpha_{\mathbf{p}}(\gamma_{\mathbf{p}}x_{\mathbf{p}} + \delta_{\mathbf{p}}) - (\alpha_{\mathbf{p}}x_{\mathbf{p}} + \beta_{\mathbf{p}})\gamma_{\mathbf{p}} = \alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}}.$$

Hence, proving $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$ is equivalent to proving $\alpha_{\mathbf{p}}\mathfrak{D}_{\mathbf{p}} - \mathfrak{N}_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$. It is the latter that we are going to do, because $\alpha_{\mathbf{p}}$, $\mathfrak{D}_{\mathbf{p}}$, $\mathfrak{N}_{\mathbf{p}}$ and $\gamma_{\mathbf{p}}$ are easier to get our hands on than $\beta_{\mathbf{p}}$ and $\delta_{\mathbf{p}}$.

Recall that our proof that

$$\det(\mathrm{rows}_{i,i+1,\dots,p}((I_p \mid C)[i+k-1:p+k])) \in \mathbb{F}[x_{\mathbf{p}\Downarrow}] \cdot x_{\mathbf{p}} + \mathbb{F}[x_{\mathbf{p}\Downarrow}]$$

proceeded by applying the Laplace expansion with respect to the first row to the matrix $\mathrm{rows}_{i,i+1,\dots,p}((I_p \mid C)[i+k-1:p+k])$. The only term involving $x_{\mathbf{p}}$ was

$$x_{\mathbf{p}} \cdot (\text{some polynomial in entries lying in } \mathbb{F}[x_{\mathbf{p}\Downarrow}]).$$

The second factor above is actually the $(1, p - i + 1)$-th cofactor of the matrix $\mathrm{rows}_{i,i+1,\dots,p}\left((I_p \mid C)\left[i + k - 1 : p + k\right]\right)$. Hence,

$$
\begin{aligned}
\alpha_{\mathbf{p}} &= \left(\text{the } (1, p - i + 1)\text{-th cofactor of } \mathrm{rows}_{i,i+1,\dots,p}\left((I_p \mid C)\left[i + k - 1 : p + k\right]\right)\right) \\
&= (-1)^{p-i} \cdot \det\left(\mathrm{rows}_{i+1,i+2,\dots,p}\left((I_p \mid C)\left[i + k - 1 : p + k - 1\right]\right)\right).
\end{aligned}
\tag{26}
$$

Similarly,

$$
\gamma_{\mathbf{p}} = \det\left(\mathrm{rows}_{i+1,i+2,\dots,p}\left(C_q \mid (I_p \mid C)\left[i + k : p + k - 1\right]\right)\right)
\tag{27}
$$

(note that we lost the sign $(-1)^{p-i}$ from (25) since it got cancelled against the $(-1)^{p-(i+1)}$ arising from the definition of a cofactor).

Now, since neither $k = 1$ nor $i = p$, $(i + 1, k - 1)$ also belongs to $\mathbf{P}$; hence, we can apply (20) to $(i + 1, k - 1)$ in lieu of (i, k), and obtain

$$
\mathfrak{N}_{(i+1,k-1)} = \det\left(\mathrm{rows}_{i+1,i+2,\dots,p}\left((I_p \mid C)\left[i + k - 1 : p + k - 1\right]\right)\right).
$$

In light of this, (26) becomes

$$
\alpha_{\mathbf{p}} = (-1)^{p-i} \cdot \mathfrak{N}_{(i+1,k-1)}.
$$

Similarly, applying (21) to $(i + 1, k - 1)$ in lieu of (i, k), rewrites (27) as

$$
\gamma_{\mathbf{p}} = (-1)^{p-(i+1)} \cdot \mathfrak{D}_{(i+1,k-1)}.
$$

Hence,

$$
\begin{aligned}
&\overbrace{\alpha_{\mathbf{p}}}^{} \qquad \mathfrak{D}_{\mathbf{p}} - \mathfrak{N}_{\mathbf{p}} \qquad \overbrace{\gamma_{\mathbf{p}}}^{} \\
&{=}{(-1)^{p-i} \cdot \mathfrak{N}_{(i+1,k-1)}} \qquad\qquad {=}{(-1)^{p-(i+1)} \cdot \mathfrak{D}_{(i+1,k-1)}} \\
&= (-1)^{p-i} \cdot \mathfrak{N}_{(i+1,k-1)} \cdot \mathfrak{D}_{\mathbf{p}} - \mathfrak{N}_{\mathbf{p}} \cdot \underbrace{(-1)^{p-(i+1)}}_{=-(-1)^{p-i}} \cdot \mathfrak{D}_{(i+1,k-1)} \\
&= (-1)^{p-i} \cdot \left(\mathfrak{N}_{(i+1,k-1)}\mathfrak{D}_{\mathbf{p}} + \mathfrak{N}_{\mathbf{p}}\mathfrak{D}_{(i+1,k-1)}\right).
\end{aligned}
$$

Thus, we can shift our goal from proving $\alpha_{\mathbf{p}}\mathfrak{D}_{\mathbf{p}} - \mathfrak{N}_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$ to proving $\mathfrak{N}_{(i+1,k-1)}\mathfrak{D}_{\mathbf{p}} + \mathfrak{N}_{\mathbf{p}}\mathfrak{D}_{(i+1,k-1)} \neq 0$.

But this turns out to be surprisingly simple: Since $\mathbf{p} = (i, k)$, we have

$$
\begin{aligned}
&\mathfrak{N}_{(i+1,k-1)}\mathfrak{D}_{\mathbf{p}} + \mathfrak{N}_{\mathbf{p}}\mathfrak{D}_{(i+1,k-1)} \\
&= \mathfrak{N}_{(i+1,k-1)}\mathfrak{D}_{(i,k)} + \mathfrak{N}_{(i,k)}\mathfrak{D}_{(i+1,k-1)} = \mathfrak{D}_{(i,k)} \cdot \mathfrak{N}_{(i+1,k-1)} + \mathfrak{N}_{(i,k)} \cdot \mathfrak{D}_{(i+1,k-1)} \\
&= \det\left((I_p \mid C)\left[0 : i \mid i + k : p + k\right]\right) \cdot \det\left((I_p \mid C)\left[1 : i + 1 \mid i + k - 1 : p + k - 1\right]\right) \\
&\qquad + \det\left((I_p \mid C)\left[1 : i \mid i + k - 1 : p + k\right]\right) \\
&\qquad\qquad \cdot \det\left((I_p \mid C)\left[0 : i + 1 \mid i + k : p + k - 1\right]\right) \\
&= \det\left((I_p \mid C)\left[0 : i \mid i + k - 1 : p + k - 1\right]\right) \cdot \det\left((I_p \mid C)\left[1 : i + 1 \mid i + k : p + k\right]\right) \tag{28}
\end{aligned}
$$

by definition and Theorem 50. On the other hand, $(i, k-1)$ and $(i+1, k)$ also belong to $\mathbf{P}$ and satisfy

$$\mathfrak{D}_{(i,k-1)} = \det\left((I_p \mid C)\left[0 : i \mid i + k - 1 : p + k - 1\right]\right)$$

and

$$\mathfrak{N}_{(i+1,k)} = \det\left((I_p \mid C)\left[1 : i + 1 \mid i + k : p + k\right]\right)$$

Hence, (28) becomes

$$\begin{aligned}
&\mathfrak{N}_{(i+1,k-1)}\mathfrak{D}_{\mathbf{p}} + \mathfrak{N}_{\mathbf{p}}\mathfrak{D}_{(i+1,k-1)} \\
&= \underbrace{\det\left((I_p \mid C)\left[0 : i \mid i + k - 1 : p + k - 1\right]\right)}_{=\mathfrak{D}_{(i,k-1)}} \cdot \underbrace{\det\left((I_p \mid C)\left[1 : i + 1 \mid i + k : p + k\right]\right)}_{=\mathfrak{N}_{(i+1,k)}} \\
&= \mathfrak{D}_{(i,k-1)} \cdot \mathfrak{N}_{(i+1,k)} \neq 0
\end{aligned}$$

by Step 2. This finishes our proof that $\mathfrak{N}_{(i+1,k-1)}\mathfrak{D}_{\mathbf{p}} + \mathfrak{N}_{\mathbf{p}}\mathfrak{D}_{(i+1,k-1)} \neq 0$, thus also that $\alpha_{\mathbf{p}}\mathfrak{D}_{\mathbf{p}} - \mathfrak{N}_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$, hence also that $\alpha_{\mathbf{p}}\delta_{\mathbf{p}} - \beta_{\mathbf{p}}\gamma_{\mathbf{p}} \neq 0$, and ultimately of the $\mathbf{P}$-triangularity of the family $(Q_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$.

Details of Step 5: Recall that our goal is to prove the existence of a matrix $\widetilde{D} \in (\mathbb{F}(x_{\mathbf{P}}))^{p\times(p+q)}$ satisfying (14). By Step 4, we know that the family $(Q_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}} \in (\mathbb{F}(x_{\mathbf{P}}))^{\mathbf{P}}$ is $\mathbf{P}$-triangular. Hence, Lemma 54 **(b)** shows that there exists a $\mathbf{P}$-triangular family $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}} \in (\mathbb{F}(x_{\mathbf{P}}))^{\mathbf{P}}$ such that every $\mathbf{q} \in \mathbf{P}$ satisfies $Q_{\mathbf{q}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = x_{\mathbf{q}}$. Applying Lemma 54 **(a)** to this family $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$, we conclude that $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$ is algebraically independent.

In Step 3, we have shown that $Q_{\mathbf{p}} = (\mathrm{Grasp}_0 (I_p \mid C)) (\mathbf{p})$ for every $\mathbf{p} \in \mathbf{P}$. Renaming $\mathbf{p}$ as $\mathbf{q}$, we rewrite this as follows:

$$Q_{\mathbf{q}} = (\mathrm{Grasp}_0 (I_p \mid C)) (\mathbf{q}) \qquad \text{for every } \mathbf{q} \in \mathbf{P}. \tag{29}$$

Now, let $\widetilde{C} \in (\mathbb{F}(x_{\mathbf{P}}))^{p\times(p+q)}$ denote the matrix obtained from $C \in (\mathbb{F}[x_{\mathbf{P}}])^{p\times(p+q)}$ by substituting $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$ for the variables $(x_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$. Since (29) is an identity between rational functions in the variables $(x_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$, we thus can substitute $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$ for the variables $(x_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$ in (29)[7], and obtain

$$Q_{\mathbf{q}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = \left(\mathrm{Grasp}_0\left(I_p \mid \widetilde{C}\right)\right)(\mathbf{q}) \qquad \text{for every } \mathbf{q} \in \mathbf{P}$$

(since this substitution takes the matrix C to $\widetilde{C}$). But since $Q_{\mathbf{q}}\left((R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}\right) = x_{\mathbf{q}}$ for every $\mathbf{q} \in \mathbf{P}$, this rewrites as

$$x_{\mathbf{q}} = \left(\mathrm{Grasp}_0\left(I_p \mid \widetilde{C}\right)\right)(\mathbf{q}) \qquad \text{for every } \mathbf{q} \in \mathbf{P}.$$

---

[7]The substitution does not suffer from vanishing denominators because $(R_{\mathbf{p}})_{\mathbf{p}\in\mathbf{P}}$ is algebraically independent.

Upon renaming $\mathbf{q}$ as $\mathbf{p}$ again, this becomes

$$x_{\mathbf{p}} = \left(\operatorname{Grasp}_0 \left(I_p \mid \widetilde{C} \right) \right) (\mathbf{p}) \qquad \text{for every } \mathbf{p} \in \mathbf{P}.$$

Hence, there exists a matrix $\widetilde{D} \in (\mathbb{F}(x_{\mathbf{P}}))^{p \times (p+q)}$ satisfying (14) (namely, $\widetilde{D} = \left(I_p \mid \widetilde{C} \right)$). This completes the proof of Proposition 49. $\qquad \square$

8 The rectangle: finishing the proofs

As promised, we now use Propositions 48 and 49 to derive our initially stated results on rectangles. First, we formulate an easy inductive consequence of Proposition 48:

Corollary 55. *Let $A \in \mathbb{K}^{p \times (p+q)}$ be a matrix. Then every $i \in \mathbb{N}$ satisfies*

$$\operatorname{Grasp}_{-i} A = R^i_{\operatorname{Rect}(p,q)} (\operatorname{Grasp}_0 A)$$

(provided that A is sufficiently generic in the sense of Zariski topology that both sides of this equality are well-defined).

Proof of Theorem 30. We need to show that $\operatorname{ord} \left(R_{\operatorname{Rect}(p,q)} \right) = p + q$. According to Proposition 35, it is enough to prove that almost every (in the Zariski sense) reduced labelling $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ satisfies $R^{p+q}_{\operatorname{Rect}(p,q)} f = f$. So let $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ be a sufficiently generic reduced labelling. In other words, f is a sufficiently generic element of $\mathbb{K}^{\operatorname{Rect}(p,q)}$ (because the reduced labellings $\mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ are being identified with the elements of $\mathbb{K}^{\operatorname{Rect}(p,q)}$). By Proposition 49, there exists a matrix $A \in \mathbb{K}^{p \times (p+q)}$ satisfying $f = \operatorname{Grasp}_0 A$. Consider this A. By Corollary 55 (applied to $i = p + q$), we have

$$\operatorname{Grasp}_{-(p+q)} A = R^{p+q}_{\operatorname{Rect}(p,q)} \left(\underbrace{\operatorname{Grasp}_0 A}_{=f} \right) = R^{p+q}_{\operatorname{Rect}(p,q)} f.$$

But Proposition 46 (applied to $j = -(p+q)$) yields

$$\operatorname{Grasp}_{-(p+q)} A = \operatorname{Grasp}_{p+q+(-(p+q))} A = \operatorname{Grasp}_0 A = f.$$

Hence, $f = \operatorname{Grasp}_{-(p+q)} A = R^{p+q}_{\operatorname{Rect}(p,q)} f$, proving the theorem. $\qquad \square$

Proof of Theorem 36. We regard the reduced labelling $f \in \mathbb{K}^{\widehat{\operatorname{Rect}(p,q)}}$ as an element of $\mathbb{K}^{\operatorname{Rect}(p,q)}$. We assume WLOG that this element $f \in \mathbb{K}^{\operatorname{Rect}(p,q)}$ is generic enough (among the reduced labellings) for Proposition 49 to apply; hence, there exists a matrix $A \in \mathbb{K}^{p \times (p+q)}$ satisfying $f = \operatorname{Grasp}_0 A$. By Corollary 55 (applied to $i + k - 1$ instead of i), we have

$$\operatorname{Grasp}_{-(i+k-1)} A = R^{i+k-1}_{\operatorname{Rect}(p,q)} \left(\underbrace{\operatorname{Grasp}_0 A}_{=f} \right) = R^{i+k-1}_{\operatorname{Rect}(p,q)} f.$$

But Proposition 47 (applied to $j = -(i + k - 1)$) yields

$$
\begin{aligned}
\left(\mathrm{Grasp}_{-(i+k-1)} A \right) ((i, k)) &= \frac{1}{\left(\mathrm{Grasp}_{-(i+k-1)+i+k-1} A \right) ((p + 1 - i, q + 1 - k))} \\
&= \frac{1}{f ((p + 1 - i, q + 1 - k))},
\end{aligned}
$$

so that

$$
f ((p + 1 - i, q + 1 - k)) = \frac{1}{\left(\mathrm{Grasp}_{-(i+k-1)} A \right) ((i, k))} = \frac{1}{\left(R_{\mathrm{Rect}(p,q)}^{i+k-1} f \right) ((i, k))},
$$

proving Theorem 36. $\qquad\square$

Proof of Theorem 32. Recall the notation $(a_0, a_1, \ldots, a_{n+1}) \flat f$ defined in Definition 18. Let $f \in \mathbb{K}^{\widehat{\mathrm{Rect}(p,q)}}$ be arbitrary. By genericity, we assume WLOG that $f(0)$ and $f(1)$ are nonzero.

Let $n = p + q - 1$, so $\mathrm{Rect}(p, q)$ is an n-graded poset. For any element $(i, k) \in \mathrm{Rect}(p, q)$, we have $i + k - 1 \in \{0, 1, \ldots, n\}$ and $1 \leqslant n - i - k + 2 \leqslant n$.

Define an $(n + 2)$-tuple $(a_0, a_1, \ldots, a_{n+1}) \in \mathbb{K}^{n+2}$ by

$$
a_r = \begin{cases} \dfrac{1}{f(0)}, & \text{if } r = 0; \\ 1, & \text{if } 1 \leqslant r \leqslant n; \\ \dfrac{1}{f(1)}, & \text{if } r = n + 1 \end{cases} \qquad \text{for every } r \in \{0, 1, \ldots, n + 1\}.
$$

Thus, $a_{n-i-k+2} = 1$ (since $1 \leqslant n - i - k + 2 \leqslant n$) and $a_0 = \dfrac{1}{f(0)}$ and $a_{n+1} = \dfrac{1}{f(1)}$.

Let $f' = (a_0, a_1, \ldots, a_{n+1}) \flat f$. Then clearly $f'(0) = 1$ and $f'(1) = 1$, i.e., f' is a reduced $\mathbb{K}$-labelling. Hence, Theorem 36 (applied to f' instead of f) yields

$$
f' ((p + 1 - i, q + 1 - k)) = \frac{1}{\left(R_{\mathrm{Rect}(p,q)}^{i+k-1} (f') \right) ((i, k))}. \tag{30}
$$

On the other hand, it is easy to see that $f'(v) = f(v)$ for every $v \in \mathrm{Rect}(p, q)$. This yields, in particular, that $f' ((p + 1 - i, q + 1 - k)) = f ((p + 1 - i, q + 1 - k))$.

But let us define an element $\widehat{a}_\kappa^{(\ell)} \in \mathbb{K}^\times$ for every $\ell \in \{0, 1, \ldots, n + 1\}$ and $\kappa \in \{0, 1, \ldots, n + 1\}$ as in Proposition 19. Then, it is easy to see that every $\kappa \in \{0, 1, \ldots, n + 1\}$ satisfies

$$
\widehat{a}_\kappa^\kappa = a_{n+1} a_0 = \frac{1}{f(0) f(1)} \tag{31}
$$

(since $a_{n+1} = \dfrac{1}{f(1)}$ and $a_0 = \dfrac{1}{f(0)}$). Proposition 19 (applied to $\ell = i + k - 1$) yields

$$R_{\text{Rect}(p,q)}^{i+k-1}\left((a_0, a_1, \ldots, a_{n+1})\, \flat\, f\right) = \left(\widehat{a}_0^{(i+k-1)}, \widehat{a}_1^{(i+k-1)}, \ldots, \widehat{a}_{n+1}^{(i+k-1)}\right) \flat \left(R_{\text{Rect}(p,q)}^{i+k-1} f\right).$$

Since $(a_0, a_1, \ldots, a_{n+1})\, \flat\, f = f'$, this rewrites as

$$R_{\text{Rect}(p,q)}^{i+k-1}\left(f'\right) = \left(\widehat{a}_0^{(i+k-1)}, \widehat{a}_1^{(i+k-1)}, \ldots, \widehat{a}_{n+1}^{(i+k-1)}\right) \flat \left(R_{\text{Rect}(p,q)}^{i+k-1} f\right).$$

Hence,

$$\left(R_{\text{Rect}(p,q)}^{i+k-1}\left(f'\right)\right)((i,k))$$
$$= \left(\left(\widehat{a}_0^{(i+k-1)}, \widehat{a}_1^{(i+k-1)}, \ldots, \widehat{a}_{n+1}^{(i+k-1)}\right) \flat \left(R_{\text{Rect}(p,q)}^{i+k-1} f\right)\right)((i,k))$$
$$= \widehat{a}_{\deg((i,k))}^{(i+k-1)} \cdot \left(R_{\text{Rect}(p,q)}^{i+k-1} f\right)((i,k))$$
$$= \widehat{a}_{i+k-1}^{(i+k-1)} \cdot \left(R_{\text{Rect}(p,q)}^{i+k-1} f\right)((i,k)) \qquad \text{(since } \deg((i,k)) = i + k - 1\text{)}$$
$$= \frac{1}{f(0)\,f(1)} \cdot \left(R_{\text{Rect}(p,q)}^{i+k-1} f\right)((i,k)).$$

Thus, (30) rewrites as

$$f'\left((p+1-i, q+1-k)\right) = \frac{1}{\dfrac{1}{f(0)\,f(1)} \cdot \left(R_{\text{Rect}(p,q)}^{i+k-1} f\right)((i,k))} = \frac{f(0)\,f(1)}{\left(R_{\text{Rect}(p,q)}^{i+k-1} f\right)((i,k))}.$$

This rewrites as

$$f\left((p+1-i, q+1-k)\right) = \frac{f(0)\,f(1)}{\left(R_{\text{Rect}(p,q)}^{i+k-1} f\right)((i,k))}$$

(since we know that $f'(v) = f(v)$ on $\text{Rect}(p,q)$), proving the theorem. $\qquad\square$

9 The $\rhd$ triangle

Having proven the main properties of birational rowmotion R on the rectangle $\text{Rect}(p,q)$, we now turn to other posets. We will spend the next three sections discussing the order of birational rowmotion on certain triangle-shaped posets obtained as subsets of the square $\text{Rect}(p,p)$. We start with the easiest case:

Definition 56. Let p be a positive integer. Define a subset $\text{Tria}(p)$ of $\text{Rect}(p,p)$ by

$$\text{Tria}(p) = \left\{(i,k) \in \{1,2,\ldots,p\}^2 \mid i \leqslant k\right\}.$$

This subset $\text{Tria}(p)$ inherits the structure of a $(2p-1)$-graded poset from $\text{Rect}(p,p)$. It has the form of a triangle as shown below.

Example 57. Below we show on the *left* the Hasse diagram of the poset $\operatorname{Rect}(4,4)$, with the elements that belong to $\operatorname{Tria}(4)$ marked by underlines; on the *right* is the Hasse diagram of the poset $\operatorname{Tria}(4)$ itself:
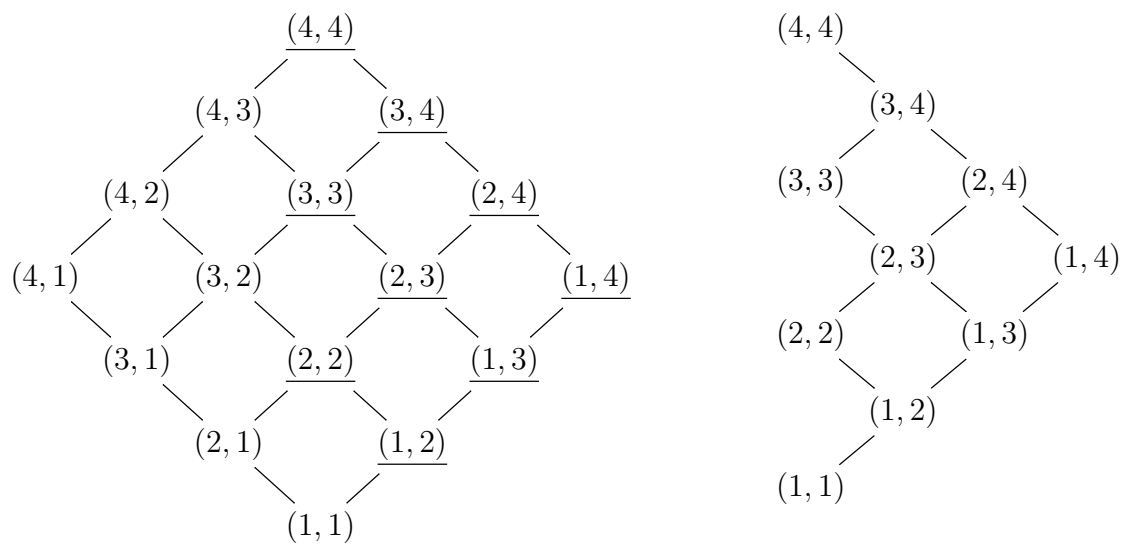
We could also consider the subset $\left\{(i,k) \in \{1,2,\ldots,p\}^2 \mid i \geqslant k\right\}$, but that would yield a poset isomorphic to $\operatorname{Tria}(p)$ and thus would not be of any further interest.

Theorem 58. *Let p be a positive integer. Let $\mathbb{K}$ be a field. Then,* $\operatorname{ord}\left(R_{\operatorname{Tria}(p)}\right) = 2p$.

As for rectangles, we get here the birational version of a known result for classical rowmotion. The poset $\operatorname{Tria}(p)$ appears in [StWi11, §6.2] as the poset of order ideals $J([2] \times [p-1])$, where the authors show that $\operatorname{ord}\left(\mathbf{r}_{\operatorname{Tria}(p)}\right) = 2p$. Theorem 58 thus shows that birational rowmotion and classical rowmotion have the same order for $\operatorname{Tria}(p)$.

In order to prove Theorem 58, we need a way to turn labellings of $\operatorname{Tria}(p)$ into labellings of $\operatorname{Rect}(p,p)$ in a rowmotion-equivariant way. It turns out that the obvious "unfolding" construction (with some fudge coefficients) works:

Lemma 59. *Let p be a positive integer. Let $\mathbb{K}$ be a field of characteristic $\neq 2$.*

(a) Let $\operatorname{vrefl} : \operatorname{Rect}(p,p) \to \operatorname{Rect}(p,p)$ *be the map sending every* $(i,k) \in \operatorname{Rect}(p,p)$ *to* (k,i). *This map* vrefl *is an involutive poset automorphism of* $\operatorname{Rect}(p,p)$. *(In intuitive terms,* vrefl *is simply reflection across the vertical axis.) We have* $\operatorname{vrefl}(v) \in \operatorname{Tria}(p)$ *for every* $v \in \operatorname{Rect}(p,p) \setminus \operatorname{Tria}(p)$.

We extend vrefl *to an involutive poset automorphism of* $\widehat{\operatorname{Rect}(p,p)}$ *by setting* $\operatorname{vrefl}(0) = 0$ *and* $\operatorname{vrefl}(1) = 1$.

(b) Define a map $\operatorname{dble} : \mathbb{K}^{\widehat{\operatorname{Tria}(p)}} \to \mathbb{K}^{\widehat{\operatorname{Rect}(p,p)}}$ *by setting*

$$(\operatorname{dble} f)(v) = \begin{cases} \dfrac{1}{2}f(1), & \text{if } v = 1; \\ 2f(0), & \text{if } v = 0; \\ f(v), & \text{if } v \in \operatorname{Tria}(p); \\ f(\operatorname{vrefl}(v)), & \text{otherwise} \end{cases}$$

for all $v \in \widehat{\mathrm{Rect}\,(p,p)}$ *for all* $f \in \mathbb{K}^{\widehat{\mathrm{Tria}(p)}}$. *This is well-defined. We have*

$$(\mathrm{dble}\, f)\,(v) = f\,(v) \qquad \textit{for every } v \in \mathrm{Tria}\,(p)\,. \tag{32}$$

Also,

$$(\mathrm{dble}\, f)\,(\mathrm{vrefl}\,(v)) = f\,(v) \qquad \textit{for every } v \in \mathrm{Tria}\,(p)\,. \tag{33}$$

(c) *We have*

$$R_{\mathrm{Rect}(p,p)} \circ \mathrm{dble} = \mathrm{dble} \circ R_{\mathrm{Tria}(p)}.$$

The coefficients $\dfrac{1}{2}$ and 2 in the definition of dble ensure that the labellings $R_{\mathrm{Rect}(p,p)} \circ$ dble and dble $\circ R_{\mathrm{Tria}(p)}$ in part **(c)** of the Lemma are equal at every element of the poset, without extraneous factors appearing in certain ranks.

Proof. The proofs of **(a)** and **(b)** are easy, following in a few lines from the definitions. The proof of **(c)** involves a few pages of rewriting formulas and case-checking, but there are no surprises. Full details are available in [GrRo14b]. $\qquad\square$

Proof of Theorem 58. Applying Lemma 26 to $2p - 1$ and $\mathrm{Tria}\,(p)$ instead of n and P, we see that $\mathrm{ord}\left(R_{\mathrm{Tria}(p)}\right)$ is divisible by $2p - 1 + 1 = 2p$. Now, if we can prove that $\mathrm{ord}\left(R_{\mathrm{Tria}(p)}\right) \mid 2p$, then we will immediately obtain $\mathrm{ord}\left(R_{\mathrm{Tria}(p)}\right) = 2p$, and Theorem 58 will be proven.

So it suffices to show that $R_{\mathrm{Tria}(p)}^{2p} = \mathrm{id}$. Since this statement boils down to a collection of polynomial identities in the labels of an arbitrary $\mathbb{K}$-labelling of $\mathrm{Tria}\,(p)$, it is clear that it is enough to prove it in the case when $\mathbb{K}$ is a field of rational functions in finitely many variables over $\mathbb{Q}$. So let us WLOG assume we are in this case; then the characteristic of $\mathbb{K}$ is $0 \neq 2$, so that we can apply Lemma 59**(c)** to get

$$R_{\mathrm{Rect}(p,p)} \circ \mathrm{dble} = \mathrm{dble} \circ R_{\mathrm{Tria}(p)}.$$

From this, it follows (by induction over k) that

$$R_{\mathrm{Rect}(p,p)}^{k} \circ \mathrm{dble} = \mathrm{dble} \circ R_{\mathrm{Tria}(p)}^{k}$$

for every $k \in \mathbb{N}$. Applied to $k = 2p$, this yields

$$R_{\mathrm{Rect}(p,p)}^{2p} \circ \mathrm{dble} = \mathrm{dble} \circ R_{\mathrm{Tria}(p)}^{2p}. \tag{34}$$

But Theorem 30 (applied to $q = p$) yields $\mathrm{ord}\left(R_{\mathrm{Rect}(p,p)}\right) = p + p = 2p$, so that $R_{\mathrm{Rect}(p,p)}^{2p} = \mathrm{id}$. Hence, (34) simplifies to

$$\mathrm{dble} = \mathrm{dble} \circ R_{\mathrm{Tria}(p)}^{2p}.$$

We can cancel dble from this equation, because dble is an injective and therefore left-cancellable map. As a consequence, we obtain $\mathrm{id} = R_{\mathrm{Tria}(p)}^{2p}$. In other words, $R_{\mathrm{Tria}(p)}^{2p} = \mathrm{id}$. This proves Theorem 58. $\qquad\square$

10 The Δ and ∇ triangles

The next kind of triangle-shaped posets is more interesting.

Definition 60. Let p be a positive integer. Define three subsets $\Delta(p)$, $\mathrm{Eq}(p)$ and $\nabla(p)$ which partition $\mathrm{Rect}(p,p)$ by

$$\Delta(p) = \left\{ (i,k) \in \{1,2,\ldots,p\}^2 \mid i+k > p+1 \right\};$$
$$\mathrm{Eq}(p) = \left\{ (i,k) \in \{1,2,\ldots,p\}^2 \mid i+k = p+1 \right\};$$
$$\nabla(p) = \left\{ (i,k) \in \{1,2,\ldots,p\}^2 \mid i+k < p+1 \right\}.$$

These subsets $\Delta(p)$, $\mathrm{Eq}(p)$ and $\nabla(p)$ inherit a poset structure from $\mathrm{Rect}(p,p)$.

Clearly, $\mathrm{Eq}(p)$ is an antichain with p elements. (The name Eq comes from "equator".) The posets $\Delta(p)$ and $\nabla(p)$ are $(p-1)$-graded posets. They have the form of a "Delta-shaped triangle" and a "Nabla-shaped triangle", respectively (whence the names).

Example 61. Here is the Hasse diagram of the poset $\mathrm{Rect}(4,4)$, where the elements belonging to $\Delta(4)$ have been underlined and the elements belonging to $\mathrm{Eq}(4)$ have been boxed:
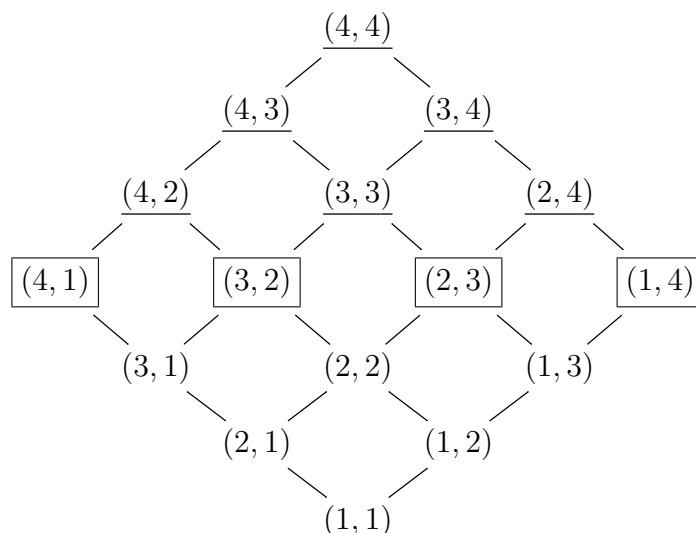
We show the Hasse diagrams of the poset $\Delta(4)$ (on the *left*) and $\nabla(4)$ (on the *right*) below:
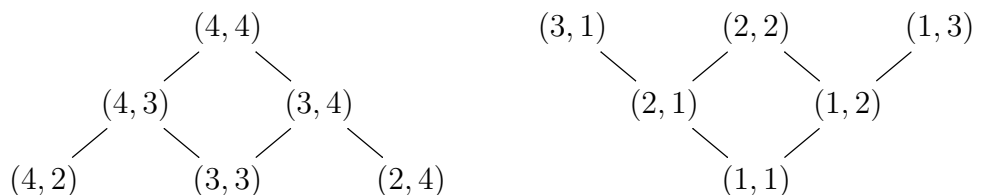
Remark 62. Let p be a positive integer. The poset $\Delta(p)$ is isomorphic to the poset $\Phi^+(A_{p-1})$ of [StWi11, §3.2].

Remark 63. For every positive integer p, we have $\nabla(p) \cong (\Delta(p))^{\mathrm{op}}$ as posets. This follows immediately from the poset antiautomorphism

$$\mathrm{hrefl} : \mathrm{Rect}(p,p) \to \mathrm{Rect}(p,p),$$
$$(i,k) \mapsto (p+1-k, p+1-i)$$

sending $\nabla(p)$ to $\Delta(p)$.

Here we are using the following notions:

Definition 64. (a) If P and Q are two posets, then a map $f : P \to Q$ is called a *poset antihomomorphism* if and only if every $p_1 \in P$ and $p_2 \in P$ satisfying $p_1 \leqslant p_2$ in P satisfy $f(p_1) \geqslant f(p_2)$ in Q. It is easy to see that the poset antihomomorphisms $P \to Q$ are precisely the poset homomorphisms $P \to Q^{\mathrm{op}}$.

(b) If P and Q are two posets, then an invertible map $f : P \to Q$ is called a *poset antiisomorphism* if and only if both f and f^{-1} are poset antihomomorphisms.

(c) If P is a poset and $f : P \to P$ is an invertible map, then f is said to be a *poset antiautomorphism* if f is a poset antiisomorphism.

We now state the main property of birational rowmotion R on the posets $\nabla(p)$ and $\Delta(p)$. The antiautomorphism above will allow us to transfer results about the order of birational rowmotion from one poset to the other.

Theorem 65. *Let p be an integer $\geqslant 1$. Let $\mathbb{K}$ be a field. For every $(i,k) \in \nabla(p)$ and every $f \in \mathbb{K}^{\widehat{\nabla(p)}}$, we have*

$$\left(R^p_{\nabla(p)} f \right)((i,k)) = f((k,i)).$$

The same holds if we replace $\nabla(p)$ everywhere with $\Delta(p)$.

Corollary 66. *Let p be an integer > 1. Then:*
(a) We have $\mathrm{ord}\left(R_{\nabla(p)} \right) \mid 2p$.
(b) If $p > 2$, then $\mathrm{ord}\left(R_{\nabla(p)} \right) = 2p$.
The same holds if we replace $\nabla(p)$ everywhere with $\Delta(p)$.

Corollary 66 (for $\Delta(p)$) is analogous to a known result for classical rowmotion. In fact, from [StWi11, Conjecture 3.6] (originally a conjecture of Panyushev, then proven by Armstrong, Stump and Thomas) and our Remark 62, it can be seen that every integer $p > 2$ satisfies $\mathrm{ord}\left(\mathbf{r}_{\Delta(p)} \right) = 2p$ (where $\mathbf{r}_P$ denotes the classical rowmotion map on the order ideals of a poset P). Also, the equivalence of these results for $\nabla(p)$ and $\Delta(p)$ follows from Remark 63 and Proposition 25).

The proof of Theorem 65 will use a mapping that transforms labellings of $\Delta(p)$ into labellings of $\mathrm{Rect}(p,p)$ in a way that is rowmotion-equivariant at least under a rather liberal condition on the labelling. This mapping is similar in its function to the mapping dble of Lemma 59, but its definition is more intricate. Thanks to a suggestion by an anonymous referee, we state a more general lemma that will specialize to the one we need.

Lemma 67. *Let p be a positive integer. Let P be a $(2p-1)$-graded finite poset. Let* $\mathrm{hrefl} : P \to P$ *be an involution such that* hrefl *is a poset antiautomorphism of P. We extend* hrefl *to an involutive poset antiautomorphism of $\widehat{P}$ by setting* $\mathrm{hrefl}\,(0) = 1$ *and* $\mathrm{hrefl}\,(1) = 0$.

Assume that every $v \in \widehat{P}$ satisfies $\deg\,(\mathrm{hrefl}\,v) = 2p - \deg v$.

Let N be a positive integer. Assume that, for every $v \in P$ satisfying $\deg v = p - 1$, there exist precisely N elements u of P satisfying $u \gtrdot v$.

Define three subsets Δ, Eq and ∇ of P by

$$\Delta = \{v \in P \ \mid \ \deg v > p\};$$
$$\mathrm{Eq} = \{v \in P \ \mid \ \deg v = p\};$$
$$\nabla = \{v \in P \ \mid \ \deg v < p\}.$$

Clearly, Δ, Eq and ∇ become subposets of P. The poset Eq is an antichain, while the posets Δ and ∇ are $(p-1)$-graded.

Assume that $\mathrm{hrefl}\,|_{\mathrm{Eq}} = \mathrm{id}$. *It is easy to see that* $\mathrm{hrefl}\,(\Delta) = \nabla$.

Let $\mathbb{K}$ be a field such that N is invertible in $\mathbb{K}$.

(a) Define a rational map $\mathrm{wing} : \mathbb{K}^{\widehat{\Delta}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ *by setting*

$$(\mathrm{wing}\, f)\,(v) = \begin{cases} f\,(v), & \text{if } v \in \Delta \cup \{1\}\,; \\ 1, & \text{if } v \in \mathrm{Eq}; \\ \dfrac{1}{\left(R_\Delta^{p - \deg v} f\right)(\mathrm{hrefl}\, v)}, & \text{if } v \in \nabla \cup \{0\} \end{cases}$$

for all $v \in \widehat{P}$ for all $f \in \mathbb{K}^{\widehat{\Delta}}$. This is well-defined.

(b) There exists a rational map $\overline{\mathrm{wing}} : \overline{\mathbb{K}^{\widehat{\Delta}}} \dashrightarrow \overline{\mathbb{K}^{\widehat{P}}}$ *such that the diagram*

$$
\begin{array}{ccc}
\mathbb{K}^{\widehat{\Delta}} & \overset{\mathrm{wing}}{\dashrightarrow} & \mathbb{K}^{\widehat{P}} \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
\overline{\mathbb{K}^{\widehat{\Delta}}} & \underset{\overline{\mathrm{wing}}}{\dashrightarrow} & \overline{\mathbb{K}^{\widehat{P}}}
\end{array}
$$

commutes.

(c) The rational map $\overline{\mathrm{wing}}$ *defined in Lemma 67 (b) satisfies*

$$\overline{R}_P \circ \overline{\mathrm{wing}} = \overline{\mathrm{wing}} \circ \overline{R}_\Delta.$$

(d) Almost every (in the sense of Zariski topology) labelling $f \in \mathbb{K}^{\widehat{\Delta}}$ satisfying $f\,(0) = N$ satisfies

$$R_P\,(\mathrm{wing}\, f) = \mathrm{wing}\,(R_\Delta f)\,.$$

The condition $f\,(0) = N$ in part **(d)** of this lemma has been made to ensure that we obtain a honest equality between $R_P\,(\mathrm{wing}\, f)$ and $\mathrm{wing}\,(R_\Delta f)$, without "correction factors" in certain ranks.

Proof of Lemma 67. We will not delve into the details of this tedious and yet straightforward proof. Parts **(a)** and **(b)** are straightforward and quick. Parts **(c)** and **(d)** can be verified label-by-label using Propositions 10 and 13 and some nasty casework (see, again, [GrRo14b]). $\qquad\square$

Example 68. Here is an example of a poset $P \neq \text{Rect}(p, p)$ to which Lemma 67 applies. Namely, the hypotheses of Lemma 67 are satisfied when $p = 5$, $N = 3$, P is the poset with Hasse diagram
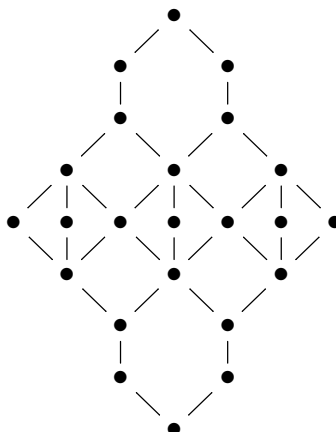
and $\text{hrefl} : P \to P$ is the reflection about the horizontal axis of symmetry.

Example 69. For the case of interest in this section, we now specify henceforth the map $\text{hrefl} : \text{Rect}(p, p) \to \text{Rect}(p, p)$ to be given by $(i, k) \in \text{Rect}(p, p) \mapsto (p + 1 - k, p + 1 - i)$. This map hrefl clearly satisfies the hypotheses of Lemma 67, where we set $P = \text{Rect}(p, p)$ and $N = 2$; we then have $\Delta = \Delta(p)$ and $\nabla = \nabla(p)$. In intuitive terms, hrefl is simply reflection across the horizontal axis, i.e., the line $\text{Eq}(p)$.

We are ready to prove the main theorem of this section.

Proof of Theorem 65. The result that we are striving to prove is a collection of identities between rational functions, hence boils down to a collection of polynomial identities in the labels of an arbitrary $\mathbb{K}$-labelling of $\Delta(p)$. Therefore, it is enough to prove it in the case when $\mathbb{K}$ is a field of rational functions in finitely many variables over $\mathbb{Q}$. So let us WLOG assume that we are in this case. Then, 2 is invertible in $\mathbb{K}$, so that we can apply Lemma 67.

Consider the maps hrefl, wing, and vrefl defined in Example 69, Lemma 67, and Lemma 59. The restrictions of vrefl to the subposets $\Delta(p)$ and $\nabla(p)$ are automorphisms of these subposets, and will also be denoted by vrefl.

Let $g \in \mathbb{K}^{\widehat{\Delta(p)}}$ be any sufficiently generic zero-free labelling of $\Delta(p)$. We need to show that $R^p_{\Delta(p)} g = g \circ \text{vrefl}$ (indeed, this is merely a restatement of Theorem 65 with f renamed as g).

Since the poset $\Delta(p)$ is $(p-1)$-graded, using Definition 18 we can find a $(p+1)$-tuple $(a_0, a_1, \ldots, a_p) \in (\mathbb{K}^\times)^{p+1}$ such that $((a_0, a_1, \ldots, a_p) \flat g)(0) = 2$ (by setting $a_0 = \dfrac{2}{g(0)}$, and

choosing all other a_i arbitrarily). Fix such a $(p+1)$-tuple, and set $f = (a_0, a_1, \ldots, a_p) \flat g$. Then, $f(0) = 2$. We are going to prove that $R_{\Delta(p)}^p f = f \circ \mathrm{vrefl}$. Until we have done this, we can forget about g; all we need to know is that f is a sufficiently generic $\mathbb{K}$-labelling of $\Delta(p)$ satisfying $f(0) = 2$.

Let $(i, k) \in \Delta(p)$ be arbitrary. Then, $i + k > p + 1$ (since $(i, k) \in \Delta(p)$). Consequently, $2p - (i + k - 1)$ is a well-defined element of $\{1, 2, \ldots, p - 1\}$. Denote this element by h. Thus, $h \in \{1, 2, \ldots, p - 1\}$ and $i + k - 1 + h = 2p$. Moreover, $(k, i) = \mathrm{vrefl}\, v \in \Delta(p)$.

Let $v = (p + 1 - k, p + 1 - i)$. Then, $v = \mathrm{hrefl}\,((i, k)) \in \nabla(p)$ (since $(i, k) \in \Delta(p)$) and $\deg v = h$. Moreover, $\mathrm{hrefl}\, v = (i, k)$.

Lemma 67 **(d)** (applied h times) yields $R_{\mathrm{Rect}(p,p)}^h (\mathrm{wing}\, f) = \mathrm{wing}\left(R_{\Delta(p)}^h f \right)$; hence,

$$\left(R_{\mathrm{Rect}(p,p)}^h (\mathrm{wing}\, f) \right)(v) = \left(\mathrm{wing}\left(R_{\Delta(p)}^h f \right) \right)(v) = \frac{1}{\left(R_{\Delta(p)}^{p - \deg v} \left(R_{\Delta(p)}^h f \right) \right)(\mathrm{hrefl}\, v)}$$

$$= \frac{1}{\left(R_{\Delta(p)}^{p - h} \left(R_{\Delta(p)}^h f \right) \right)(\mathrm{hrefl}\, v)} \qquad (\text{since } \deg v = h)$$

$$= \frac{1}{\left(R_{\Delta(p)}^p f \right)((i, k))}. \qquad (35)$$

But Theorem 32 yields

$$\left(R_{\mathrm{Rect}(p,p)}^h (\mathrm{wing}\, f) \right)((p + 1 - k, p + 1 - i))$$
$$= \frac{\left(R_{\mathrm{Rect}(p,p)}^h (\mathrm{wing}\, f) \right)(0) \cdot \left(R_{\mathrm{Rect}(p,p)}^h (\mathrm{wing}\, f) \right)(1)}{\left(R_{\mathrm{Rect}(p,p)}^{i+k-1} \left(R_{\mathrm{Rect}(p,p)}^h (\mathrm{wing}\, f) \right) \right)((k, i))}.$$

Since $(p + 1 - k, p + 1 - i) = v$ and

$$R_{\mathrm{Rect}(p,p)}^{i+k-1} \left(R_{\mathrm{Rect}(p,p)}^h (\mathrm{wing}\, f) \right) = R_{\mathrm{Rect}(p,p)}^{2p} (\mathrm{wing}\, f) = \mathrm{wing}\, f,$$

this equality rewrites as

$$\left(R_{\mathrm{Rect}(p,p)}^h (\mathrm{wing}\, f) \right)(v) = \frac{\left(R_{\mathrm{Rect}(p,p)}^h (\mathrm{wing}\, f) \right)(0) \cdot \left(R_{\mathrm{Rect}(p,p)}^h (\mathrm{wing}\, f) \right)(1)}{(\mathrm{wing}\, f)((k, i))}.$$

By Corollary 12 and the definition of wing this simplifies to

$$\left(R_{\mathrm{Rect}(p,p)}^h (\mathrm{wing}\, f) \right)(v) = \frac{1}{(\mathrm{wing}\, f)((k, i))}.$$

Compared with (35), this yields $\dfrac{1}{\left(R_{\Delta(p)}^p f \right)((i, k))} = \dfrac{1}{(\mathrm{wing}\, f)((k, i))}$. Taking inverses in

this equality, we get

$$\left(R^p_{\Delta(p)} f\right)((i,k)) = (\text{wing } f)((k,i)) = f\left(\underbrace{(k,i)}_{=\text{vrefl}(i,k)}\right)$$
$$= f(\text{vrefl}(i,k)) = (f \circ \text{vrefl})((i,k)).$$

Now we have shown this for **every** $(i,k) \in \Delta(p)$, hence that $R^p_{\Delta(p)} f = f \circ \text{vrefl}$. Next recall that $f = (a_0, a_1, \ldots, a_p) \flat g$. Hence,

$$R^p_{\Delta(p)} f = R^p_{\Delta(p)}((a_0, a_1, \ldots, a_p) \flat g) = (a_0, a_1, \ldots, a_p) \flat \left(R^p_{\Delta(p)} g\right) \qquad (36)$$

by Corollary 21. On the other hand, $f = (a_0, a_1, \ldots, a_p) \flat g$ yields

$$f \circ \text{vrefl} = ((a_0, a_1, \ldots, a_p) \flat g) \circ \text{vrefl} = (a_0, a_1, \ldots, a_p) \flat (g \circ \text{vrefl}) \qquad (37)$$

(this is easy to check directly using the definitions of $\flat$ and vrefl, since every $v \in \widehat{\Delta(p)}$ satisfies $\deg(\text{vrefl } v) = \deg v$). In light of (36) and (37), the equality $R^p_{\Delta(p)} f = f \circ$ vrefl becomes $(a_0, a_1, \ldots, a_p) \flat \left(R^p_{\Delta(p)} g\right) = (a_0, a_1, \ldots, a_p) \flat (g \circ \text{vrefl})$. We can cancel the "$(a_0, a_1, \ldots, a_p) \flat$" from both sides of this equation (since all a_i are nonzero), and thus obtain $R^p_{\Delta(p)} g = g \circ \text{vrefl}$. This proves Theorem 65 for $\Delta(p)$.

It is now straightforward to obtain the results for $\nabla(p)$ using the poset antiautomorphism hrefl of Rect (p,p) defined in Remark 63, which restricts to a poset antiisomorphism hrefl : $\nabla(p) \to \Delta(p)$, that is, to a poset isomorphism hrefl : $\nabla(p) \to (\Delta(p))^{\text{op}}$. Details appear in [GrRo14b]. $\qquad \square$

The proof of Corollary 66 is now a simple exercise (or can be looked up in [GrRo14b]).

11 The quarter-triangles

We have now studied the order of birational rowmotion on all four triangles (two of which are isomorphic as posets) which are obtained by cutting the rectangle Rect (p,p) along one of its diagonals. But we can also cut Rect (p,p) along **both** diagonals into four smaller triangles. These are isomorphic in pairs, and we will analyze them now. The following definition is an analogue of Definition 60 but using Tria (p) instead of Rect (p,p):

**Definition 70.** Let p be a positive integer. Define three subsets NEtri (p), Eqtri (p) and SEtri (p) of Tria (p) by

$$\text{NEtri}(p) = \{(i,k) \in \text{Tria}(p) \mid i + k > p + 1\};$$
$$\text{Eqtri}(p) = \{(i,k) \in \text{Tria}(p) \mid i + k = p + 1\};$$
$$\text{SEtri}(p) = \{(i,k) \in \text{Tria}(p) \mid i + k < p + 1\}.$$

These subsets $\mathrm{NEtri}\,(p)$, $\mathrm{Eqtri}\,(p)$ and $\mathrm{SEtri}\,(p)$ inherit a poset structure from $\mathrm{Tria}\,(p)$. In the following, we will consider $\mathrm{NEtri}\,(p)$, $\mathrm{Eqtri}\,(p)$ and $\mathrm{SEtri}\,(p)$ as posets using this structure.

Clearly, $\mathrm{Eqtri}\,(p)$ is an antichain. The posets $\mathrm{NEtri}\,(p)$ and $\mathrm{SEtri}\,(p)$ are $(p-1)$-graded posets having the form of right-angled triangles.

Example 71. Here is the Hasse diagram of the poset $\mathrm{Tria}\,(4)$, where the elements belonging to $\mathrm{NEtri}\,(4)$ have been underlined and the elements belonging to $\mathrm{Eqtri}\,(4)$ have been boxed:
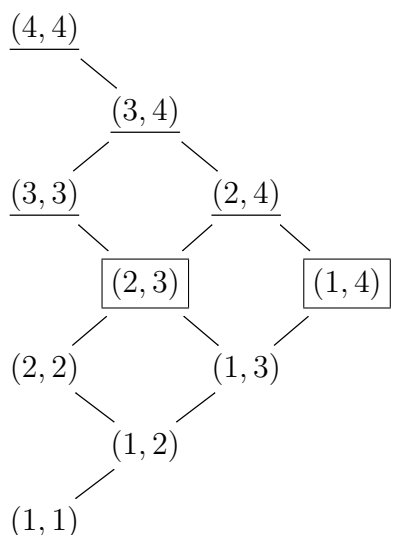
Next we display the Hasse diagrams of the poset $\mathrm{NEtri}\,(4)$ (on the *left*) and $\mathrm{SEtri}\,(4)$ (on the *right*):

Remark 72. Let p be an **even** positive integer. The poset $\mathrm{NEtri}\,(p)$ is isomorphic to the poset $\Phi^+\left(B_{p/2}\right)$ of [StWi11, §3.2]. (For odd p, the poset $\mathrm{NEtri}\,(p)$ does not seem to appear in [StWi11, §3.2].)

The following conjecture has been verified using Sage for small values of p:

Conjecture 73. Let p be an integer > 1. Then, $\mathrm{ord}\left(R_{\mathrm{SEtri}(p)}\right) = p$ and $\mathrm{ord}\left(R_{\mathrm{NEtri}(p)}\right) = p$.

In the case when p is odd, we can prove this conjecture using the same approach that was used to prove Theorem 58 (see [GrRo14b] for details):

Theorem 74. *Let p be an odd integer > 1. Then,* $\mathrm{ord}\left(R_{\mathrm{SEtri}(p)}\right) = p$ *and* $\mathrm{ord}\left(R_{\mathrm{NEtri}(p)}\right) = p$.

However, this reasoning fails in the even-p case (although the order of **classical** row-motion is again known to be p in the even-p case – see [StWi11, Conjecture 3.6]).

Nathan Williams suggested that the following generalization of Conjecture 73 might hold:

Conjecture 75. Let p be an integer > 1. Let $s \in \mathbb{N}$. Let $\mathrm{NEtri}'(p)$ be the subposet $\{(i,k) \in \mathrm{NEtri}(p) \mid k \geqslant s\}$ of $\mathrm{NEtri}(p)$. Then, $\mathrm{ord}\left(R_{\mathrm{NEtri}'(p)}\right) \mid p$.

This conjecture has been verified using Sage for all $p \leqslant 7$. Williams (based on a philosophy from his thesis [Will13]) suspects there could be a birational map between $\mathbb{K}^{\widehat{\mathrm{NEtri}'(p)}}$ and $\mathbb{K}^{\widehat{\mathrm{Rect}(s-1,p-s+1)}}$ which commutes with the respective birational rowmotion operators for all $s > \dfrac{p}{2}$; this, if shown, would obviously yield a proof of Conjecture 75. This already is an interesting question for classical rowmotion; a bijection between the antichains (and thus between the order ideals) of $\mathrm{NEtri}'(p)$ and those of $\mathrm{Rect}(s-1, p-s+1)$ was found by Stembridge [Stem86, Theorem 5.4], but does not commute with classical rowmotion.

12   Negative results

Generally, it is not true that if P is an n-graded poset, then $\mathrm{ord}(R_P)$ is necessarily finite. When $\mathrm{char}\,\mathbb{K} = 0$, the authors have proven the following[8]:

- If P is the poset $\{x_1, x_2, x_3, x_4, x_5\}$ with relations $x_1 < x_3$, $x_1 < x_4$, $x_1 < x_5$, $x_2 < x_4$ and $x_2 < x_5$ (this is a 5-element 2-graded poset), then $\mathrm{ord}(R_P) = \infty$.

- If P is the "chain-link fence" poset $/\backslash/\backslash/\backslash$ (that is, the subposet $\{(i,k) \in \mathrm{Rect}(4,4) \mid 5 \leqslant i + k \leqslant 6\}$ of $\mathrm{Rect}(4,4)$), then $\mathrm{ord}(R_P) = \infty$.

- If P is the Boolean lattice $[2] \times [2] \times [2]$, then $\mathrm{ord}(R_P) = \infty$.

The situation seems even more hopeless for non-graded posets.

13   The root system connection

A question naturally suggesting itself is: What is it that makes certain posets P have finite $\mathrm{ord}(R_P)$, while others have not? Can we characterize the former posets? It might be too optimistic to expect a full classification, given that our examples are already rather diverse (skeletal posets, rectangles, triangles, posets like that in Remark 33). As a first step (and inspired by the general forms of the Zamolodchikov conjecture), we were tempted to study posets arising from Dynkin diagrams. It appears that, unlike in the Zamolodchikov conjecture, the interesting cases are not those having P be a product of Dynkin diagrams, but those having P be a positive root poset of a root system, or a parabolic quotient

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[8]See the ancillary files of [GrRo14b] for an outline of the (rather technical) proofs.

thereof. The idea is not new, as it was already conjectured by Panyushev [Pan08, Conjecture 2.1] and proven by Armstrong, Stump and Thomas [AST11, Theorem 1.2] that if W is a finite Weyl group with Coxeter number h, then classical rowmotion on the set $J(\Phi^+(W))$ (where $\Phi^+(W)$ is the poset of positive roots of W) has order h or $2h$ (along with a few more properties, akin to our "reciprocity" statements)[9].

In the case of birational rowmotion, the situation is less simple. Specifically, the following can be said about positive root posets of crystallographic root systems (as considered in [StWi11, §3.2])[10]:

- If $P = \Phi^+(A_n)$ for $n \geqslant 2$, then $\operatorname{ord}(R_P) = 2(n+1)$. This is just the assertion of Corollary 66. Note that for $n = 1$, the order $\operatorname{ord}(R_P)$ is 2 instead of $2(1+1) = 4$.

- If $P = \Phi^+(B_n)$ for $n \geqslant 1$, then Conjecture 73 claims that $\operatorname{ord}(R_P) = 2n$. Note that $\Phi^+(B_n) \cong \Phi^+(C_n)$.

- We have $\operatorname{ord}(R_P) = 2$ for $P = \Phi^+(D_2)$, and we have $\operatorname{ord}(R_P) = 8$ for $P = \Phi^+(D_3)$. However, $\operatorname{ord}(R_P) = \infty$ in the case when $P = \Phi^+(D_4)$. This should not come as a surprise, since $\Phi^+(D_4)$ has a property that none of the $\Phi^+(A_n)$ or $\Phi^+(B_n) \cong \Phi^+(C_n)$ have, namely an element covered by three other elements. On the other hand, the finite orders in the $\Phi^+(D_2)$ and $\Phi^+(D_3)$ cases can be explained by $\Phi^+(D_2) \cong \Phi^+(A_1 \times A_1) \cong$ (two-element antichain) and $\Phi^+(D_3) \cong \Phi^+(A_3)$.

Nathan Williams has suggested that the behavior of $\Phi^+(A_n)$ and $\Phi^+(B_n) \cong \Phi^+(C_n)$ to have finite orders of R_P could generalize to the "positive root posets" of the other "coincidental types" H_3 and $I_2(m)$ (see, for example, Table 2.2 in [Will13]). And indeed, computations in Sage have established that $\operatorname{ord}(R_P) = 10$ for $P = \Phi^+(H_3)$, and we also have $\operatorname{ord}(R_P) = \operatorname{lcm}(2, m)$ for $P = \Phi^+(I_2(m))$ (this is a very easy consequence of Lemma 26).

It seems that minuscule heaps, as considered e.g. in [RuSh12, §6], also lead to small $\operatorname{ord}(R_P)$ values. Namely:

- The heap $P_{w_0^J}$ of type E_6 in [RuSh12, Figure 8 (b)] satisfies $\operatorname{ord}(R_P) = 12$.

- The heap $P_{w_0^J}$ of type E_7 in [RuSh12, Figure 9 (b)] seems to satisfy $\operatorname{ord}(R_P) = 18$ (this was verified on numerical examples, as the poset is too large for efficient general computations).

(These two posets also appear as posets corresponding to the "Cayley plane" and the "Freudenthal variety" in [ThoYo07, p. 2].)

Various other families of posets related to root systems (minuscule posets, d-complete posets, rc-posets, alternating sign matrix posets) remain to be studied.

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[9]Neither [Pan08] nor [AST11] work directly with order ideals and rowmotion, but instead they study antichains of the poset $\Phi^+(W)$ (which are called "nonnesting partitions" in [AST11]) and an operation on these antichains called Panyushev complementation. There is, however, a simple bijection between the set of antichains of a poset P and the set $J(P)$, and the conjugate of Panyushev complementation with respect to this bijection is precisely classical rowmotion.

[10]We refer to [StWi11, Definition 3.4] for notations.

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