A remark on the tournament game

Dennis Clemens *
Institut für Mathematik
Technische Universität Hamburg-Harburg
21073 Hamburg, Germany
dennis.clemens@tuhh.de

Mirjana Mikalački †
Department of Mathematics and Informatics
Faculty of Sciences, University of Novi Sad
Novi Sad, Serbia
mirjana.mikalacki@dmi.uns.ac.rs

Submitted: Mar 29, 2015; Accepted: Aug 20, 2015; Published: Sep 4, 2015
Mathematics Subject Classifications: 05C20, 05C57, 91A24

Abstract

We study the Maker-Breaker tournament game played on the edge set of a given graph G. Two players, Maker and Breaker, claim unclaimed edges of G in turns, while Maker additionally assigns orientations to the edges that she claims. If by the end of the game Maker claims all the edges of a pre-defined goal tournament, she wins the game. Given a tournament T_k on k vertices, we determine the threshold bias for the $(1 : b)$ T_k-tournament game on K_n. We also look at the $(1 : 1)$ T_k-tournament game played on the edge set of a random graph $\mathcal{G}_{n,p}$ and determine the threshold probability for Maker's win. We compare these games with the clique game and discuss whether a random graph intuition is satisfied.

Keywords: positional games; Maker-Breaker; tournament

1 Introduction

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of the subsets of X. Let a and b be two positive integers. In the $(a : b)$ Maker-Breaker positional game $(X, \mathcal{F})$ two players, Maker and Breaker, take turns in claiming previously unclaimed elements of X, with Maker going first. In each turn, Maker claims a unclaimed elements and then Breaker claims b unclaimed elements of X. The game is played until all the elements of X are claimed. Maker wins the game if she claims all the elements of some $F \subseteq \mathcal{F}$ by the end of the game.

*Research was initiated at the FU Berlin, when the author was supported by DFG, project SZ 261/1-1.
†Research partly supported by Ministry of Science and Technological Development, Republic of Serbia, and Provincial Secretariat for Science, Province of Vojvodina.

Otherwise, Breaker wins. If Maker can win against any strategy of Breaker then the game is said to be a *Maker's win*. Otherwise, the game is said to be a *Breaker's win*. The set X is referred to as the *board* of the game, while the elements of $\mathcal{F}$ are referred to as the *winning sets*. The values a and b are called *biases* of Maker, and Breaker, respectively. The most basic case of these games are *unbiased games*, where $a = b = 1$.

In this paper, we focus on Maker-Breaker *graph games*, i.e., games where the board is the edge set of a given graph G. In these games Maker's aim is to create a graph consisting only of edges claimed by her that contains some predefined graph theoretic structure. For example, in the k*-clique game* (or sometimes abbreviated just as *clique game* when the value of k is not crucial), Maker's goal is to create a graph that contains a clique of order at least k. We denote this game by $(E(G), \mathcal{K}_k)$.

Here, we study a variant of the clique game - the T*-tournament game* $(E(G), \mathcal{K}_T)$. In the T-tournament game, introduced by Beck in [2], the goal graph is a tournament T, i.e. a complete graph where each edge has an *orientation*. That is, in T we have exactly one arc between each pair of vertices. Before the game starts, the tournament T is fixed. Then Maker and Breaker in turns claim previously unclaimed edges of G. When Maker claims an edge of G, she immediately chooses and fixes one of the two possible orientations for that edge. Contrary, Breaker never orients his edges and, in particular, when Maker oriented an edge between two vertices x and y, Breaker cannot claim the edge xy anymore. Finally, if Maker's graph contains a copy of the given tournament T by the end of the game, Maker wins. Otherwise, Breaker does.

Games on $\mathbf{K_n}$. Very well-studied graph games are the ones where $G = K_n$ is the complete graph on n vertices. Erdős and Selfridge [6] initiated the study of the largest value of k, $k_c = k_c(n)$, such that Maker can win the k_c-clique game on K_n and they were able to prove that $k_c \leqslant 2(1 - o(1)) \log_2 n$. Indeed, it turns out that in $(1 : 1)$ Maker-Breaker clique game on K_n, Maker has a strategy to occupy a clique of size $(2 - o(1)) \log_2(n)$, as shown by Beck [2], and therefore $k_c = 2(1 - o(1)) \log_2 n$ holds. The most interesting fact about this result is that it shows an intriguing relation between games and random graphs here, referred to as the *random graph intuition* or *probabilistic intuition*. To be precise, if both players played randomly throughout the game, then Maker's graph would be distributed as a random graph with n vertices and $\lceil \frac{1}{2}\binom{n}{2} \rceil$ edges, which is well known to have clique size $(2 - o(1)) \log_2(n)$ with high probability, see e.g. [1]. That is, for most values of k, a randomly played k-clique game on K_n typically has the same winner as the deterministic game played by two intelligent players.

For the tournament game we can ask the same question, as initiated by Beck [2]. Motivated by the study of a randomly played T-tournament game, Beck conjectured that the largest value of k, $k_t = k_t(n)$, for which Maker can win in the T-tournament game, for any tournament T on (at most) k vertices, is of size $(1 - o(1)) \log_2 n$. However, as the first author together with Gebauer and Liebenau [5] showed, the truth is twice as large as the conjectured value, i.e., $k_t = (2 - o(1)) \log_2 n$. This, in particular, tells us two things. Opposite to the clique game, the tournament game does not satisfy the random graph intuition mentioned above. Secondly, since the two values, k_c and k_t, are very close to each other, it does not make a big difference for Maker whether she needs to build a graph with or without orientations in an unbiased game on K_n.

In the following, we want to find out whether we have similar observations in case we fix k to be a constant, while changing either the bias of Breaker or the board of the game. We start with biased games, in order to give more power to Breaker. Chvátal and Erdős [4] observed that Maker-Breaker games are *bias monotone*, meaning that if the $(1 : b)$ game $(X, \mathcal{F})$ is a Breaker's win, then the $(1 : b+1)$ game is also a Breaker's win. Having this in mind, it thus becomes interesting to find the unique *threshold bias* $b_{\mathcal{F}}(n) = b_{\mathcal{F}}$, which is the largest non-negative integer such that for every $b \leqslant b_{\mathcal{F}}$ the $(1 : b)$ game is a Maker's win. For the k-clique game on K_n, Bednarska and Łuczak [3] showed that the threshold bias is $b_{\mathcal{K}_k} = \Theta(n^{\frac{2}{k+1}})$. Naturally, one may wonder what happens with the tournament game, and whether in this case orientations of the edges make things more complicated for Maker. We show that, for every tournament T of order k, the threshold bias of the T-tournament game $(E(K_n), \mathcal{K}_T)$ is of the same order as in the k-clique game.

Proposition 1. *Let T be a tournament on $k \geqslant 3$ vertices, then the threshold bias for the T-tournament game on K_n is $b_{\mathcal{K}_T} = \Theta(n^{\frac{2}{k+1}})$.*

Games on random boards. Another way to give Breaker more power in positional games is to play unbiased graph games on a random graph, as introduced by Stojaković and Szabó [9]. The idea behind this approach is to make the board sparser before the game starts by randomly eliminating edges, so that some of the winning sets no longer exist. We look at the random graph model $\mathcal{G}_{n,p}$, which is obtained from the complete graph on n vertices by removing each edge independently with probability $1 - p$.

Now, if an unbiased game $(E(K_n), \mathcal{F})$ is a Maker's win, then we are curious about finding the *threshold probability* $p_{\mathcal{F}}$ such that for $p = \omega(p_{\mathcal{F}})$ the game $(E(\mathcal{G}_{n,p}), \mathcal{F})$ is a Maker's win asymptotically almost surely (i.e. with probability tending to 1 as n tends to infinity and abbreviated *a.a.s.* in the rest of the paper), and for $p = o(p_{\mathcal{F}})$, the game $(E(\mathcal{G}_{n,p}), \mathcal{F})$ is a.a.s. a Breaker's win.

When the k-clique game is played on $\mathcal{G}_{n,p}$, Stojaković and Szabó [9] showed that for $k = 3$, in the *triangle* game, $p_{\mathcal{K}_3} = n^{-\frac{5}{9}}$ and for $k \geqslant 4$, it holds that $n^{-\frac{2}{k+1}-\varepsilon} \leqslant p_{\mathcal{K}_k} \leqslant n^{-\frac{2}{k+1}}$. Müller and Stojaković [8] recently proved that for all $k \geqslant 4$ the threshold probability is indeed $p_{\mathcal{K}_k} = n^{-\frac{2}{k+1}}$. This again underlines an intriguing relation between games and random graphs, again referred to as the *probabilistic intuition*. Indeed, what we can observe here in case $k \geqslant 4$ (and also holds for several other natural graph games) is that the threshold probability for Maker's win in the $(1 : 1)$ game $(E(\mathcal{G}_{n,p}), \mathcal{F})$ is of the same order of magnitude as the inverse of the threshold bias $b_{\mathcal{F}}$ in the $(1 : b)$ game $(E(K_n), \mathcal{F})$. The triangle game is the only exception in this regard, as here Maker a.a.s. can win also for probabilities below the so-called critical probability $1/b_{\mathcal{K}_3}$.

We show that the tournament game behaves similarly to the clique game when played on $\mathcal{G}_{n,p}$. So, even when played on a sparse graph $\mathcal{G}_{n,p}$, creating a graph with oriented edges is not much more difficult for Maker than creating a graph without oriented edges. For the tournaments on k vertices, $k \geqslant 4$, we show the following, which also supports the probabilistic intuition.

Proposition 2. *Let T be a tournament on $k \geqslant 4$ vertices, then the threshold probability for winning the T-tournament game on $\mathcal{G}_{n,p}$ is $n^{-\frac{2}{k+1}}$.*

So again, for $k \geqslant 4$, the outcome of the game does not depend much on the choice of the tournament T on k vertices, i.e., on the way the edges of the goal tournament are oriented. However, our next theorem states that the tournament on three vertices behaves differently. In case T is the acyclic triangle T_A, we obtain the same threshold probability as in the triangle game on $\mathcal{G}_{n,p}$. But, in case T is the cyclic triangle T_C, the threshold probability is closer to the critical probability $1/b_{\mathcal{K}_3}$.

Theorem 3. *The threshold probability for winning the unbiased T_A-tournament game on $\mathcal{G}_{n,p}$ is $p_{\mathcal{K}_{T_A}} = n^{-\frac{5}{9}}$, while for the unbiased T_C-tournament game this threshold probability is $p_{\mathcal{K}_{T_C}} = n^{-\frac{8}{15}}$.*

Notation and terminology. Our graph-theoretic notation is standard and follows that of [10]. In particular, we use the following. For a graph G, $V(G)$ and $E(G)$ denote its sets of vertices and edges respectively, $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For disjoint sets $A, B \subseteq V(G)$, let $E(A, B)$ denote the set of edges of G with one endpoint in A and one endpoint in B. Given two vertices, x and y, an undirected edge is denoted by xy, while (x, y) is a directed edge with orientation from vertex x towards vertex y. If an edge is unclaimed by any of the players we call it *free*. For a vertex $x \in V(G)$, $N(x) = \{u \in V(G) : ux \in E(G)\}$ denotes the set of neighbours of the vertex x in G. We let $d(x) = |N(x)|$ denote the degree of vertex x in graph G. The minimum and maximum degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. The density of a graph G is defined as $d(G) = \frac{e(G)}{v(G)}$, while its *maximum density* is $m(G) = \max_{H \subseteq G} d(H)$.

Let $n, k \in \mathbb{N}$ be positive integers. Then with $T_{n,k}$ we denote the *Turán graph* with n vertices and k vertex classes. That is, its vertex set $V(T_{n,k}) = [n]$ comes with a partition $V(T_{n,k}) = V_1 \cup \ldots \cup V_k$ such that $\Big||V_i| - |V_j|\Big| \leqslant 1$ for all $1 \leqslant i < j \leqslant k$, and such that its edge set is $E(T_{n,k}) = \{vw \mid v \in V_i,\ w \in V_j,\ 1 \leqslant i < j \leqslant k\}$. Moreover, let a graph G be given together with a partition $P = V_1 \cup V_2 \cup \ldots \cup V_k$ of its vertex set. For a given graph H on at most k vertices we then say that a subgraph $G' \subseteq G$ is a *good copy* of H in G with respect to (w.r.t.) the partition P, if $G' \cong H$ and $|V(G') \cap V_i| \leqslant 1$ for every $i \in [k]$.

Let $p \in [0, 1]$ and moreover let $M \in [e(T_{n,r})]$. Then with $\mathcal{G}(T_{n,k}, p)$ we denote the random graph model obtained from $T_{n,k}$ by deleting each edge of $T_{n,k}$ independently with probability $1 - p$. That is, $\mathcal{G}(T_{n,k}, p)$ is the probability space of all subgraphs G of $T_{n,k}$, where the probability for a subgraph to be chosen is $p^{e(G)}(1 - p)^{e(T_{n,k}) - e(G)}$. Similarly, with $\mathcal{G}(T_{n,k}, M)$ we denote the probability space of all subgraphs G of $T_{n,k}$ with M edges, together with the uniform distribution. $Bin(n, p)$ denotes the binomial distribution, i.e.

the distribution of the number of successes among n independent experiments, where in each experiment we have success with probability p. Moreover, let us write $X \sim Bin(n,p)$ if X is a random variable with distribution $Bin(n,p)$.

Finally, P_k denotes the *path* on k vertices, i.e. $V(P) = \{v_1, \ldots, v_k\}$ and $E(P_k) = \{v_i v_{i+1} : 1 \leqslant i \leqslant k-1\}$, while C_k denotes the *cycle* on k vertices, which is obtained form P_k by adding the edge $v_k v_1$. $W_k = (V,E)$ is called a *k-wheel*, if it is obtained from the cycle C_k by adding one further vertex z which is made adjacent to every vertex of C_k. The special vertex z is called the *center* of W_k.

Throughout the paper ln stands for the natural logarithm.

Organization of the paper. The rest of the paper is organized as follows. At first we collect some useful results in the Preliminaries. In Section 3 we prove Proposition 1 and Proposition 2. Finally, in Section 4 we prove Theorem 3.

2 Preliminaries

As indicated earlier, we will refer to the following result due to Bednarska and Łuczak [3].

Theorem 4 (Corollary from Theorem 1 in [3])**.** *Let $k \geqslant 3$. Then, there is a constant $c = c(k) > 0$ such that for large enough n and every $b \geqslant cn^{\frac{2}{k+1}}$, Breaker has a winning strategy in the clique game $(E(K_n), \mathcal{K}_k)$.*

The following estimate is usually referred to as a Chernoff inequality [7].

Lemma 5 (Theorem 2.1 in [7])**.** *Let $X \sim Bin(n,p)$ and $\lambda = \mathbf{E}(X) = np$. Then for $t \geqslant 0$, it holds that $Pr(X \geqslant \mathbf{E}(X) + t) \leqslant \exp\Big(-\frac{t^2}{2\lambda} + \frac{t^3}{6\lambda^2}\Big)$.*

As indicated above, we will consider the random graph models $\mathcal{G}(T_{n,k}, p)$ and $\mathcal{G}(T_{n,k}, M)$. For this, we will make use of some general results about random sets.

Following [7], let Γ be a set of size $N \in \mathbb{N}$. For $p \in [0,1]$, we let Γ_p denote the probability space of all subsets $A \subseteq \Gamma$, where the probability of choosing A is $p^{|A|}(1-p)^{|\Gamma \setminus A|}$. Moreover, for $M \in [N]$, we let Γ_M denote the probability space of all subsets $A \subseteq \Gamma$ of size M, together with the uniform distribution. In case we choose a random set A according to the model Γ_p, we shortly write $A \sim \Gamma_p$. Similarly, we write $A \sim \Gamma_M$, when A is chosen according to the uniform model Γ_M.

One important fact about the two models above is that in many cases they are closely related to each other when $p \sim \frac{M}{N}$; see Section 1.4 in [7]. In particular, we will make use of the following two statements, which help us to transfer results from one model to the other.

Lemma 6 (Pittel's Inequality, Equation (1.6) in [7])**.** *Let Γ be a set of size N, let $M \in [N]$, and $p = \frac{M}{N} \in [0,1]$. Let $\mathcal{P}$ be a family of subsets of Γ. Moreover, let $H_p \sim \Gamma_p$ and*

$H_M \sim \Gamma_M$, *then*

$$Pr(H_M \notin \mathcal{P}) \leqslant 3\sqrt{M} \cdot Pr(H_p \notin \mathcal{P}).$$

Lemma 7 (Corollary 1.16 (iii) in [7]). *Let* Γ *be a set of size* N *and let* $M \in [N]$. *Let* $\delta > 0$ *be such that* $0 \leqslant (1+\delta)\frac{M}{N} \leqslant 1$, *and let* $p = (1+\delta)\frac{M}{N}$. *Let* $\mathcal{P}$ *be a family of subsets of* Γ. *Moreover, let* $H_p \sim \Gamma_p$ *and* $H_M \sim \Gamma_M$, *then*

$$Pr(H_M \in \mathcal{P}) \to 1 \text{ implies } Pr(H_p \in \mathcal{P}) \to 1.$$

Later we want to know whether a certain random graph contains a copy of a fixed graph with high probability. In this regard, we make use of the following two theorems.

Theorem 8 (Theorem 2.18 (ii) in [7]). *Let* Γ *be a set,* $p \in [0,1]$ *and let* $H \sim \Gamma_p$. *Let* $\mathcal{S}$ *be a family of subsets of* Γ. *Moreover, for every* $A \in \mathcal{S}$ *let* I_A *be the indicator variable which is 1 if* $A \subseteq H$, *and* 0 *otherwise. Finally, let* $X = \sum_{A \in S} I_A$ *be the random variable counting the number of elements of* $\mathcal{S}$ *that are contained in* H. *Then*

$$Pr(X = 0) \leqslant \exp\Big(- \frac{\mathbf{E}(X)^2}{\sum_{A \in \mathcal{S}} \sum_{\substack{B \in \mathcal{S} \\ A \cap B \neq \varnothing}} \mathbf{E}(I_A I_B)} \Big).$$

Theorem 9 (Theorem 3.4 in [7]). *Let* H *be a graph, and let* X_H *denote random variable counting the number of copies of* H *in a random graph* $G \sim \mathcal{G}_{n,p}$. *Then, as* n *tends to infinity, we have*

$$Pr(X_H > 0) \to \begin{cases} 0 & \text{if } p \ll n^{-\frac{1}{m(H)}} \\ 1 & \text{if } p \gg n^{-\frac{1}{m(H)}}. \end{cases}$$

3 Most tournaments behave like cliques

The main idea for the proof of the propositions is as follows: Let G be the graph on which the game is to be played. Let T be the goal tournament with vertices $v_1, \ldots, v_k$. Then, before the game starts Maker splits the vertex set of G into k parts $V_1, \ldots, V_k$ with $\Big||V_i| - |V_j|\Big| \leqslant 1$ for all $1 \leqslant i < j \leqslant k$, and she identifies each class V_i with the vertex v_i according to the following rule: Whenever Maker claims an edge between some classes V_i and V_j, she always chooses the orientation of this edge according to the orientation of the edge $v_i v_j$ in T. Because of this identification, it then remains to show that Maker has a strategy for the usual Maker-Breaker game on G to occupy a good copy of K_k in K_n w.r.t. the partition $P = V_1 \cup \ldots \cup V_k$.

In order to show that Maker has such a strategy for this game, we will make use of results from [7], and follow the proof ideas from [3, 9]. As most parts are proven analogously to results in the aforementioned publications, we rather keep our argument short and, whenever possible, we refer back to the known results. At first, analogously to Theorem 3.9 in [7], we bound the probability that a random graph $G \sim \mathcal{G}(T_{n,k}, p)$ does not contain a copy of K_k.

Claim 10. *Let $k \geqslant 3$ be a positive integer. Then there is a constant $c_1 = c_1(k) > 0$ such that for every large enough n the following is true: If $n^{-\frac{2}{k+1}} \leqslant p \leqslant 4n^{-\frac{2}{k+1}}$ and if X denotes the random variable counting the number of copies of K_k in a random graph $G \sim \mathcal{G}(T_{n,k}, p)$, then $Pr(X = 0) \leqslant \exp(-c_1 n^2 p)$.*

Proof Let $G \sim \mathcal{G}(T_{n,k}, p)$. Let $\mathcal{S}$ be the family of copies of K_k in $T_{n,k}$. For each such copy $C_i \in \mathcal{S}$ let I_{C_i} be the indicator variable which is 1 if and only if $C_i \subseteq G$. By Theorem 8,

$$Pr(X = 0) \leqslant \exp\Big(- \frac{(\mathbb{E}(X))^2}{\sum_{C_1} \sum_{C_2:\ E(C_1) \cap E(C_2) \neq \varnothing} \mathbb{E}(I_{C_1} I_{C_2})} \Big).$$

The denominator in the above expression can be bounded from above by

$$\begin{aligned}
\sum_{t=2}^{k} \sum_{C_1 \in \mathcal{S}} \sum_{\substack{C_2 \in \mathcal{S}: \\ C_1 \cap C_2 \cong K_t}} p^{2\binom{k}{2} - \binom{t}{2}} &\leqslant \sum_{t=2}^{k} n^{2k-t} p^{2\binom{k}{2} - \binom{t}{2}} \\
&= \Theta(\mathbb{E}(X)^2) \cdot \sum_{t=2}^{k} n^{-t} p^{-\binom{t}{2}} \\
&= \Theta(\mathbb{E}(X)^2 \cdot n^{-2} p^{-1}) \sum_{t=2}^{k} \left(n^{-1} p^{-\frac{t+1}{2}} \right)^{t-2} \\
&= \Theta(\mathbb{E}(X)^2 \cdot n^{-2} p^{-1}),
\end{aligned}$$

where in the last equality we use that $p = \Theta(n^{-\frac{2}{k+1}})$. Thus, the claim follows. □

Corollary 11. *Let $k \geqslant 3$ be a positive integer. Then there is a constant $c_1' = c_1'(k) > 0$ such that for every large enough n the following is true: If $M = \lfloor n^{2-\frac{2}{k+1}} \rfloor$ and if X' denotes the random variable counting the number of copies of K_k in a random graph $G \sim \mathcal{G}(T_{n,k}, M)$, then $Pr(X' = 0) \leqslant \exp(-c_1' M)$.*

Proof Set $p = \frac{M}{e(T_{n,k})}$ and observe that $n^{-\frac{2}{k+1}} \leqslant p \leqslant 4n^{-\frac{2}{k+1}}$. The statement now follows by Claim 10 and Lemma 6. □

Corollary 12. *Let $k \geqslant 3$ be a positive integer. Then there is a constant $\delta = \delta(k) > 0$ such that for every large enough n and $M = 2\lfloor n^{2-\frac{2}{k+1}} \rfloor$, a random graph $G \sim \mathcal{G}(T_{n,k}, M)$ satisfies the following property a.a.s.: Every subgraph of G with at least $\lfloor (1-\delta)M \rfloor$ edges contains a copy of K_k.*

Proof We proceed analogously to [3]. Let $\delta > 0$ such that $\delta - \delta \log(\delta) < c_1'/3$, with c_1' from Corollary 11, and count the number of pairs (H, H') where H is a subgraph of $T_{n,k}$ with M edges and where $H' \subseteq H$ is a subgraph with $\lfloor (1-\delta)M \rfloor$ edges that does not contain a copy of K_k. Then using Corollary 11 (and simplifying the notation slightly by ignoring floor signs) we obtain that the number of such pairs is at most

$$\exp(-\frac{c_1'M}{2})\binom{e(T_{n,r})}{(1-\delta)M}\binom{e(T_{n,r})-(1-\delta)M}{\delta M}$$

$$\begin{aligned}
&\leqslant \exp(-\frac{c_1'M}{2})\binom{M}{\delta M}\binom{e(T_{n,r})}{M}\\
&\leqslant \exp\Big(-\frac{c_1'M}{2}+\delta M(1-\log(\delta))\Big)\binom{e(T_{n,r})}{M}\\
&= o(1)\binom{e(T_{n,r})}{M}.
\end{aligned}$$

□

Using this last corollary, we can start proving the existence of Maker strategies. The following claim is an analogue statement to Theorem 19 in [9], and thus its proof is analogous to [9].

Claim 13. *Let $k \geqslant 3$ and n be positive integers. Then there is a constant $c_2 = c_2(k) > 0$ such that for every $M \geqslant c_2^{-1}n^{2-\frac{2}{k+1}}$, every $1 \leqslant b \leqslant c_2 M n^{-2+\frac{2}{k+1}}$, for a random graph $G \sim \mathcal{G}(T_{n,k}, M)$ the following a.a.s. holds: Maker has a strategy to occupy a copy of K_k in the $(1:b)$ Maker-Breaker game on G.*

Proof Choose $\delta = \delta(G)$ according to Corollary 12 and let $c_2 = \delta/10$. Maker's strategy is as follows: in each of her moves she chooses an edge from G uniformly at random among all edges from G that have not been claimed so far by herself. If she chooses an edge that is not claimed by Breaker so far, she claims this edge. Otherwise, Maker declares her move as a failure and skips it. Similar to [9], we consider the first $M' := 2\lfloor n^{2-\frac{2}{k+1}}\rfloor \leqslant \frac{\delta}{2}\cdot\frac{1}{b+1}M$ rounds of the game. As only a $\frac{\delta}{2}$-fraction of all edges are claimed in these rounds, the probability for a failure is at most $\frac{\delta}{2}$ in each round. So, the number of failures can be "upper bounded" by a binomial random variable $X \sim Bin(M', \frac{\delta}{2})$, which by Chernoff's inequality (Lemma 5) satisfies $Pr(X \geqslant 2\mathbf{E}(X)) \leqslant \exp(-\frac{\mathbf{E}(X)}{3}) = o(1)$. That is, the number of failures will be at most $\delta M'$ a.a.s. Thus, Maker a.a.s. creates a graph $H \setminus R$ with $H \sim \mathcal{G}(T_{n,k}, M')$ and $e(R) \leqslant \delta M'$, against any strategy of Breaker, which by Corollary 12 a.a.s. contains a copy of K_k. Thus, a.a.s. Breaker cannot have a strategy to prevent copies of K_k, and as either Maker or Breaker needs to have a winning strategy, the claim follows. □

Corollary 14. *Let $k \geqslant 3$ and n be positive integers Then there is a constant $c_3 = c_3(k) > 0$ such that for every $p \geqslant c_3 n^{-\frac{2}{k+1}}$ and $G \sim \mathcal{G}(T_{n,k}, p)$ the following a.a.s. holds: Maker has a strategy to occupy a copy of K_k in the unbiased Maker-Breaker game on G.*

Proof The statement follows immediately from Corollary 13 and Lemma 7, where we choose $\mathcal{P}$ to be the family of all graphs $G \subseteq T_{n,k}$ for which Maker has a strategy to occupy a copy of K_k in the unbiased Maker-Breaker game on $E(G)$. □

Finally, we can prove the two propositions.

Proof of Proposition 1. Let T be the tournament, with $k \geqslant 3$ vertices, of which Maker aims to create a copy on K_n. By Theorem 4, we know that there is a constant $c > 0$ such that for large enough n and for every $b \geqslant cn^{\frac{2}{k+1}}$, Breaker has a strategy to prevent cliques of order k. Using this strategy, Breaker wins the T-tournament game on K_n. Now, let $c_2 = c_2(k)$ be given according to Claim 13, and let $M = e(T_{n,k})$, $b = 0.25c_2n^{\frac{2}{k+1}}$. We now apply Claim 13, in which case we note that the game is played on $T_{n,k}$ rather than on a random graph. The claim implies that Maker has a strategy to occupy a copy of K_k in the $(1 : b)$ Maker-Breaker game on $T_{n,k}$, which at the same time tells us that she has a strategy to occupy a good copy of K_k in the game on K_n w.r.t. the partition $P = V_1 \cup \ldots V_k$. But, as we argued earlier, this also gives Maker a strategy for the $(1 : b)$ T-tournament game on K_n. □

Proof of Proposition 2. Let T be the tournament, with $k \geqslant 4$ vertices, of which Maker aims to create a copy in an unbiased game on $G \sim \mathcal{G}_{n,p}$. By Theorem 1.1 in [8], we know that there is a constant $c > 0$ such that for $p \leqslant cn^{-\frac{2}{k+1}}$, Breaker a.a.s. has a strategy to block cliques of order k in the unbiased Maker-Breaker game on G, which again gives a winning strategy for Breaker in the T-tournament game on G. Now, let $p \geqslant c_3n^{-\frac{2}{k+1}}$, with $c_3 = c_3(k)$ from Corollary 14. Before sampling the random graph $G \sim \mathcal{G}_{n,p}$ fix a partition $P = V_1 \cup \ldots \cup V_k = [n]$ as before. Then, after sampling $G \sim \mathcal{G}_{n,p}$, we know that the subgraph induced by those edges which intersect two different parts V_i and V_j is sampled like a random graph $F \sim \mathcal{G}(T_{n,k}, p)$. According to Corollary 14, Maker a.a.s. has a strategy to occupy a copy of K_k in $F \subseteq G$, and therefore it follows, analogously to the previous proof, that Maker a.a.s. has a strategy to create a copy T in the unbiased tournament game on G. □

4 The triangle case

In the following we prove **Theorem 3**.

For the acyclic triangle T_A, the result can be obtained from [9] as follows: For $p \ll n^{-\frac{5}{9}}$ Breaker a.a.s. has a strategy to prevent triangles in the unbiased Maker-Breaker game on $G \sim \mathcal{G}_{n,p}$. Applying such a strategy in the T_A-tournament game as Breaker obviously blocks acyclic triangles. For $p \gg n^{-\frac{5}{9}}$ a.a.s. Maker has a strategy to gain an undirected triangle in the unbiased Maker-Breaker game on $G \sim \mathcal{G}_{n,p}$. In the T_A-game, Maker now can proceed as follows. She fixes an arbitrary ordering $\{v_1, \ldots, v_n\}$ of $V(G)$ before the game starts. Then she applies the mentioned strategy of Maker for gaining an undirected triangle, where she always chooses orientations from vertices of smaller index to vertices of larger index. This way, every triangle claimed by her will be an acyclic triangle, and thus she wins.

Thus, from now on, we can restrict the problem to the discussion of the cyclic triangle T_C. To show that $n^{-\frac{8}{15}}$ is the threshold probability for the existence of a winning strategy for Maker in the T_C-tournament game on $G \sim \mathcal{G}_{n,p}$, we will study Maker's and Breaker's strategy separately.

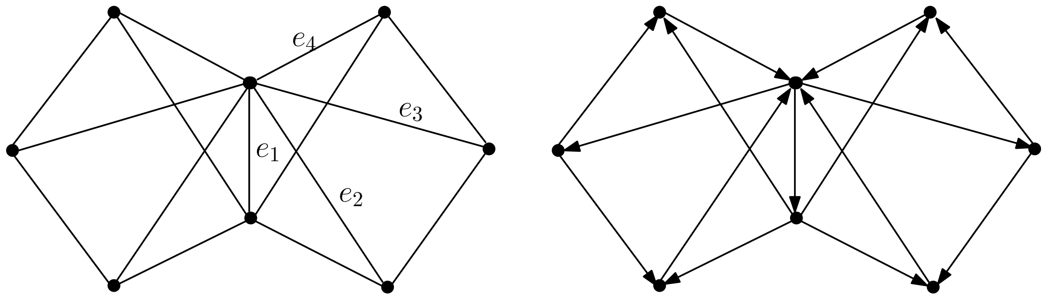

Figure 1: Graph H without and with orientation.

We start with **Maker's strategy**. Let $p \gg n^{-\frac{8}{15}}$. Then, by Theorem 9, a.a.s. $G \sim \mathcal{G}_{n,p}$ contains the graph H, presented in the left half of Figure 1, as $m(H) = \frac{15}{8}$. As indicated in the right half of the same figure, its edges can be oriented in such a way that each triangle has a cyclic orientation, and thus, it is enough to prove that Maker has a strategy to claim an undirected triangle in the unbiased Maker-Breaker game on H. Her strategy is as follows. At first she claims the edge e_1, as indicated in the figure. By symmetry, we can assume that afterwards Breaker claims an edge which is on the "left side" of e_1. Then in the next moves, as long as she cannot close a triangle, Maker claims the edges e_2, e_3 and e_4, always forcing Breaker to block an edge which could close a triangle, and Maker will surely be able to complete a triangle in the next round.

Now, let $p \ll n^{-\frac{8}{15}}$. We are going to show that a.a.s. there exists a **Breaker's strategy** which blocks copies of T_C, when playing on $G \sim \mathcal{G}_{n,p}$. We start with some preparations. Amongst others, we will consider *triangle collections*, as studied in [9].

Definition 15. Let $G = (V, E)$ be some graph without isolated vertices. Further, let $T(G) = (V_T, E_T)$ be the graph where $V_T = \{H \subseteq G : \; H \cong K_3\}$ is the set of all triangles in G, and $E_T = \{H_1H_2 : \; E(H_1) \cap E(H_2) \neq \varnothing\}$ is the (binary) relation on V_T of having a common edge. Then:

- G is called *very basic* if $T(G)$ is a subgraph of a copy of K_3^+ (triangle plus a pending edge), or a subgraph of a copy of P_k with $k \in \mathbb{N}$.
- G is called *basic* if there are distinct edges $e_1, e_2 \in E(G)$ such that $G - e_i$ is very basic for both $i \in \{1, 2\}$.
- G is a *triangle collection* if every edge of G is contained in some triangle and $T(G)$ is connected.

If G is a triangle collection we further call it a *bunch* (of triangles) if we can find triangles $F_1, \dots F_r \in V_T$ covering all edges of G with the property that $|V(F_i) \setminus \cup_{j<i} V(F_j)| = 1$ and $|E(F_i) \setminus \cup_{j<i} E(F_j)| \geqslant 2$ for every $i \in [r]$.

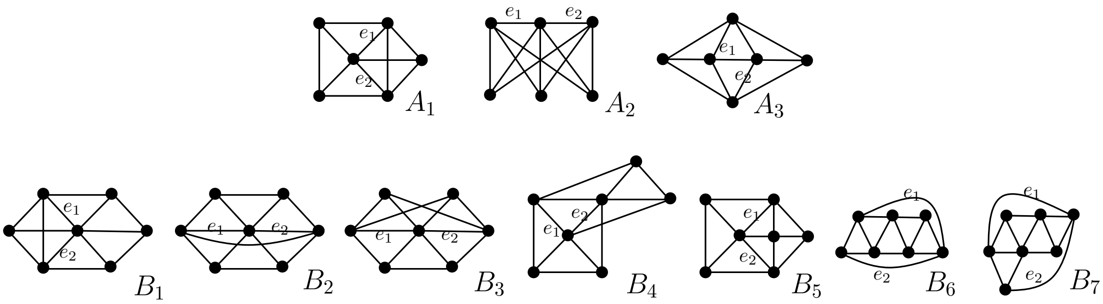

Figure 2: Basic triangle collections.

Note that every collection on a given number n of vertices, contains a bunch on the same number of vertices with at least $2n-3$ edges. Figure 2 shows some collections that are easily checked to be basic. For each of the graphs, the edges e_1 and e_2 indicated in the figure satisfy the condition from the definition of basic graphs. Moreover, the following observation is easily verified.

Observation 16. *Let $G = (V, E)$. Maker has a strategy to create a triangle (a copy of T_C) on G if and only if G contains a collection C such that she has a strategy to create a triangle (a copy of T_C) on C.*

In the following we show now that Breaker can prevent Maker from occupying a triangle when playing on basic graphs. This also ensures a winning strategy for Breaker in the corresponding T_C-tournament game. We start with the following proposition.

Proposition 17. *Let $G = (V, E)$ be very basic, then Breaker can block every triangle in the unbiased Maker-Breaker game on $E(G)$, even if Maker is allowed to claim two edges in the very first round.*

Proof Without loss of generality (abbreviated *W.l.o.g.* in the rest of the paper) we can assume that $T(G) \cong P_k$ for some k, or $T(G) \cong K_3^+$, with $T(G)$ as given in Definition 15. We further can assume that Maker in the first round claims two edges $f_1, f_2 \in E(G)$ that participate in triangles of G. If $T(G) \cong P_k$ then observe that there is an ordering $F_1, \dots, F_k$ of the vertices in $T(G)$, such that $f_1 \in E(F_1)$, and $|V(F_i) \setminus \cup_{j<i} V(F_j)| = 1$, and $|E(F_i) \setminus \cup_{j<i} E(F_j)| = 2$ for every $2 \leqslant i \leqslant k$. To see this one just has to start the sequence with a triangle F_1 containing f_1, and to extend the sequence along the path-like structure of $T(G)$. Finally, let $A_1 := E(F_1) \setminus \{f_1\}$ and $A_i := E(F_i) \setminus \cup_{j<i} E(F_j)$ for every $i \in [k] \setminus \{1\}$. These sets are pairwise disjoint, have cardinality 2 and satisfy $A_i \subseteq E(F_i)$ for each $i \in [k]$. That is, Breaker can block triangles by an easy pairing strategy. (In particular, for his first move, Breaker claims the unique edge f for which there is an $i \in [k]$

with $A_i = \{f_2, f\}$.) If $T(G) \cong K_3^+$, then it can be shown that G contains exactly four triangles and that one can find an ordering $F_1, \dots F_k$ (with $k = 4$) with the properties from the previous case. Indeed, G needs to be a copy of the graph presented in Figure 3. So, Breaker wins similarly. □

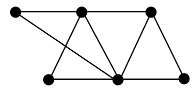

Figure 3: $T(G) \cong K_3^+$

Corollary 18. *Let $G = (V, E)$ be basic, then Breaker can block every triangle in the unbiased Maker-Breaker game on $E(G)$.*

Proof Let e_1, e_2 be the edges given by the definition of a basic graph. Breaker's strategy is to claim e_1 or e_2 in the first round. Afterwards, the game reduces to the graph $G - e_i$ for some $i \in [2]$, where Maker claims 2 edges, before Breaker claims his first edge. Now, since $G - e_i$ is very basic for both $i \in \{1, 2\}$, Breaker then succeeds by the previous proposition. □

We further observe the following two statements which can be checked by easy case distinctions.

Observation 19. *Breaker has a strategy to prevent cyclic triangles in an unbiased game on $E(K_4)$, even if Maker is allowed to claim and orient two edges in her first turn.*

Observation 20. *Breaker has a strategy to prevent cyclic triangles in an unbiased game on $E(W_4)$, even if Maker is allowed to claim and orient two edges in her first turn, as long as not both edges are incident with the center vertex of W_4.*

Now, using the previous statements we will show that for $p \ll n^{-\frac{8}{15}}$ a.a.s. every collection C in $G \sim G_{n,p}$ is such that Breaker has a strategy to prevent cyclic triangles in an unbiased game on C. It follows then by Observation 16 that a.a.s. Breaker wins on G. To do so, we start with the following propositions, motivated by [9], which helps to restrict the set of collections we need to consider.

Proposition 21. *Let $p \ll n^{-\frac{8}{15}}$, then a.a.s. every triangle collection C in $G \sim \mathcal{G}_{n,p}$ satisfies $m(C) < \frac{15}{8}$.*

Proof Each collection C on at least 25 vertices contains a bunch B on exactly 25 vertices with

$$d(B) = \frac{e(B)}{v(B)} \geqslant \frac{2v(B) - 3}{v(B)} > \frac{15}{8}.$$

Since there are only finitely many such bunches and each of them a.a.s. does not appear in G according to Theorem 9, together with the union bound we obtain that a.a.s. each collection in G lives on at most 25 vertices. Since there are only finitely many collections with at most 25 vertices, we also know by the same reason that a.a.s. each collection in G on at most 25 vertices needs to have maximum density smaller than $\frac{15}{8}$. □

Proposition 22. *Let C be a triangle collection with $m(C) < \frac{15}{8}$ such that Maker has a strategy to create a cyclic triangle in an unbiased game on C, but there is no such strategy for any collection $C' \subset C$. Then the following properties hold:*

(a) $5 \leqslant v(C) \leqslant 7$*,*

(b) $e(C) = 2v(C) - 1$*,*

(c) $\delta(C) \geqslant 3$*,*

(d) C is not basic.

Proof Property (d) obviously holds, using Corollary 18. Moreover, (c) follows immediately. Indeed, if there were a vertex v with $d_C(v) \leqslant 2$, then Breaker could prevent cycles on $C - v$ by the minimality condition on C, and cycles containing v by simply pairing the edges incident with v (if there exist two such edges), a contradiction. Furthermore, $v(C) \geqslant 5$ is needed, according to Observation 19. Now, let B be a bunch contained in C with $v(C)$ vertices, then $e(C) > e(B)$, since $\delta(B) = 2 < \delta(C)$. As such a bunch contains at least $2v(B)-3$ edges, it follows that $e(C) \geqslant e(B)+1 \geqslant 2v(C)-2$. Furthermore $e(C) \leqslant 2v(C)-1$, since otherwise $m(C) \geqslant 2$. If $e(C) = 2v(C)-1$, then together with $m(C) < \frac{15}{8}$, we deduce that $v(C) \leqslant 7$. Otherwise, we have $e(C) = 2v(C) - 2$ and $e(C) = e(B) + 1$. Analogously to the proof of Theorem 23 in [9] it then follows that C can only be a wheel; for completeness let us include the argument here: Let $E(C) \setminus E(B) = \{v_1v_2\}$. By the definition of a bunch, we can find triangles $F_1, \dots F_r$ in B covering all edges of B with the property that $|V(F_i) \setminus \cup_{j<i} V(F_j)| = 1$ and $|E(F_i) \setminus \cup_{j<i} E(F_j)| \geqslant 2$ for every $i \in [r]$. As $e(B) = e(C)-1 = 2v(B)-3$ it then follows that $r = v(C)-2$ and $|E(F_i) \setminus \cup_{j<i} E(F_j)| = 2$ for every $i \in [r] \setminus \{1\}$, as otherwise $e(B) > 3 + 2(r-1) = 2v(C) - 3$, a contradiction. Thus, for every $i \in [r] \setminus \{1\}$, F_i needs to share exactly one edge with $\cup_{j<i} F_j$. From this, we can conclude that B needs to contain at least two vertices of degree 2. However, as $\delta(C) \geqslant 3$ and $E(C) \setminus E(B) = \{v_1v_2\}$, we know that v_1 and v_2 must be the only vertices in B of degree 2. Now, by the definition of a triangle collection, v_1v_2 needs to be part of a triangle in C. Thus, there needs to be a vertex v_3 such that $v_1v_3, v_3v_2 \in E(B)$. But this is only possible if v_3 belongs to every triangle F_i, $i \in [r]$, and thus, C needs to be a wheel. Now, to finish the proof, observe that Breaker can always prevent triangles in an unbiased game on a wheel by a simple pairing strategy, a contradiction to our assumption. □

So, the goal will be to show that there exists no collection C which satisfies all the conditions given in Proposition 22.

Lemma 23. *If a collection C satisfies (a) - (d) from Proposition 22, then either C is isomorphic to K_5^- (K_5 minus one edge) or C is isomorphic to one of the graphs S_i, $1 \leqslant i \leqslant 4$, given in Figure 4.*

Proof If $v(C) = 5$, then $e(C) = 9$, by Property (b), and the statement follows obviously. So, let $v(C) \neq 5$. We will show now that a collection satisfying (a) - (c) either is isomorphic

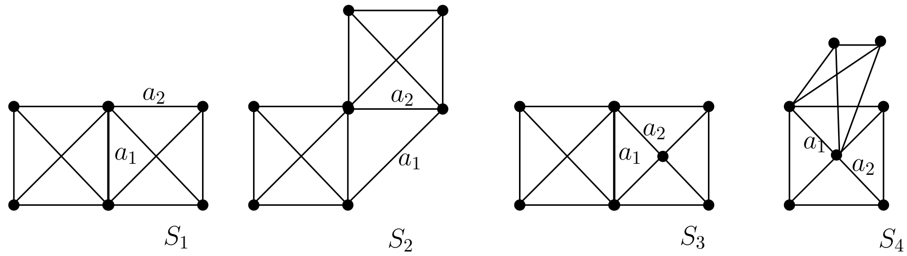

Figure 4: Special collections.

to one of the collections S_i, or it is isomorphic to one of the basic collections A_i or B_i from Figure 2, thus contradicting Property (d).

Let us start with $v(C) = 6$. Assume first that C contains a subgraph $H \cong K_4$ and let $\{x, y\} = V(C) \setminus V(H)$. With $e(C) = 11$ and $\delta(C) \geqslant 3$ we conclude $xy \in E(C)$, and by the definition of a collection it follows that x and y have a common neighbour $v_1 \in V(H)$. Because of (c), we further have $xv_2 \in E(C)$ for some $v_2 \in V(H) \setminus \{v_1\}$. Now, if $yv_2 \in E(C)$, then $C \cong S_1$, otherwise by (c) we have $yv_3 \in E(C)$ for some $v_3 \in V(H) \setminus \{v_1, v_2\}$ and so $C \cong A_1$.
Assume then that C does not contain a clique of order 4. We still find a subgraph $H' \subseteq C$ with four vertices $V(H') = \{v_1, v_2, v_3, v_4\}$ and five edges, say $v_1v_3 \notin E(H)$, as C is a triangle collection and therefore needs to contain two intersecting triangles. Moreover, since C is a triangle collection, there needs to be some $x \in V(C) \setminus V(H')$ that is part of the same triangle as an edge e from H'. Let y be the unique vertex in $V(C) \setminus (V(H') \cup \{x\})$.

Assume first that $e = v_2v_4$. We know then that $\{x, v_1, v_3\}$ is an independent set in C, since otherwise we would have a 4-clique in C. Using (c), we conclude that $N(x) = \{v_2, v_4, y\}$. With (b) we obtain that y needs to have exactly four neighbours, but not both v_2 and v_4 can be neighbours since otherwise there we find a copy of K_4 in C. Therefore, $N(y) = \{x, v_1, v_3, v_i\}$ for some $i \in \{2, 4\}$, which gives $C \cong A_2$.

Assume then that $e \neq v_2v_4$ and w.l.o.g. $e = v_3v_4$ by symmetry of H'. If $v_1x \in E(C)$, it then follows that $d(y) = 3$, since (b) and (c) need to hold; moreover, $C[V(C) \setminus \{y\}] \cong W_4$ where v_4 represents the center of the wheel. In case $v_4y \in E(C)$, we can only have $C \cong A_2$, as C does not contain a 4-clique; and in case $v_4y \notin E(C)$, we can assume that $N(y) = \{v_1, v_2, v_3\}$ (because of the symmetry of the 4-wheel), which yields $C \cong A_3$. If otherwise $v_1x \notin E(C)$, then, since there is no 4-clique in C, we immediately obtain $d(y) = 4$ and $v_1, x \in N(y)$, as $e(C) = 11$ and $\delta(C) \geqslant 3$. Moreover, $v_4 \notin N(y)$, since we otherwise would obtain a 4-clique, independently of the choice of the fourth neighbour of y. Thus, we conclude $N(y) = \{v_1, v_2, v_3, x\}$ and $C \cong A_3$.

Now, let $v(C) = 7$. We distinguish three cases.

Case 1. Assume that C contains a subgraph $H \cong K_4$. Let $\{x, y, z\} = V(C) \setminus V(H) =: V'$. If V' were an independent set, then with (c), we would conclude that $e(C) \geqslant 15$, in contradiction to (b). Thus, it follows that $\{x, y, z\}$ is not an independent set, w.l.o.g.

$xy \in E(C)$. By the definition of a collection it further follows that x and y have a common neighbour – the vertex z or some vertex $v \in V(H)$.

Assume first that $z \in N(x) \cap N(y)$. By $\delta(C) \geqslant 3$ each vertex in V' needs to have at least one neighbour in $V(H)$. If there were a matching of size 3 between V' and $V(H)$, then by (b), one of the matching edges could not be part of a triangle, a contradiction. If all the three vertices have a common neighbour in $V(H)$, then one easily deduces $C \cong S_2$. Otherwise, by symmetry we can assume that there is a vertex $v_1 \in V(H)$ such that $v_1x, v_1y \in E(C)$ and $v_1z \notin E(C)$, and moreover, $v_2z \in E(C)$ for some $v_2 \in V(H) \setminus \{v_1\}$. Now, let $\{v_3, v_4\} = V(H) \setminus \{v_1, v_2\}$. To ensure that v_2z belongs to some triangle in C, we finally need to have exactly one of the edges from $\{v_3z, v_4z, v_2x, v_2y\}$ to be an edge in C. The first two edges however do not result in a triangle collection, while for the other two edges we get $C \cong S_3$.

Assume then that $z \notin N(x) \cap N(y)$, but $v \in N(x) \cap N(y)$ for some $v \in V(H)$. Because of (b) and (c), either $xz \in E(C)$ or $yz \in E(C)$, w.l.o.g. say $xz \in E(C)$ and $yz \notin E(C)$. As $\delta(C) \geqslant 3$, we then immediately get $yw \in E(C)$ for some $w \in V(H) \setminus \{v\}$. Moreover, we then need two other edges incident with z besides xz, of which one is zv to ensure that xz belongs to a triangle. If the second edge is zw, then $C \cong S_4$; otherwise $C \cong B_1$.

Case 2. Assume that C does not contain a clique of order 4, but there is some $H \subseteq C$ with $H \cong W_4$. Let $\{x, y\} = V(C) \setminus V(H) =: V'$ and let z be the unique vertex with $d_H(z) = 4$. By (b) and (c), it follows that $xy \in E(C)$, and since C is a collection, there is a common neighbour of x and y in $V(H)$.

Assume first that $z \in N(x) \cap N(y)$. As $\delta(C) \geqslant 3$, both vertices x and y have another neighbour in $V(H) \setminus \{z\}$, however there cannot be a second common neighbour, since there is no 4-clique in C. One easily checks that $C \cong B_2$ or $C \cong B_3$ follows.

Assume then that $z \notin N(x) \cap N(y)$, but $v \in N(x) \cap N(y)$ for some $v \in V(H) \setminus \{z\}$. If $xz \in E(C)$ (or $yz \in E(C)$), we then need $yw \in E(C)$ (or $xw \in E(C)$) for some $w \in N_H(v) \setminus \{z\}$ to ensure that $e(C) = 13$ and $\delta(C) \geqslant 3$ holds while C is a triangle collection. This gives $C \cong B_4$. Otherwise, we have $z \notin N(x) \cup N(y)$. In this case, let w' to be the unique vertex of H not belonging to $N(v) \cup \{v\}$. Then we also have $w' \notin N(x) \cup N(y)$. Indeed, if we had $yw' \in E(C)$ say, then as yw' needs to be part of some triangle and as $d(x) \geqslant 3$ and $e(C) = 13$, we would need $xw' \in E(C)$, in which case it is easily checked that C is not a triangle collection. So, we can assume that $xv_1 \in E(C)$ for some $v_1 \in V(H) \setminus \{v, w', z\}$, and $yv_1 \notin E(C)$, because C does not have a 4-clique. Finally, since $\delta(C) \geqslant 3$, we need $v_2y \in E(C)$ for the unique vertex $v_2 \in V(H) \setminus \{v, w', z, v_1\}$, i.e. $C \cong B_5$.

Case 3. Finally assume that C neither contains a 4-clique nor a 4-wheel. It is easy to check that $C_0 \subseteq C$ (see Figure 5): Indeed, as C is a triangle collection, there needs to be a subgraph C' which consists of two triangles that intersect in some edge e'. Now, if we had $C_0 \nsubseteq C$, then each of the three vertices in $V(C) \setminus V(C')$ would need to form a triangle together with the edge e'. However, this then leads to a graph which cannot satisfy (b) and (c) at the same time, a contradiction.

Thus, we can fix a subgraph C_0 (with notation of vertices as given in Figure 5) and by the assumption of this case we further have $v_1v_3, v_1v_4, v_3v_5 \notin E(C)$. Since C is a triangle collection, we find a vertex $x \in V' := V(C) \setminus V(C_0)$ which belongs to a triangle that also contains an edge $e \in E(C_0)$. Let $\{y\} = V' \setminus \{x\}$. By symmetry of C_0 we may assume that $e \in \{v_2v_5, v_4v_5, v_1v_5, v_1v_2\}$.

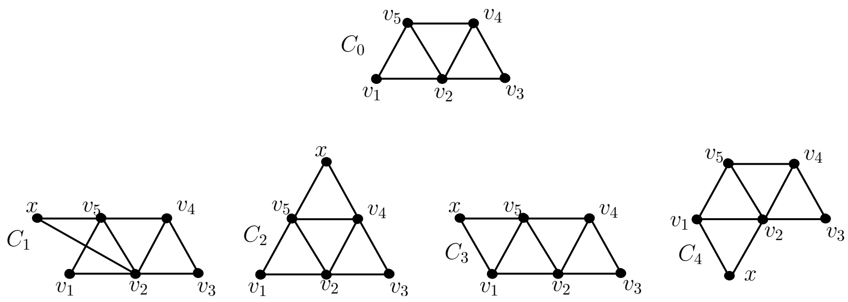

Figure 5: Subgraphs.

Assume first that $e = v_2v_5$ were possible, i.e. $C_1 \subseteq C$. Then we conclude that $v_1, v_3, v_4 \notin N(x)$, as otherwise we had a copy of K_4 or W_4, in contradiction to the assumption of Case 3. Thus, every edge in $E(C) \setminus E(C_1)$ would need to be incident with y. Because of (b) and (c) we then had that $d(y) = 4$ and $v_1y, v_3y, xy \in E(C)$. Since these three edges would need to belong to triangles, we further would need $yv_2 \in E(C)$, which would create a 4-wheel on $V(C) \setminus \{v_3, v_4\}$ with center v_2, again in contradiction to our assumption.

So, as next assume that $e = v_4v_5$ were possible, i.e. $C_2 \subseteq C$. Then analogously every edge in $E(C) \setminus E(C_2)$ would need to be incident with y, and $d(y) = 4$ and $\{v_1, v_3, x\} \subseteq N(y)$, because of (b) and (c). But then, independently of what the fourth neighbour of y is, one of the edges v_1y, v_3y, xy could not belong to a triangle, again a contradiction.

As third, assume that $e = v_1v_5$, i.e. $C_3 \subseteq C$. By the assumption of Case 3, every edge in $E(C) \setminus (E(C_3) \cup \{xv_3\})$ needs to be incident with y. If $xv_3 \notin E(C)$, then we have $d(y) = 4$ and $xy, v_3y \in E(C)$, because of $e(C) = 13$ and $\delta(C) \geqslant 3$. Depending on how the other two edges incident with y are chosen, we either obtain a contradiction by creating a 4-clique or a 4-wheel, or we see that $C \cong B_6$. So, let $xv_3 \in E(C)$. Then $d(y) = 3$, by (b) and (c), and to have xv_3 in a triangle, we need $yx, yv_3 \in E(C)$. It follows that $C \cong B_6$, if $yv_1 \in E(C)$ or $yv_4 \in E(C)$, or $C \cong B_7$, if $yv_2 \in E(C)$ or $yv_5 \in E(C)$.

As last, assume that $e = v_1v_2$, i.e. $C_4 \subseteq C$. If $xv_3 \in E(C)$ were possible, then we had $d(y) = 3$ because of $e(C) = 13$ and $\delta(C) \geqslant 3$. But then, depending on the three edges incident with y, we would get a 4-clique or a 4-wheel in C, or we would find an edge which is not contained in a triangle, a contradiction. So, we can assume that $xv_3 \notin E(C)$. Then, by (b), (c) and the assumption of Case 3, we deduce that $d(y) = 4$ and $yx, yv_3 \in E(C)$. If $yv_2 \in E(C)$ were also an edge of C, then for any choice of the fourth edge incident with y, we would create a 4-clique or a 4-wheel in C. That is, we can assume that $yv_2 \notin E(C)$. But then we need $v_1y, v_4y \in E(C)$ to ensure that yx and yv_3 belong to triangles, which yields $C \cong B_7$. □

Lemma 24. *For any collection given by Lemma 23, Breaker has a strategy to prevent cyclic triangles.*

Proof If $C \cong S_i$ for some i, note that C is covered by two (not necessarily disjoint) graphs $C(1)$, $C(2)$, plus at most one additional edge if $C \cong S_2$, where each of the $C(i)$ is isomorphic to K_4 or W_4. Choose edges a_1 and a_2 as indicated in Figure 4. In his first move, Breaker claims the edge a_1 if Maker did not orient it before; otherwise he claims the edge a_2. Afterwards, Breaker plays on $C(1)$ and $C(2)$ separately, meaning: each time Maker orients an edge of $C(i)$, Breaker claims an edge of $C(i)$ if there remains one. Now, using Proposition 17 and Observation 19, Breaker can do this in a way such that he prevents cyclic triangles on each $C(i)$, and therefore in C.

Finally, we need to look at the case when $C \cong K_5^-$. By an easy case analysis, it can be proven that Breaker has a strategy to prevent cyclic triangles on C. We give a sketch in the following. Let $V(C) = X \cup Y$ with $X = \{v_1, v_2, v_3\}$ and $Y = \{v_4, v_5\}$, and let $E(C) = \binom{X}{2} \cup \{xy : x \in X, y \in Y\}$.

Case 1. Maker orients an edge in $E(X, Y)$ in her first turn.

W.l.o.g. let $e = v_1v_4 \in E(X, Y)$ be the edge to which Maker gives an orientation in her first move. Then Breaker's strategy is to delete the edge v_1v_2. Note that $C - \{v_1v_2\}$ is isomorphic to the 4-wheel W_4, here with center v_3, and Maker's first arc is not incident with v_3. Thus, Breaker can win by Observation 20.

Case 2. Maker orients an edge inside $E(X)$ in her first turn.

W.l.o.g. let Maker's first oriented edge be (v_1, v_2). Then Breaker's first move will be to delete the edge v_2v_4. Afterwards, Breaker's second move will depend on Makers second move, as follows:

If Maker orients (v_1, v_3) or (v_3, v_2) for her second move, then Breaker claims v_2v_5 and afterwards he wins by an easy pairing strategy, with the pairs $\{v_1v_4, v_3v_4\}$ and $\{v_1v_5, v_3v_5\}$.

If Maker for her second move chooses one of the arcs (v_1, v_4), (v_4, v_1), (v_3, v_4), (v_4, v_3), (v_1, v_5), (v_5, v_2), (v_2, v_3) and (v_3, v_5), then Breaker for his second move claims the edge v_1v_3. As he claims v_2v_4 and v_1v_3 then, the only triplets on which Maker could create a triangle are $\{v_1, v_2, v_5\}$ and $\{v_2, v_3, v_5\}$. In either of the cases it is easy to check that from now on Breaker can prevent cyclic triangles.

If Maker for her second move chooses (v_2, v_5) or (v_5, v_3), then Breaker claims v_1v_5 for his second move. Afterwards there remain three triplets on which Maker still could create a triangle, namely $\{v_1, v_3, v_4\}$, $\{v_1, v_2, v_3\}$ and $\{v_2, v_3, v_5\}$. To block a triangle on $\{v_1, v_3, v_4\}$, Breaker can consider a pairing $\{v_1v_4, v_3v_4\}$. For the other two triplets it is easy to check then that Breaker can prevent cyclic triangles, since the orientation which v_2v_3 needs, to create a cyclic triangle, is different for these two remaining triplets.

If Maker for her second move chooses (v_3, v_1), then Breaker needs to claim v_2v_3. Afterwards there remain three triplets on which Maker still could create a triangle, namely $\{v_1, v_3, v_4\}$, $\{v_1, v_2, v_5\}$ and $\{v_1, v_3, v_5\}$. To block a triangle on $\{v_1, v_3, v_4\}$, Breaker can consider a pairing $\{v_1v_4, v_3v_4\}$. For the other two triplets it again is easy to check that Breaker can prevent cyclic triangles, since the orientation which v_1v_5 needs, to create a cyclic triangle, is different for these two triplets.

Finally, if Maker for her second move chooses (v_5, v_1), then Breaker needs to claim v_2v_5. Afterwards there remain three triplets on which Maker still could create a triangle, namely $\{v_1, v_3, v_4\}$, $\{v_1, v_2, v_3\}$ and $\{v_1, v_3, v_5\}$. To block a triangle on $\{v_1, v_3, v_4\}$, Breaker can consider a pairing $\{v_1v_4, v_3v_4\}$. For the other two triplets it again is easy to check that Breaker can prevent cyclic triangles, since the orientation which v_1v_3 needs, to create a cyclic triangle, is different for these two triplets. □

To summarize, we have shown now that for $p \ll n^{-\frac{8}{15}}$, a.a.s. Breaker can prevent cyclic triangles in the tournament game on $G \sim \mathcal{G}_{n,p}$. Indeed, by Proposition 22, Lemma 23 and Lemma 24, we know that there exists no collection C with $m(C) < \frac{15}{8}$ on which Maker has a strategy to create a copy of T_C. By Proposition 21 we however know that for $p \ll n^{-\frac{8}{15}}$ a random graph $G \sim \mathcal{G}_{n,p}$ a.a.s. only contains such collections, and using Observation 16 we thus conclude that a.a.s. Maker does not have a winning strategy when playing on $G \sim \mathcal{G}_{n,p}$, which at the same time guarantees a winning strategy for Breaker. □

References

[1] N. Alon, and J. H. Spencer. *The Probabilistic Method.* Wiley-Interscience Series in Discrete Mathematics and Optimization, third edition, 2008.

[2] J. Beck. *Combinatorial Games: Tic-Tac-Toe Theory.* Cambridge University Press, 2008.

[3] M. Bednarska, and T. Łuczak. Biased positional games for which the random strategies are nearly optimal. *Combinatorica*, 20:477–488, 2000.

[4] V. Chvátal, and P. Erdős. Biased positional games, *Annals of Discrete Mathematics*, 2:221–228, 1978.

[5] D. Clemens, H. Gebauer, and A. Liebenau. The random graph intuition for the tournament game. submitted, arXiv:1307.4229 [math.CO]

[6] P. Erdős, and J. Selfridge. On a combinatorial game. *Journal of Combinatorial Theory Series B*, 14:298–301, 1973.

[7] S. Janson, T. Łuczak, and A. Ruciński. *Random Graphs.* John Wiley & Sons, Inc., 2000.

[8] T. Müller, and M. Stojaković. A threshold for the Maker-Breaker clique game. *Random Structures & Algorithms*, 45:318–341, 2014.

[9] M. Stojaković, and T. Szabó. Positional games on random graphs. *Random Structures & Algorithms*, 26:204–223, 2005.

[10] D. B. West. *Introduction to Graph Theory.* Prentice Hall, 2001.