Arithmetic Properties of a Restricted Bipartition Function

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Abstract

A bipartition of n is an ordered pair of partitions (λ, μ) such that the sum of all of the parts equals n. In this article, we concentrate on the function $c_5(n)$, which counts the number of bipartitions (λ, μ) of n subject to the restriction that each part of μ is divisible by 5. We explicitly establish four Ramanujan type congruences and several infinite families of congruences for $c_5(n)$ modulo 3.

Keywords: bipartition, congruence

1 Introduction

In a series of papers [4, 5, 6], Chan studied the arithmetic properties of the cubic partition function $a(n)$, which is defined by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}.$$

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Throughout the paper, we adopt the following standard q-series notation

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1}).$$

In [4], Chan proved that

Theorem 1. *For $n \geqslant 0$,*

$$a(3n+2) \equiv 0 \pmod 3. \tag{1}$$

Later, Kim [13] gave a combinatorial interpretation of the congruence (1). Furthermore, Chan [5] showed that

Theorem 2. *For $k \geqslant 1$ and $n \geqslant 0$,*

$$a(3^k n + s_k) \equiv 0 \pmod{3^{k+\delta(k)}},$$

where s_k is the reciprocal modulo 3^k of 8 and $\delta(k) = 1$ if k is even, and 0 otherwise.

Chan and Toh [7] also established the following nice congruence, which was also discovered by Xiong [20] independently.

Theorem 3. *If $k \geqslant 1$ and $n \geqslant 0$, then*

$$a(5^k n + t_k) \equiv 0 \pmod{5^{\lfloor k/2 \rfloor}},$$

where t_k is the reciprocal modulo 5^k of 8.

Inspired by the work of Ramanujan on the standard partition function $p(n)$, Chan[5] asked whether there are any other congruences of the following form

$$a(\ell n + k) \equiv 0 \pmod{\ell},$$

where ℓ is prime and $0 \leqslant k < \ell$. Later, Sinick [18] answered Chan's question in the negative by considering the following restricted bipartition function:

$$\sum_{n=0}^{\infty} c_N(n) q^n = \frac{1}{(q;q)_\infty (q^N;q^N)_\infty}. \tag{2}$$

A bipartition of n is an ordered pair of partitions (λ, μ) such that the sum of all of the parts equals n. Then we know that $c_N(n)$ counts the number of bipartitions (λ, μ) of n subject to the restriction that each part of μ is divisible by N. Recently, bipartitions with certain restrictions on each partition have been investigated by many authors, see [3, 8, 9, 10, 11, 12, 14, 15, 16, 17, 19] for instance.

In this paper, we investigate the bipartition function $c_5(n)$ from an arithmetic point of view in the spirit of Ramanujan's congruences for the standard partition function $p(n)$.

2 Ramanujan type congruences for $c_5(n)$

We first introduce a useful lemma which will be used later.

Lemma 4. *We have*

$$(q;q)_\infty^2(q^5;q^5)_\infty^2 \equiv (q^3;q^3)_\infty^4 + q(q^3;q^3)_\infty^2(q^{15};q^{15})_\infty^2 - q^2(q^{15};q^{15})_\infty^4 \pmod 3. \tag{3}$$

Proof. From [1, p.28, Entry 1.6.2], we see that

$$\begin{aligned}(q;q)_\infty^2(q^5;q^5)_\infty^2 &= (\psi^2(q) - q\psi^2(q^5))(\psi^2(q) - 5q\psi^2(q^5)) \\ &\equiv \psi(q)\psi(q^3) - q^2\psi(q^5)\psi(q^{15}) \pmod 3, \end{aligned} \tag{4}$$

where

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}.$$

Invoking [2, p.49, Corollary (ii)], we have

$$\psi(q) = A(q^3) + q\psi(q^9), \tag{5}$$

where

$$A(q) = (-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty.$$

Substituting (5) into (4), we find that

$$\begin{aligned}(q;q)_\infty^2(q^5;q^5)_\infty^2 &\equiv \psi(q^3)\left(A(q^3) + q\psi(q^9)\right) \\ &\quad - q^2\psi(q^{15})\left(A(q^{15}) + q^5\psi(q^{45})\right) \pmod 3 \\ &= \psi(q^3)A(q^3) - q^2\psi(q^{15})A(q^{15}) \\ &\quad + q\left(\psi(q^3)\psi(q^9) - q^6\psi(q^{15})\psi(q^{45})\right).\end{aligned}$$

On the other hand, applying (4) with q replaced by q^3 yields that

$$(q^3;q^3)_\infty^2(q^{15};q^{15})_\infty^2 \equiv \psi(q^3)\psi(q^9) - q^6\psi(q^{15})\psi(q^{45}) \pmod 3.$$

Therefore, we arrive at

$$(q;q)_\infty^2(q^5;q^5)_\infty^2 \equiv \psi(q^3)A(q^3) - q^2\psi(q^{15})A(q^{15}) + q(q^3;q^3)_\infty^2(q^{15};q^{15})_\infty^2 \pmod 3. \tag{6}$$

In addition, it is easy to see that

$$\psi(q)A(q) \equiv (q;q)_\infty^4 \pmod 3.$$

Utilizing the above congruence in (6), we complete the proof of (3). ☐

With Lemma 4 in hand, we now move to the dissections of the generating function for $c_5(n)$ modulo 3.

Theorem 5. *We have*

$$\sum_{n=0}^{\infty} c_5(3n)q^n \equiv \frac{(q^3;q^3)_\infty}{(q^5;q^5)_\infty} \pmod 3, \tag{7}$$

$$\sum_{n=0}^{\infty} c_5(3n+1)q^n \equiv (q;q)_\infty(q^5;q^5)_\infty \pmod 3, \tag{8}$$

$$\sum_{n=0}^{\infty} c_5(3n+2)q^n \equiv -\frac{(q^{15};q^{15})_\infty}{(q;q)_\infty} \pmod 3. \tag{9}$$

Proof. From (2), we can easily deduce that

$$\sum_{n=0}^{\infty} c_5(n)q^n \equiv \frac{(q;q)_\infty^2(q^5;q^5)_\infty^2}{(q^3;q^3)_\infty(q^{15};q^{15})_\infty} \pmod 3.$$

Applying Lemma 4, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} c_5(n)q^n &\equiv \frac{(q^3;q^3)_\infty^3}{(q^{15};q^{15})_\infty} + q(q^3;q^3)_\infty(q^{15};q^{15})_\infty - q^2\frac{(q^{15};q^{15})_\infty^3}{(q^3;q^3)_\infty} \\ &\equiv \frac{(q^9;q^9)_\infty}{(q^{15};q^{15})_\infty} + q(q^3;q^3)_\infty(q^{15};q^{15})_\infty - q^2\frac{(q^{45};q^{45})_\infty}{(q^3;q^3)_\infty} \pmod 3,\end{aligned}$$

frow which we get

$$\sum_{n=0}^{\infty} c_5(3n)q^{3n} \equiv \frac{(q^9;q^9)_\infty}{(q^{15};q^{15})_\infty} \pmod 3,$$

$$\sum_{n=0}^{\infty} c_5(3n+1)q^{3n+1} \equiv q(q^3;q^3)_\infty(q^{15};q^{15})_\infty \pmod 3,$$

$$\sum_{n=0}^{\infty} c_5(3n+2)q^{3n+2} \equiv -q^2\frac{(q^{45};q^{45})_\infty}{(q^3;q^3)_\infty} \pmod 3,$$

simplification upon which yields the desired results. □

The following is a consequence of Theorem 5.

Corollary 6. *We have*

$$\sum_{n=0}^{\infty} c_5(9n+7)q^n \equiv -(q^3;q^3)_\infty(q^{15};q^{15})_\infty \pmod 3. \tag{10}$$

Proof. By (8), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} c_5(3n+1)q^n &\equiv (q^3;q^3)_\infty(q^{15};q^{15})_\infty \times \frac{1}{(q;q)_\infty^2(q^5;q^5)_\infty^2} \pmod 3\\ &= (q^3;q^3)_\infty(q^{15};q^{15})_\infty \times \left(\sum_{n=0}^{\infty} c_5(3n)q^{3n}\right.\\ &\quad \left.+\sum_{n=0}^{\infty} c_5(3n+1)q^{3n+1}+\sum_{n=0}^{\infty} c_5(3n+2)q^{3n+2}\right)^2.\end{aligned}$$

Extracting those terms on each side for which the powers of q are of the form $3n+2$, dividing by q^2, and replacing q^3 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} c_5(9n+7)q^n &\equiv (q;q)_\infty(q^5;q^5)_\infty \times \left(\left(\sum_{n=0}^{\infty} c_5(3n+1)q^n\right)^2\right.\\ &\quad \left.+2\sum_{n=0}^{\infty} c_5(3n)q^n \sum_{n=0}^{\infty} c_5(3n+2)q^n\right) \pmod 3.\end{aligned}$$

It follows from Theorem 5 that

$$\begin{aligned}\sum_{n=0}^{\infty} c_5(9n+7)q^n &\equiv (q;q)_\infty(q^5;q^5)_\infty \times \left((q;q)_\infty^2(q^5;q^5)_\infty^2 - 2\frac{(q^3;q^3)_\infty(q^{15};q^{15})_\infty}{(q^5;q^5)_\infty(q;q)_\infty}\right)\\ &\equiv -(q^3;q^3)_\infty(q^{15};q^{15})_\infty \pmod 3.\end{aligned}$$

This completes the proof. □

We now establish four Ramanujan type congruences for $c_5(n)$.

Theorem 7. *For all $n \geqslant 0$,*

$$c_5(15n+6) \equiv 0 \pmod 3, \tag{11}$$

$$c_5(15n+10) \equiv 0 \pmod 3, \tag{12}$$

$$c_5(15n+12) \equiv 0 \pmod 3, \tag{13}$$

$$c_5(15n+13) \equiv 0 \pmod 3. \tag{14}$$

Proof. Recall that Euler's pentagonal number theorem [2, p.36, Entry 22]

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}. \tag{15}$$

Substituting (15) into (7), we have

$$\sum_{n=0}^{\infty} c_5(3n)q^n \equiv \frac{1}{(q^5;q^5)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{3n(3n+1)/2} \pmod 3.$$

Extracting those terms on each side whose power of q is of the form $5n+2$ or $5n+4$, and employing the fact that there exist no integers n such that $3n(3n+1)/2$ is congruent to 2 or 4 modulo 5, we get

$$\sum_{n=0}^{\infty} c_5(15n+6)q^{5n+2} \equiv \sum_{n=0}^{\infty} c_5(15n+12)q^{5n+4} \equiv 0 \pmod 3,$$

which means that

$$c_5(15n+6) \equiv c_5(15n+12) \equiv 0 \pmod 3.$$

Similarly, from (8) and the fact that there are no integers n with $n(3n+1)/2$ being congruent to 3 or 4 modulo 5, it is not hard to obtain

$$c_5(15n+10) \equiv c_5(15n+13) \equiv 0 \pmod 3.$$

This concludes the proof. □

3 Two infinite families of congruences for $c_5(n)$

We start with investigating a generalization of the congruences (8) and (10).

Theorem 8. *For $\alpha \geqslant 1$, we have*

$$\sum_{n=0}^{\infty} c_5\left(3^{2\alpha-1}n + \frac{3^{2\alpha-1}+1}{4}\right) q^n \equiv (-1)^{\alpha+1}(q;q)_\infty(q^5;q^5)_\infty \pmod 3, \tag{16}$$

$$\sum_{n=0}^{\infty} c_5\left(3^{2\alpha}n + \frac{3^{2\alpha+1}+1}{4}\right) q^n \equiv (-1)^{\alpha}(q^3;q^3)_\infty(q^{15};q^{15})_\infty \pmod 3. \tag{17}$$

Proof. We proceed by induction on α. The case $\alpha = 1$ corresponds to the congruences (8) and (10).

Assume that

$$\sum_{n=0}^{\infty} c_5\left(3^{2\alpha}n + \frac{3^{2\alpha+1}+1}{4}\right) q^n \equiv (-1)^{\alpha}(q^3;q^3)_\infty(q^{15};q^{15})_\infty \pmod 3$$

is true for some fixed integer $\alpha \geqslant 1$. Since the terms appearing on the right side of the above congruence are powers of q^3, we have

$$\sum_{n=0}^{\infty} c_5\left(3^{2\alpha}(3n) + \frac{3^{2\alpha+1}+1}{4}\right) q^{3n} \equiv (-1)^{\alpha}(q^3;q^3)_\infty(q^{15};q^{15})_\infty \pmod 3,$$

which yields that

$$\sum_{n=0}^{\infty} c_5\left(3^{2\alpha+1}n+\frac{3^{2\alpha+1}+1}{4}\right)q^n \equiv (-1)^{\alpha+2}(q;q)_\infty(q^5;q^5)_\infty \pmod 3.$$

Now we suppose that

$$\sum_{n=0}^{\infty} c_5\left(3^{2\alpha-1}n+\frac{3^{2\alpha-1}+1}{4}\right)q^n \equiv (-1)^{\alpha+1}(q;q)_\infty(q^5;q^5)_\infty \pmod 3$$

is true for some fixed integer $\alpha \geqslant 1$, to which applying the same argument as in the proof of Corollary 6 yields that

$$\sum_{n=0}^{\infty} c_5\left(3^{2\alpha}n+\frac{3^{2\alpha+1}+1}{4}\right)q^n \equiv (-1)^{\alpha+2}(q^3;q^3)_\infty(q^{15};q^{15})_\infty \pmod 3.$$

The proof is complete. □

As a consequence of Theorem 8, we have the following result.

Corollary 9. *If $\alpha \geqslant 1$ and $n \geqslant 0$,*

$$c_5\left(3^{2\alpha+1}n+\frac{7\times 3^{2\alpha}+1}{4}\right) \equiv 0 \pmod 3, \tag{18}$$

$$c_5\left(3^{2\alpha+1}n+\frac{11\times 3^{2\alpha}+1}{4}\right) \equiv 0 \pmod 3. \tag{19}$$

Proof. Note that all the terms on the right hand side of (17) are of the form q^{3n}. We can immediately obtain (18) and (19) by equating the coefficients of q^{3n+1} and q^{3n+2} on both sides of (17). □

4 More infinite families of congruences for $c_5(n)$

To establish new congruences for $c_5(n)$, we need the following lemma.

Lemma 10. *Let*

$$\sum_{n=0}^{\infty} b(n)q^n = (q;q)_\infty(q^5;q^5)_\infty. \tag{20}$$

Then, for a given prime $p \geqslant 5$ with $\left(\frac{-5}{p}\right) = -1$, we have

$$\sum_{n=0}^{\infty} b\left(pn+\frac{p^2-1}{4}\right)q^n = (q^p;q^p)_\infty(q^{5p};q^{5p})_\infty. \tag{21}$$

Proof. Applying Euler's pentagonal number theorem, we have

$$\sum_{n=0}^{\infty} b(n)q^n = \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{m(3m+1)/2+5n(3n+1)/2}. \tag{22}$$

We now consider

$$\frac{m(3m+1)}{2} + \frac{5n(3n+1)}{2} \equiv \frac{p^2-1}{4} \pmod{p},$$

namely,

$$(6m+1)^2 + 5(6n+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-5}{p}\right) = -1$, we deduce that

$$6m+1 \equiv 6n+1 \equiv 0 \pmod{p}.$$

If $p \equiv 1 \pmod{6}$, then $m \equiv n \equiv \frac{p-1}{6} \pmod{p}$. Let

$$m = kp + \frac{p-1}{6} \text{ and } n = lp + \frac{p-1}{6},$$

we have

$$m(3m+1)/2 + 5n(3n+1)/2 = (p^2-1)/4 + p^2(3k^2+k)/2 + 5p^2(3l^2+l)/2.$$

If $p \equiv -1 \pmod{6}$, then $m \equiv n \equiv \frac{-p-1}{6} \pmod{p}$. Let

$$m = -kp - \frac{p+1}{6} \text{ and } n = -lp - (p+1)/6,$$

we also have

$$m(3m+1)/2 + 5n(3n+1)/2 = (p^2-1)/4 + p^2(3k^2+k)/2 + 5p^2(3l^2+l)/2.$$

Extracting the terms whose power of q is congruent to $\frac{p^2-1}{4}$ modulo p from (22), and employing the above analysis, we obtain

$$\sum_{n=0}^{\infty} b\left(pn + \frac{p^2-1}{4}\right) q^{pn+\frac{p^2-1}{4}} = \sum_{k,l=-\infty}^{\infty} (-1)^{k+l} q^{(p^2-1)/4+p^2(3k^2+k)/2+5p^2(3l^2+l)/2},$$

which can be simplified to

$$\sum_{n=0}^{\infty} b\left(pn + \frac{p^2-1}{4}\right) q^n = \sum_{k,l=-\infty}^{\infty} (-1)^{k+l} q^{p(3k^2+k)/2+5p(3l^2+l)/2}.$$

Applying Euler's pentagonal number theorem again, we derive that

$$\sum_{n=0}^{\infty} b\left(pn + \frac{p^2-1}{4}\right) q^n = (q^p;q^p)_\infty (q^{5p};q^{5p})_\infty,$$

which completes the proof. □

Based on Lemma 10, we can easily obtain the following congruence.

Theorem 11. *If $p \geqslant 5$ is a prime with $\left(\frac{-5}{p}\right) = -1$, we have*

$$\sum_{n=0}^{\infty} c_5\left(3pn + \frac{3p^2+1}{4}\right) q^n \equiv (q^p;q^p)_\infty (q^{5p};q^{5p})_\infty \pmod 3. \tag{23}$$

Proof. It follows from (8) and (20) that

$$c_5(3n+1) \equiv b(n) \pmod 3.$$

Applying Lemma 10, we deduce that

$$\sum_{n=0}^{\infty} c_5\left(3\left(pn + \frac{p^2-1}{4}\right)+1\right) q^n \equiv (q^p;q^p)_\infty (q^{5p};q^{5p})_\infty \pmod 3,$$

which finishes the proof. □

One can generalize the above congruence to the form as we show below.

Theorem 12. *Given a prime $p \geqslant 5$ with $\left(\frac{-5}{p}\right) = -1$, then for all $\alpha \geqslant 1$, we have*

$$\sum_{n=0}^{\infty} c_5\left(3p^{2\alpha-1}n + \frac{3p^{2\alpha}+1}{4}\right) q^n \equiv (q^p;q^p)_\infty (q^{5p};q^{5p})_\infty \pmod 3. \tag{24}$$

Proof. The proof follows by induction on α. The case $\alpha = 1$ is given in Theorem 11. Assuming the result holds for a positive integer $\alpha = t$, namely,

$$\sum_{n=0}^{\infty} c_5\left(3p^{2t-1}n + \frac{3p^{2t}+1}{4}\right) q^n \equiv (q^p;q^p)_\infty (q^{5p};q^{5p})_\infty \pmod 3.$$

Choosing those terms on each side whose power of q is of the form pn, and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} c_5\left(3p^{2t}n + \frac{3p^{2t}+1}{4}\right) q^n \equiv (q;q)_\infty (q^5;q^5)_\infty \pmod 3,$$

which implies that

$$c_5\left(3p^{2t}n + \frac{3p^{2t}+1}{4}\right) \equiv b(n) \pmod 3.$$

Furthermore, from Lemma 10 we see that

$$\sum_{n=0}^{\infty} c_5\left(3p^{2t}\left(pn + \frac{p^2-1}{4}\right) + \frac{3p^{2t}+1}{4}\right) q^n \equiv (q^p;q^p)_\infty (q^{5p};q^{5p})_\infty \pmod 3,$$

which upon simplification completes the induction on α. □

As an immediate consequence of Theorem 12, we obtain the following infinite families of congruences for $c_5(n)$.

Corollary 13. *Given a prime $p \geqslant 5$ with $\left(\frac{-5}{p}\right) = -1$, if $\alpha \geqslant 1$ and $n \geqslant 0$, we have*

$$c_5\left(3p^{2\alpha}n + 3p^{2\alpha-1}i + \frac{3p^{2\alpha}+1}{4}\right) \equiv 0 \pmod{3}, \tag{25}$$

where $i = 1, 2, \ldots, p-1$.

Proof. Collecting those terms on each side of (24) for which the powers of q are of the form $pn + i$, dividing by q^i, and replacing q^p by q, we obtain that for $i = 1, 2, \ldots, p-1$,

$$\sum_{n=0}^{\infty} c_5\left(3p^{2\alpha-1}(pn+i) + \frac{3p^{2\alpha}+1}{4}\right) q^n \equiv 0 \pmod{3},$$

which proves the claim in the corollary. □

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