Strong Turán stability

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Abstract

We study maximal K_{r+1}-free graphs G of almost extremal size—typically, $e(G) = \mathrm{ex}(n, K_{r+1}) - O(n)$. We show that any such graph G must have a large amount of 'symmetry': in particular, all but very few vertices of G must have twins. (Two vertices u and v are *twins* if they have the same neighbourhood.) As a corollary, we obtain a new, short proof of a theorem of Simonovits on the structure of extremal K_{r+1}-free graphs of chromatic number at least k for all fixed $k \geqslant r \geqslant 2$.

Keywords: Forbidden subgraph; stability; saturation

1 Introduction

Let $T_{n,r}$ denote the Turán graph on n vertices with r partition classes of size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$ each, and put $t_{n,r} := e(T_{n,r})$. From Turán's theorem we know that $t_{n,r}$ maximises the size of a K_{r+1}-free graph of order n. One of the best-known extensions of Turán's theorem is the Erdős–Simonovits stability theorem, which says, in particular, that a K_{r+1}-free graph on n vertices and $t_{n,r} - o(n^2)$ edges can be turned into $T_{n,r}$ by adding or removing $o(n^2)$ edges. To phrase it qualitatively, a K_{r+1}-free graph whose size is close to being extremal looks essentially like the extremal graph. This behaviour has become known as *stability* and has been extensively studied in various discrete structures.

In this paper we are concerned with different aspects of Turán stability. More concretely, we shall study K_{r+1}-free graphs G with $e(G) = t_{n,r} - O(n)$ or $e(G) = t_{n,r} - O(n \log n)$. This is much closer to the Turán threshold than the range of the Erdős–Simonovits stability theorem and makes it possible to observe different aspects of stability. Our results can therefore be viewed as a part of a larger programme of studying the

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'phase transition' of K_{r+1}-free graphs near the Turán threshold that has been emphasized by Simonovits.

First, in Section 2 we give a new, short proof of a theorem on the maximum size of a K_{r+1}-free graph of chromatic number at least $r+1$. This result was proved in the case $r=2$ by Andrásfai, Erdős, and Gallai [8]. The general case was first explicitly proved by Brouwer [7], although implicitly it follows from earlier work of Simonovits [15]. It has also been re-discovered several times [4, 9, 11].

Let

$$h(n,r) = \begin{cases} t_{n,r} - \left\lfloor \frac{n}{r} \right\rfloor + 1, & n \geqslant 2r+1, \\ t_{n,r} - 2, & r+3 \leqslant n \leqslant 2r, \end{cases} \tag{1.1}$$

and note that the second case is vacuous if $r=2$.

Theorem 1.1. *If $n \geqslant r+3$, then every K_{r+1}-free graph of order n and size at least $h(n,r)+1$ is r-colourable.*

Unlike the Erdős–Simonovits theorem, which says that a K_{r+1}-free graph on sufficiently many edges is *approximately* r-partite, this theorem gives a condition for a K_{r+1}-free graph to *actually* be r-partite.

A natural generalisation of Theorem 1.1 would be to find the maximal number of edges in a graph G with $|V(G)| = n$, $\omega(G) = r$ and $\chi(G) \geqslant k$ for every $k \geqslant r+1$. It is easy to see that the extremal number is of order $t_{n,r} - O(n)$: for instance, take the disjoint union of a Turán graph $T_{n',r}$ and a finite order graph G' with $\omega(G') = r$ and $\chi(G') \geqslant k$. Note that determining the constant in the linear term asymptotically as $k \to \infty$ is essentially equivalent to determining the asymptotic behaviour of the Ramsey numbers $R(r+1,k)$. (This connection is discussed further in Remark 4.14.) We determine the constant exactly in the first open case, $k = r+2$; see Theorem 4.8.

Another interesting problem is to determine the structure of the extremal K_{r+1}-free graphs of chromatic number at least k. One simple way to construct such graphs (more efficiently than the trivial construction given above) is the following: take a finite order graph G' with $\omega(G') = r$ and $\chi(G') = k$, and blow up an r-clique of G' in a way that maximises the number of edges. Let us call a graph (or, more precisely, a graph sequence) *simple* if it is a blow-up of a bounded order graph. It is natural to ask whether the extremal graph must be simple. This was answered in the affirmative by Simonovits for $r=2$ in [16] and (as part of a more general result) for arbitrary r in [17].

Theorem 1.2. *For each $r \geqslant 2$ and each $k \geqslant r$, there exists $m(k,r)$ such that if G is an extremal K_{r+1}-free graph with chromatic number at least k, then G is a blow-up of a graph G' with $|V(G')| \leqslant m(k,r)$.*

In other words, for every r and k, the sequence of extremal graphs G for $\omega(G) \leqslant r$ and $\chi(G) \geqslant k$ is simple.

We shall use Theorem 1.2 to show that for any r and k, there is a quantity $\Lambda_r(k)$ that determines the constant in the linear term in the extremal number for K_{r+1}-free graphs with chromatic number at least k. Thus, computing $\Lambda_r(k)$ is equivalent to determining the size of an extremal graph up to an additive constant. Moreover, our results show that there exists a finite time algorithm that determines the extremal size exactly.

Recall that a graph G is called *maximal H-free* or *H-saturated* if it is H-free but adding any edge to G would create a copy of H as a subgraph. For $H = K_{r+1}$ the corresponding saturated graphs are also called *$(r+1)$-saturated.* In Section 3 we suggest a new generalisation of Theorem 1.1, namely the study of K_{r+1}-saturated graphs on many edges; note that the extremal K_{r+1}-free graph for a given chromatic number is a special case. In the spirit of Theorem 1.2, we prove sharp bounds on how large $e(G)$ must be in order to guarantee that G is simple. Perhaps surprisingly, the thresholds for $r = 2$ and for $r \geqslant 3$ are substantially different, with the proof being very short in the former case and more involved in the latter.

Theorem 1.3. *For every $c > 0$, there exists a constant $m_2(c)$ such that every 3-saturated graph G on n vertices with $e(G) > t_{n,2} - cn$ is a blow-up of a graph on at most m_2 vertices.*

Let $r \geqslant 3$. For every $\varepsilon > 0$, there exists a constant $m_r(\varepsilon)$ such that every $(r+1)$-saturated graph G on n vertices with $e(G) > t_{n,r} - (2-\varepsilon)n/r$ is a blow-up of a graph on at most m_r vertices.

In both cases, the result is sharp—see Examples 3.5 and 3.7.

Taking this study further, we obtain a sharp threshold for a maximal K_{r+1}-free graph to have a single pair of *twin* vertices (that is, vertices with identical neighbourhoods). Clearly, this threshold must be lower than the bound in Theorem 1.3. We consider the following theorem to be the main result of this paper.

Theorem 1.4. *For every $r \geqslant 2$ there exists a constant $c > 0$ such that if n is sufficiently large, then every $(r+1)$-saturated graph G on n vertices with $e(G) \geqslant t_{n,r} - cn \log n$ has a pair of twin vertices.*

This result is sharp up to the value of the constant c—see Examples 3.9 and 3.11.

Note that unlike Theorem 1.3, in this case the bounds are similar for all values of r, though the proof is still much shorter in the case $r = 2$ (see Proposition 3.8). As a corollary of Theorem 1.4, we obtain a new, short proof of Theorem 1.2.

We consider one more way in which an $(r+1)$-saturated graph may be 'close' to $T_{n,r}$, namely, by having a large complete r-partite subgraph. We shall show that if $r \geqslant 3$ and if c is large enough, then there exist $(r+1)$-saturated graphs of order n with more than $t_{n,r} - cn$ edges that are not simple (see Example 3.7). However, for all $c > 0$, every 4-saturated graph with more than $t_{n,3} - cn$ edges must contain a large complete tripartite subgraph.

Theorem 1.5. *For every $c > 0$, every 4-saturated graph G on n vertices with $e(G) > t_{n,3} - cn$ contains a complete tripartite graph on $(1 - o(1))n$ vertices.*

Let us note that the corresponding minimal degree (rather then graph size) conditions for properties such as low chromatic number in a K_{r+1}-free graph have been extensively studied. These questions will not be in the scope of our discussion. For a discussion of these results, see, e.g., the survey [14, Section 2.4] and the references therein.

The rest of this paper is organised as follows. In Section 2, we prove Theorem 1.1 and classify the extremal graphs. In Section 3, we prove Theorems 1.3, 1.4 and 1.5. In Section 4, we prove Theorem 1.2. We then use this result to show that the quantity $\Lambda_r(k)$ mentioned above determines the size of an extremal K_{r+1}-free graph of chromatic number at least k up to an additive constant. In Section 5, we state open problems arising from our work.

2 A new proof of Theorem 1.1

In this section we prove Theorem 1.1 and classify the extremal K_{r+1}-free, non-r-colourable graphs. As mentioned above, Theorem 1.1 has been proved several times. The proof that we shall give is similar to that of Hanson and Toft [9], in that both arguments use Zykov symmetrization (described below). However, our proof is otherwise rather different to the one in [9], and it is shorter than any of the proofs of Theorem 1.1 of which we are aware.

The following construction shows that the bound in Theorem 1.1 is tight: take a copy of $T_{n-1,r}$ on partition classes $V_1, \ldots, V_r$. Add a new vertex u and connect it to each vertex in $V_3, \ldots, V_r$ and to one vertex from each of V_1 and V_2; call them v_1 and v_2. Lastly, remove the edge v_1v_2. For $n > 2r$, if we take V_1 and V_2 to be the smallest partition classes, this construction achieves the bound of Theorem 1.1. On the other hand, for $r \geqslant 3$ and $r+3 \leqslant n \leqslant 2r$ this construction does not work, since the obtained graph will be r-colourable. Instead, the extremal construction in this case is achieved by taking V_1 and V_2 to be the largest partition classes (of size 2), for a total of $t_{n,r} - 2$ edges. In each case, we call the resulting graph $G_{n,r}$. It is easy to verify that $G_{n,r}$ is K_{r+1}-free and is not r-colourable. Finally, for $n \leqslant r+2$, every K_{r+1}-free graph on n vertices is r-colourable. Note that in general the extremal graphs are not unique; we shall discuss this after the proof of Theorem 1.1.

As mentioned above, in the proof of Theorem 1.1, we shall want to apply the *Zykov symmetrization*, defined as follows. Given a graph G and independent vertices $u, v \in V(G)$, define $Z_{u,v}(G)$ to be the graph obtained by replacing u with a twin of v. That is, we delete all edges incident to u and insert edges between u and the neighbours of v instead. Note that 'being twins' is an equivalence relation, giving rise to *twin classes*.

It is easy to see that $\omega(Z_{u,v}(G)) = \omega(G \setminus \{u\})$ and $\chi(Z_{u,v}(G)) = \chi(G \setminus \{u\})$, and, as a consequence,

$$\omega(G) - 1 \leqslant \omega\big(Z_{u,v}(G)\big) \leqslant \omega(G) \tag{2.1}$$

and

$$\chi(G) - 1 \leqslant \chi\big(Z_{u,v}(G)\big) \leqslant \chi(G). \tag{2.2}$$

Thus, if $\deg(u) < \deg(v)$, replacing G with $Z_{u,v}(G)$ increases $e(G)$, does not increase $\omega(G)$ and decreases $\chi(G)$ by at most 1. Similarly, if $\deg(u) = \deg(v)$, then we may apply either $Z_{u,v}$ or $Z_{v,u}$, with the same effects on $\omega(G)$ and $\chi(G)$, while keeping $e(G)$ unchanged. Let us call the Zykov symmetrization $Z_{u,v}$ *increasing* or an *IZS* if $\deg(u) \leqslant \deg(v)$. The following lemma is due to Zykov himself [18] and leads to his well-known proof of Turán's theorem. For the sake of self-containment we shall recall its short proof here.

Lemma 2.1. *If $\omega(G) \leqslant r$, then there exists a sequence of increasing Zykov symmetrizations transforming G into a complete s-partite graph for some $s \leqslant r$.*

Proof. Observe that every pair of twin classes forms either an empty or a complete bipartite graph. In the former case we can repeatedly apply an IZS and merge the two classes into one. We continue doing so until there are no missing edges between vertices from different twin classes, after which the obtained graph G' is complete s-partite, where s is the number of twin classes remaining. By (2.1), $\omega(G') \leqslant \omega(G) \leqslant r$, so we must have $s \leqslant r$, which proves the lemma. $\square$

Proof of Theorem 1.1. We shall show that any non-r-colourable, K_{r+1}-free graph G must have at most as many edges as $G_{n,r}$.

Step 1: We first use the Zykov symmetrization. Recall that the initial graph G satisfies $\chi(G) > r$, and by (2.2), with each IZS the chromatic number decreases by at most 1. Therefore, by Lemma 2.1, it suffices to prove that $e(G) \leqslant h(n,r)$ for every graph G such that $\omega(G) \leqslant r$, $\chi(G) = r+1$ and $\chi(Z_{u,v}(G)) = r$ for some increasing $Z_{u,v}$. The latter implies that $\chi(G \setminus \{u\}) = r$, which in turn means that G can be properly $(r+1)$-coloured such that u is the only vertex with its colour.

So from now on let us assume that $V(G)$ can be split into $r+1$ independent sets $V_1, \ldots, V_r$, and $\{u\}$. Observe that for each i, u must have a neighbour $v_i \in V_i$, for otherwise we could add u to some V_i and so r-colour G. Furthermore, at least one edge between some v_i and v_j is missing, for otherwise the v_i and u would induce a copy of K_{r+1}.

Step 2: We now apply a series of edge switches as follows. If two neighbours of u in different partition classes, say $v \in V_i$ and $w \in V_j$, are not adjacent and u has another neighbour in either V_i or V_j, say $v' \in V_i$, then we remove the edge uv and add the edge vw. At this point it does not matter what happens to $\chi(G)$. However, it is crucial that after this switch the resulting graph $\tilde{G}$ is K_{r+1}-free. Indeed, because $G[V \setminus \{u\}]$ is r-partite, any copy F of K_{r+1} in $\tilde{G}$ must contain u. This means that F cannot contain v, which implies that a copy of K_{r+1} must already be present in G, a contradiction.

Continue the switches for as long as possible; the procedure will terminate since the degree of u decreases after each switch. Once no more switches are possible, we end up with a graph G' such that u has precisely one neighbour in two of the partition classes, say $v_1 \in V_1$ and $v_2 \in V_2$, with no edge between v_1 and v_2.

Step 3: Now add all missing edges between every V_i and V_j with $i \neq j$ except for $v_1 v_2$, and between u and every V_i with $i \geqslant 3$. The obtained graph G is $(r+1)$-chromatic and contains no K_{r+1}. Moreover, its size is maximised if the sizes of $V_1, \ldots, V_r$ are as close as possible, resulting in $e(G) = h(n,r)$. $\square$

As was mentioned earlier, in general, the graph $G_{n,r}$ described above is not the unique extremal example. We shall characterise the extremal graphs. (They were also characterised in some of the earlier papers cited above). We let

$$s = s(n,r) = \lfloor n/r \rfloor. \tag{2.3}$$

Given ℓ, $1 \leqslant \ell \leqslant s-1$, let $G_{n,r}^{(\ell)}$ be the graph obtained from $G_{n,r}$ as follows: let $W \subset V_1$ with $|W| = \ell$, and, for each $w \in W$, add the edge uw and remove the edge $v_2 w$. (Note that $G_{n,r} = G^{(1)}_{n,r}$.) If $n = kr + 2$ for some $k \geqslant 2$, then V_1 and V_2 have different sizes. Without loss of generality, let $|V_1| = |V_2|+1 = k+1$. In this case, we may also modify $G_{n,r}$ by connecting u to a set $W' \subset V_2$ with $|W'| = \ell$ and disconnecting v_1 from all elements of W'. Let $G'^{(\ell)}_{n,r}$ denote the resulting graph and observe that $G'^{(\ell)}_{n,r} \ncong G^{(\ell)}_{n,r}$. Note that for $1 \leqslant \ell \leqslant s-1$, both $G^{(\ell)}_{n,r}$ and $G'^{(\ell)}_{n,r}$ are $(r+1)$-chromatic and K_{r+1}-free.

Theorem 2.2. *Let $r \geqslant 2$ and $n \geqslant r+3$. Let $h(n,r)$ and s be as in* (1.1) *and* (2.3)*, respectively. If G is a K_{r+1}-free graph of order n and size $h(n,r)$ that is not r-colourable, then there exists some $1 \leqslant \ell \leqslant s-1$ such that $G \cong G^{(\ell)}_{n,r}$, or, if $n = kr+2$ for some $k \geqslant 2$, then there exists some $1 \leqslant \ell \leqslant s-1$ such that either $G \cong G^{(\ell)}_{n,r}$ or $G \cong G'^{(\ell)}_{n,r}$.*

In the proof of Theorem 1.1, we provided an algorithm for transforming any non-r-colourable K_{r+1}-free graph into the graph $G_{n,r}$ without decreasing its size. We classify the extremal graphs by carefully examining this procedure in the case where the size of the graph never increases. The proof of Theorem 2.2 is given in the Appendix.

3 Clique-saturated graphs

In this section we shall prove a number of stability results for $(r+1)$-saturated graphs near the Turán threshold, including Theorems 1.3, 1.4 and 1.5.

We shall make frequent use of the following result of Andrásfai, Erdős and Sós [5].

Theorem 3.1. *Let $r \geqslant 2$. If a graph G on n vertices is K_{r+1}-free and not r-colourable, then there exists $v \in V(G)$ such that*

$$\deg(v) \leqslant \frac{3r-4}{3r-1}\, n.$$

We shall also often use the following immediate corollary of Theorem 3.1.

Corollary 3.2. *There exists a function $g(r,c)$ such that the vertex set of every K_{r+1}-free graph G of order n with $e(G) \geqslant t_{n,r} - cn$ can be split into a set F with $|F| \leqslant g(r,c)$ and an r-partite graph on $V(G) \setminus F$.*

Proof. Take F to be the set of vertices of G with degree at most $\frac{3r-4}{3r-1}n$. This set must be of bounded size, otherwise $e(G) \leqslant t_{n-|F|,r} + \frac{3r-4}{3r-1}n|F| < t_{n,r} - cn$, a contradiction. The remaining vertices induce, by Theorem 3.1, an r-partite graph. □

3.1 Finite-size reductions

Here we prove Theorem 1.3, which gives the minimum number of edges that guarantees that an $(r+1)$-saturated graph is simple.

We begin by proving Theorem 1.3 in the case $r = 2$.

Theorem 3.3. *For every $c \geqslant 0$ there exists $m_2(c)$ such that every 3-saturated graph G on n vertices with $e(G) > t_{n,2} - cn$ is a blow-up of a (triangle-free) graph H with $|V(H)| \leqslant m_2$.*

Proof. If G is bipartite, then because it is 3-saturated, it must be complete bipartite, and we are done. If G is not bipartite, then by Corollary 3.2 it is composed of a large bipartite graph $G_b = (U, W, E_b)$ and an exceptional vertex set V_e with $|V_e| \leqslant g(2, c)$. Now, partition the vertices of U and W according to their V_e-neighbourhoods: for every $X \subset V_e$, define

$$U_X := \{u \in U \,:\, N_{V_e}(u) = X\},$$

and W_X analogously. Let $u \in U$ and $w \in W$, and let $X = N_{V_e}(u)$ and $Y = N_{V_e}(w)$, so that $u \in U_X$ and $w \in W_Y$. If $X \cap Y = \emptyset$, then u and w must be adjacent, since G is 3-saturated. On the other hand, if $X \cap Y \neq \emptyset$, there can be no edge between u and w, as it would create a triangle. Hence, the neighbourhoods of u and w are completely determined by their V_e-neighbourhoods, meaning that for any $X \subset V_e$, any two vertices $u_1, u_2 \in U_X$ are twins (the same holds in W_X). Since there are at most $2^{|V_e|}$ possible V_e-neighbourhoods, we conclude that G has at most $|V_e| + 2 \cdot 2^{|V_e|}$ twin classes. Thus, the statement of the theorem holds with $m_2(c) = g(2, c) + 2 \cdot 2^{g(2,c)}$. ☐

As observed earlier, extremal triangle-free, ($\geqslant k$)-chromatic graphs are 3-saturated. As was mentioned in the Introduction, it is easy to construct a triangle-free, ($\geqslant k$)-chromatic graph with $t_{n,2} - c_k n$ edges. Thus, as an immediate corollary of Theorem 3.3 we obtain Theorem 1.2 (Simonovits' Theorem) for $r = 2$.

Corollary 3.4. *For each $k \geqslant 2$ there exists a constant $m(k, 2)$ such that if G is an extremal triangle-free, ($\geqslant k$)-chromatic graph, then G is a blow-up of a graph G' with $|V(G')| \leqslant m(k, 2)$.*

The following construction demonstrates that the bound of Theorem 3.3 is sharp in the following sense: given a function $f(n)$ that tends to infinity (no matter how slowly), there exist 3-saturated graphs G with $e(G) = t_{n,2} - nf(n)$, yet with an unbounded number of twin classes.

Example 3.5. We may assume that $f(n) < \frac{1}{2}\log_2 n$. Let S be a set of $f(n)$ vertices, let U and W be disjoint sets of $2^{f(n)}$ vertices each, and divide the rest of the vertices equally into two sets U' and W'. We give different vertices of U distinct neighbourhoods in S and do the same for vertices in W: for each $I \subset S$, let u_I be the vertex in U with $N_S(u_I) = I$, and define w_I similarly. Join u_I and w_J if and only if I and J are disjoint. Finally, add all edges between U' and W', between U' and W, and between U and W'. It is not hard to see that the resulting graph G is 3-saturated. Also, G has at least $2^{f(n)+1} + f(n)$ distinct neighbourhoods.

Since $f(n) < \frac{1}{2}\log_2 n$, we obtain

$$\begin{aligned} e(G) &> |U'||W'| + |U'||W| + |U||W'| > t_{n-f(n),2} - 2^{2f(n)} \\ &> t_{n,2} - \frac{nf(n)}{2} - 2^{2f(n)} > t_{n,2} - nf(n), \end{aligned}$$

as claimed.

Now we prove Theorem 1.3 for all $r \geqslant 3$.

Theorem 3.6. *For every $r \geqslant 3$ and every $\varepsilon > 0$ there exists $m_r(\varepsilon)$ such that every $(r+1)$-saturated graph G on n vertices with $e(G) > t_{n,r} - (2-\varepsilon)n/r$ is a blow-up of a (K_{r+1}-free) graph H with $|V(H)| \leqslant m_r$.*

Proof. Let F be the set of vertices of G with degree at most $(1 - 1/r - \varepsilon/(2r^2+3))n$. By arguing as in the proof of Corollary 3.2, we find that F has bounded size and $G[V \setminus F]$ is r-partite; call its partition classes $V_1, \ldots, V_r$. We claim that for any set $X \subset F$ with $k \leqslant r-2$ vertices, the common neighbourhood of the vertices in X contains at least $\varepsilon n/(r^2+1)$ vertices from each of some $r-k-1$ partition classes. Indeed, if $X \subset F$ does not satisfy the claim, then

$$\sum_{x \in X} \deg(x) \leqslant kn - (k+2)\left(\frac{n}{r} - \frac{\varepsilon n}{r^2+1}\right) + O(1).$$

Because there are at most $\binom{k}{2} = O(1)$ edges between vertices of X and $k \leqslant r-2$, we have

$$\begin{aligned} e(G \setminus X) &> t_{n,r} - \frac{2-\varepsilon}{r}n - \frac{k(r-1)-2}{r}n - \frac{k+2}{r^2+1}\varepsilon n + O(1) \\ &= t_{n,r} - \frac{k(r-1)}{r}n + \frac{r^2+1-r(k+2)}{r(r^2+1)}\varepsilon n + O(1) \\ &> t_{n-k,r}, \end{aligned}$$

a contradiction.

Given $X \subset F$ with $k \leqslant r-2$ vertices, call the $k+1$ partition classes in which the vertices of X have the smallest number of common neighbours *X-small* and the remaining ones *X-big*. By the assumption on degrees in $V \setminus F$, if v_1 and v_2 are vertices in X-small classes that are both in the common neighbourhood of X, then they cannot be adjacent: v_1 and v_2 have at most $2\varepsilon n/(2r^2+3) + O(1) < \varepsilon n/(r^2+1)$ total non-neighbours in each X-big class V_i, so they must have a common neighbour in $N_{V_i}(X)$. Thus, if v_1 and v_2 were adjacent, then v_1, v_2, the vertices of X and their common neighbour in each of the $r-k-1$ X-big classes would induce a copy of K_{r+1}.

Now consider two arbitrary non-adjacent vertices in different partition classes, say $v_1 \in V_1$ and $v_2 \in V_2$. Let $J = N_F(v_1) \cap N_F(v_2)$. We shall show that any two vertices $w_1 \in V_1$ and $w_2 \in V_2$ with $N_F(w_1) \cap N_F(w_2) = J$ are also not adjacent. This will imply that the neighbourhood of every vertex in $V \setminus F$ is determined by its F-neighbourhood, resulting in a finite number of twin classes.

Since G is $(r+1)$-saturated, adding the edge v_1v_2 would create a copy of K_{r+1}. Hence, there exist sets $X \subset F$ and $Y \subset V \setminus F$ such that if the edge v_1v_2 were present, then $\{v_1, v_2\} \cup X \cup Y$ would induce a copy of K_{r+1}. If $|X| = r-1$, then we are done immediately, as adding the edge w_1w_2 would, using X, also create a copy of K_{r+1}. So we may assume that $|X| = k \leqslant r-2$ and apply the above split into X-big and X-small classes.

Denote the vertices of Y by $v_3, \ldots, v_{r-k+1}$; different vertices must lie in different V_i. Since the number of big classes is $r-k-1$, two of the vertices $v_1, \ldots, v_{r-k+1}$ must be in small classes. As shown above, these two vertices cannot be adjacent, and the only way this can happen is if v_1 and v_2 are in small classes. In this case, w_1 and w_2 are also in X-small classes, and are therefore not adjacent, as claimed. $\square$

The following construction shows that Theorem 3.6 is sharp up to a $o(n)$ error term, which means that the thresholds for an $(r+1)$-saturated graph to be simple are truly different in the cases $r = 2$ and $r \geqslant 3$.

Example 3.7. Let $n \in \mathbb{N}$, let $m = (1/2)\log_2 n$ and let $M = \binom{m}{m/2}$; note that $M < \sqrt{n}$. Take the Turán graph $T_{n-1,r}$ and let $V_1, \ldots, V_r$ denote its partition classes. Let $W_1 \subset V_1$, $W_2 \subset V_2$ and $W_3 \subset V_3$ with $|W_1| = M$ and $|W_2| = |W_3| = m$. Add a new vertex v to G and join it to all of the vertices of the W_i and to all of the vertices of V_j for $j \notin \{1, 2, 3\}$. Remove all edges between different W_i. The resulting graph G' satisfies

$$e(G') \geqslant t_{n-1,r} - 2mM - m^2 + \left\lfloor \frac{r-3}{r}(n-1) \right\rfloor + M + 2m = t_{n,r} - \frac{2n}{r} + o(n).$$

Now we add a matching between W_2 and W_3. Also, for each $w \in W_1$ we select a subset $U_w \subset W_2$ of size $m/2$ such that different vertices of W_1 correspond to distinct subsets. We then join w to U_w in W_2 and to $W_3 \setminus N_{W_3}(U_w)$ in W_3. (Observe that we have added only $m + mM = o(n)$ edges.)

It is easy to check that the obtained graph G is $(r+1)$-saturated. Moreover, no vertices in W_1 are twins, so G has an unbounded number of twin classes.

3.2 Twin-free saturated graphs

Now we prove Theorem 1.4, which gives an upper bound on the number of edges in a twin-free, $(r+1)$-saturated graph.

We begin by proving Theorem 1.4 in the case $r = 2$.

Proposition 3.8. *For each $\varepsilon > 0$, if n is sufficiently large, then every 3-saturated graph G of order n with $e(G) > n^2/4 - (1/20 - \varepsilon)n \log_2 n$ contains a pair of twins.*

Proof. The argument is similar to the proof of Theorem 3.3. Let G be as in the statement of the proposition. By Theorem 3.1, we can produce a bipartite subgraph of G by iteratively removing a set F of m vertices of degree at most $2n/5 = n/2 - n/10$. Hence, each vertex removed increases the average degree of the remaining graph by $1/10 + o(1)$. As $G \setminus F$ is triangle-free, it has average degree at most $|V(G) \setminus F|/2$, and so the bound on $e(G)$ implies that $m < (1 - \varepsilon')\log_2 n$, i.e., that $(n - m)/2 > 2^m$.

Let V_1 and V_2 be the partition classes of $G \setminus F$. By the same argument as in the proof of Theorem 3.3, the neighbourhoods of vertices in V_1 and V_2 are determined by their neighbourhoods in F. Then the bound on m implies that two vertices of the larger partition class will have the same F-neighbourhood, which means that they are twins. $\square$

Now we show that Proposition 3.8 is best possible up to a constant factor in the $n \log_2 n$-term.

Example 3.9. Fix m and let $n = 2m + 4\log_2 m$. We build a graph G on n vertices as follows. Let S_1, S_2, U_1, U_2, B_1 and B_2 be pairwise disjoint sets of vertices with $|B_i| = m$ and $|S_i| = |U_i| = \log_2 m$ for $i = 1, 2$. Add all edges between S_1 and S_2, between U_1 and U_2 and between B_1 and B_2. Give different vertices of B_1 distinct neighbourhoods in S_2, and similarly for B_2 and S_1. Place matchings between U_1 and S_1 and between U_2 and S_2. Finally, if $u_1 \in U_1$ and $s_1 \in S_1$ are adjacent, then we join u_1 to all vertices of $B_2 \setminus N_{B_2}(s_1)$, and similarly for each $u_2 \in U_2$ and its neighbour $s_2 \in S_2$.

It is easy to see that G is twin-free and 3-saturated. Furthermore, each vertex in $B := B_1 \cup B_2$ has m neighbours in B and $\log_2 m$ neighbours outside. Thus,

$$e(G) > m^2 + 2m\log_2 m = t_{n,2} - \big(1 + o(1)\big)n\log_2 n.$$

Next we prove Theorem 1.4 for every $r \geqslant 3$.

Proof of Theorem 1.4. Let G be an $(r+1)$-saturated graph on n vertices with no twins. Our aim is to show that, provided n is sufficiently large, $e(G) \leqslant t_{n,r} - c'n\log n$ for some constant $c'(r)$ ¿ 0.

We may assume that G is not r-partite: as observed earlier, if G is r-partite, then it must be complete r-partite, which implies that every vertex has a twin. Let F_1 be the set of vertices of degree at most $(3r-4)n/(3r-1)$ given by Theorem 3.1. Note that $G[V \setminus F_1]$ is r-partite and $|F_1| \leqslant c_1 \log n$ for any constant $c_1 > 0$ (otherwise $e(G) \leqslant t_{n-|F_1|,r} + \frac{3r-4}{3r-1}n|F_1| \leqslant t_{n,r} - c_2 n\log n$ for some positive constant c_2 and we are done). In particular, we may assume that

$$\big|F_1\big| \leqslant \frac{1}{2(r-2)}\log_2 n. \tag{3.1}$$

Let $V_1, \ldots, V_r$ be the colour classes of $G[V \setminus F_1]$; each of the V_i must have $(1/r - o(1))n$ vertices, or else $e(G) < t_{n,r} - \omega(n^2)$, and we are done.

We partition the vertices of V_1 according to their neighbourhoods in F_1. Let W_1 be an arbitrary partition class and let $A_1 \subset F_1$ denote the common neighbourhood of the vertices of W_1. If $|W_1| > 1$, we may assume that $A_1 \neq \emptyset$, for otherwise, if $w \in W_1$ and $v \in V_i$ for some $i \neq 1$, then we may add the edge wv to G without creating a copy of K_{r+1}. However, G is assumed to be $(r+1)$-saturated, which means that all such edges are already present; consequently, all vertices in W_1 are twins, a contradiction.

Let H_1 be the $(r-1)$-uniform hypergraph with vertex set $V_2 \cup \cdots \cup V_r$ consisting of all cliques $(v_1, \ldots, v_{r-1})$ such that $v_1, \ldots, v_{r-1}$ are all adjacent to some $a \in A_1$. Let C_1 be a minimum vertex cover of H_1. If $|C_1| \geqslant \frac{1}{r^2}\log_2|W_1|$, then we stop. Otherwise, we set $F_2 = A_1 \cup C_1$ and partition the vertices of W_1 according to their neighbourhoods in F_2.

We continue the process inside each partition class as follows. After the jth partition, we consider sets W_j, each of which has common neighbourhood $A_j \subset F_j$. Once again, if $|W_j| > 1$, then we may assume that $A_j \neq \emptyset$. We let H_j be the $(r-j)$-uniform hypergraph with vertex set $V_2 \cup \cdots \cup V_r$ consisting of all cliques $(v_1, \ldots, v_{r-j})$ such that $v_1, \ldots, v_{r-j}$ form an r-clique with some $a_1, \ldots, a_j \in A_j$. We let C_j be a minimum vertex cover of H_j. If $|C_j| < \frac{1}{r^2}\log_2|W_j|$, then we set $F_{j+1} = A_j \cup C_j$ and continue. Otherwise, we stop.

We shall show that this process must stop after at most $r-2$ steps. Observe that H_{r-2} is a graph. Suppose for a contradiction that $|C_{r-2}| < \frac{1}{r^2}\log_2|W_{r-2}|$. (In fact, our argument shows that $|C_{r-2}| \geqslant \log_2|W_{r-2}|$.) We have assumed that none of the vertices in W_{r-2} are twins, but our assumption on $|C_{r-2}|$ means that there must exist $w_1, w_2 \in W_{r-2}$ such that $N_{C_{r-2}}(w_1) = N_{C_{r-2}}(w_2)$. Hence, there exists $s \notin C_{r-2}$ such that $sw_1 \in E(G)$ but $sw_2 \notin E(G)$. Because G is $(r+1)$-saturated, there exists a set K of $r-1$ vertices such that if we added the edge sw_2 to G, then s, w_2, and the vertices of K would induce a copy of K_{r+1}. Observe that A_{r-2} contains exactly $r-2$ vertices of K. Indeed, if $|A_{r-2} \cap K| \leqslant r-3$, then some edge of H_{r-3} would be disjoint from F_{r-2}, contradicting the definition of F_{r-2}. On the other hand, if A_{r-2} contained all $r-1$ vertices of K, then $\{s, w_1\} \cup K$ would induce a copy of K_{r+1} in G, which is again a contradiction.

Let s' be the vertex of K that is not contained in A_{r-2}. Note that this implies that $s'w_1 \notin E(G)$. Then the fact that w_2 is adjacent to $s' \in K$ and our assumption that $N_{C_{r-2}}(w_1) = N_{C_{r-2}}(w_2)$ mean that $s' \notin C_{r-2}$. However, the definition of s' also implies that ss' is an edge in H_{r-2} with no vertices in C_{r-2}, a contradiction.

Thus, for some $j \leqslant r-2$, we have $\tau(H_j) \geqslant \frac{1}{r^2}\log_2|W_j|$, where τ denotes the size of a minimum vertex cover. It is well known that if H is a t-uniform hypergraph, then $\tau(H) \leqslant t\nu(H)$ (simply remove the vertices of a maximum matching), which means that we have $\nu(H_j) \geqslant c\log_2|W_j|$. Let M be a maximum matching of H_j and let $(v_1, \ldots, v_{r-j}) \in M$. For each $w \in W_j$, one of the edges $wv_1, \ldots, wv_{r-j}$ is absent from G, because by the definition of H_j, there exist vertices $a_1, \ldots, a_j \in A_j$ that form a clique of size r with the v_i. Thus, there are at least $c|W_j|\log_2|W_j|$ non-edges between W_j and $V(H_j)$.

The procedure above defines a partition $\mathcal{W}$ of V_1. In view of the argument above, we wish to bound $\sum_{W\in\mathcal{W}}|W|\log_2|W|$ from below. We begin by observing that, by Jensen's inequality,

$$\sum_{W\in\mathcal{W}} \frac{|V_1|}{|W|}\log_2\frac{|V_1|}{|W|} \geqslant -\log_2|\mathcal{W}|,$$

which is equivalent to the inequality

$$\sum_{W\in\mathcal{W}} |W|\log_2|W| \geqslant |V_1|\log_2\frac{|V_1|}{|\mathcal{W}|}. \tag{3.2}$$

Because $|V_1| = (1/r - o(1))n$, in order to obtain the claimed upper bound on $e(G)$, it is therefore enough to show that there exists $\varepsilon > 0$ such that $|\mathcal{W}| \leqslant n^{1-\varepsilon}$.

Recall that at each step j of the partition process, we refined W_{j-1} by considering vertices with the same neighbourhood in $F_j := A_{j-1} \cup C_{j-1}$. Thus, for all $j > 1$, we have

$$|F_j| = |A_{j-1}| + |C_{j-1}| \leqslant |F_{j-1}| + |C_{j-1}|. \tag{3.3}$$

If $W \in \mathcal{W}$ is such that we do not partition W after step j, then $|C_{j-1}| < \frac{1}{r^2}\log_2|W_{j-1}|$. Thus, if we iterate the upper bound on $|F_j|$ in (3.3) and apply (3.1), we find that

$$|\mathcal{W}| \leqslant 2^{\sum_{i=1}^{r-2}|F_i|} \leqslant 2^{(r-2)|F_1| + \sum_{i=1}^{r-3}(r-2-i)|C_i|} < 2^{\frac{1}{2}\log_2 n + \frac{(r-2)^2}{2r^2}\log_2 n} \leqslant n^{1-\varepsilon}, \tag{3.4}$$

which is what we wanted.

It follows from (3.2) and (3.4) that

$$e(G) \leqslant t_{n,r} - c|V_1| \log_2 \frac{|V_1|}{|\mathcal{W}|} \leqslant t_{n,r} - c'\varepsilon n \log_2 n,$$

as claimed. This completes the proof. □

Observe that in the proof of Theorem 1.4, we did not need to assume that G was twin-free, only that it contained a twin-free independent set of size cn for some $c > 0$. Thus, Theorem 1.4 (or, more precisely, its proof) has the following corollary.

Corollary 3.10. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if G is an $(r+1)$-saturated graph of order n and $e(G) \geqslant t_{n,r} - \delta n \log n$, then at least $n - \varepsilon n$ vertices of G have twins.*

When we apply Corollary 3.10, we shall only use that if G is $(r+1)$-saturated and has a twin-free set of size cn, then $e(G) \leqslant t_{n,r} - nf(n)$, where $f(n)$ tends to infinity with n.

Now we show that Theorem 1.4 is best possible up to the value of the constant c.

Example 3.11. The construction is similar to Example 3.7. For n sufficiently large, we construct a twin-free, $(r+1)$-saturated graph on n vertices as follows. Let H be the disjoint union of $T_{n-r,r}$ and r isolated vertices $u_1, \ldots, u_r$. Let $V_1, \ldots, V_r$ denote the colour classes of the copy of $T_{n-r,r}$.

Let m be a quantity to be defined later and let $M = \binom{m}{m/2}$. We partition $V_1 \cup \cdots \cup V_r$ into three families of sets $\{W_1^{(i)}\}_{i=1}^r$, $\{W_2^{(i)}\}_{i=1}^r$ and $\{W_3^{(i)}\}_{i=1}^r$ such that for each i, we have $W_1^{(i)} \subset V_i$, $W_2^{(i)} \subset V_{i+1}$ and $W_3^{(i)} \subset V_{i+2}$ (where the addition is modulo r), as well as that $|W_1^{(i)}| = M$ and $|W_2^{(i)}| = |W_3^{(i)}| = m$. It follows that

$$n = r(M + 2m + 1). \tag{3.5}$$

Because $m = o(M)$, (3.5) implies that

$$M \sim n/r, \tag{3.6}$$

which in turn implies that

$$m \sim \log_2 n. \tag{3.7}$$

Now we modify H in order to make it twin-free and maximally K_{r+1}-free. For each i, $i = 1, \ldots, r$, we modify $H[W_1^{(i)} \cup W_2^{(i)} \cup W_3^{(i)}]$ as in Example 3.7. Then we connect u_i to all vertices of $W_1^{(i)} \cup W_2^{(i)} \cup W_3^{(i)}$ and to all vertices of each V_k, $k \notin \{i, i+1, i+2\}$ (mod r). Finally, we add a maximal set of edges among the u_i in order to saturate the graph.

Let G denote the resulting graph. It is easy to check that G is both $(r+1)$-saturated and twin-free. Moreover,

$$e(G) = t_{n-r,r} - r(2Mm + m^2 - Mm - m) + r\Big(\frac{r-3}{r}(n-r) + 2m + M\Big) + e\big(G[\{u_1, \ldots, u_r\}]\big).$$

It then follows from (3.5), (3.7) and (3.6) that

$$e(G) = t_{n,r} - r(Mm + m^2 - m) + O(n) = t_{n,r} - rMm + O(n) = t_{n,r} - n\log_2 n + O(n),$$

which is what we wanted to show.

The results of this subsection show that the threshold for the property that an $(r+1)$-saturated graph G has a pair of twins is $e(G) = t_{n,r} - \Theta(n\log_2 n)$. We have not attempted to locate the threshold precisely, and leave this as an open problem.

3.3 Large complete r-partite subgraphs

In this section, we prove Theorem 1.5, which says that if $c > 0$, then every 3-saturated graph of order n with at least $t_{n,3} - cn$ edges contains an almost-spanning complete tripartite subgraph. We shall need two basic facts.

Lemma 3.12. *If G is triangle-free tripartite graph on (m,m,m) vertices then $e(G) \leqslant t_{3m,3} - \frac{1}{4}m^2$.*

Proof. Observe that if G has average degree at most $3m/2$, then we are done. So, letting V_1, V_2 and V_3 denote the colour classes of G, we may assume that there exists a vertex, say $v \in V_1$, with at least $m/2$ neighbours in each of V_2 and V_3. As G is triangle-free, $N(v)$ must be an independent set, and so $e(G) \leqslant t_{3m,3} - \frac{1}{4}m^2$, as claimed. □

As an immediate consequence, we obtain:

Lemma 3.13. *If G is triangle-free tripartite graph on (a,b,c) vertices, where $a \leqslant b \leqslant c$, then $e(G) \leqslant t_{a+b+c,3} - \frac{1}{4}\left\lfloor \frac{b}{a} \right\rfloor a^2$.*

Proof. Simple observe that $T_{a+b+c,3}$ contains $\lfloor b/a \rfloor$ edge-disjoint copies of $T_{3a,3}$ and apply Lemma 3.12. □

(Let us note that the maximum size of a triangle-free tripartite graph is studied in detail in [6].)

Now we are ready to give the proof of Theorem 1.5.

Proof of Theorem 1.5. By Corollary 3.2, we may assume that there exists a set $V_e \subset V$ with $|V_e| \leqslant M = M(c)$ such that $G[V \setminus V_e]$ is tripartite. Let V_1, V_2 and V_3 be the partition classes of $V \setminus V_e$. For each $v \in V_e$, define A_v, B_v and C_v to be its neighbourhoods in V_1, V_2 and V_3 such that $|A_v| \leqslant |B_v| \leqslant |C_v|$.

First note that for every $v \in V_e$, $G[A_v \cup B_v \cup C_v]$ is triangle-free and tripartite. It follows from Lemma 3.12 that $|A(v)| = O(\sqrt{n})$, for otherwise $e(G) \leqslant t_{n,3} - \omega(n)$, a contradiction.

Next we pick a (large) constant C and split V_e into 'small' and 'large' vertices: that is, we set $V_e = V_s \cup V_\ell$, where $v \in V_s$ if $|A_v| < C$ and $v \in V_\ell$ otherwise. Notice that if $v \in V_\ell$, then by Lemma 3.13 we have $|B_v| \leqslant c'n$, where

$$c' = \left(\frac{4c}{C} + o(1)\right).$$

Now consider the set

$$W := V \setminus \left(V_e \cup \bigcup_{v \in V_e} A_v \cup \bigcup_{v \in V_\ell} B_v \right).$$

Putting $W_i := W \cap V_i$, we have that $|W_i| > n(1/3 - c'M) + o(n)$ for each i. Let $U = V_e \cup \bigcup_{v \in V_s} A_v$ and note that $|U| \leqslant (C+1)M$. We now partition the vertices of W into a finite number of classes according to their neighbourhoods in U. Let $w_1 \in W_1$ and $w_2 \in W_2$. We claim that whether w_1 and w_2 are adjacent depends solely on their neighbourhoods in U. If $N_U(w_1) \cap N_U(w_2)$ is not independent, then $w_1 \not\sim w_2$, for otherwise we would have a copy of K_4. On the other hand, if $N_U(w_1) \cap N_U(w_2)$ is independent but w_1 and w_2 are not adjacent, then there exist $v \in V_e$ and $u \in V_3$ such that w_1, w_2, v and u would induce a copy of K_4 if the edge $w_1 w_2$ were added. By the definition of W, this can only happen if $v \in V_s$ and $u \in A_v$. But then $u, v \in U$, so $N_U(w_1) \cap N_U(w_2)$ is not independent, a contradiction. This proves the claim.

To summarise, W can be split into at most $3 \cdot 2^{|U|} \leqslant 3 \cdot 2^{(C+1)M}$ classes such that each pair of classes induces either an empty or a complete bipartite graph. Now consider only those classes in each W_i that are of size at least $2\sqrt{(M+c)n}$. Each pair of them belonging to different W_i must form a complete bipartite graph, otherwise $e(G) \leqslant t_{n,3} - cn$, a contradiction. Hence, their union forms a complete tripartite graph with at least

$$|W_i| - 2^{(C+1)M} \cdot 2\sqrt{(M+c)n} > n(1/3 - c'M) + o(n)$$

vertices in each colour class. Since C was arbitrary and $c' \to 0$ as $C \to \infty$, this gives a complete tripartite graph on $(1 - o(1))n$ vertices, which is what we wanted. □

We do not have corresponding results for K_{r+1}-saturated graphs with $r \geqslant 4$, and so we leave this as an open problem.

4 Extremal Graphs for the Chromatic Turán Problem

In this section we shall apply Theorem 1.4 to give a new proof of Theorem 1.2 for $r \geqslant 3$ that is much shorter than the original proof by Simonovits [17]. Recall that for $r = 2$ this result was proved in Corollary 3.4. Before we embark on the proof of Theorem 1.2 for arbitrary r, we need to introduce some notation.

Let G be a graph with $\omega(G) = r$ and let C be an r-clique in G such that the quantity

$$\sum_{v \in C} \deg(v)$$

is maximised. Define

$$\Lambda_r(G) := (r-1)|V(G)| - \sum_{v \in C} \deg(v). \tag{4.1}$$

The above expression can also be written as

$$\Lambda_r(G) = \sum_{v \in V(G)} \big(r - 1 - \deg_C(v)\big). \tag{4.2}$$

Due to our assumption that G is K_{r+1}-free, the right hand side of (4.2) is non-negative, whereby it has a well-defined minimum over all graphs G with $\omega(G) = r$ and $\chi(G) \geqslant k$:

$$\Lambda_r(k) := \min_{\omega(G)=r, \chi(G) \geqslant k} \Lambda_r(G).$$

We define $C_{k,r}$ to be the minimal order of a graph realising $\Lambda_r(k)$.

Recall from Section 2 that if u, $v \in V(G)$ then the Zykov symmetrization $Z_{u,v}(G)$ replaces u with a twin of v. In Section 2, we required that u and v not be adjacent. Here we extend the notion of $Z_{u,v}$ to the case when u and v are adjacent as follows: we make u a twin of v and remove the edge uv.

Proof of Theorem 1.2. Let G be an extremal K_{r+1}-free graph on n vertices with chromatic number at least k. Suppose for a contradiction that, as $n \to \infty$, the number of twin classes in G also tends to infinity.

As was pointed out in the Introduction, it is immediate that there exists a constant $c = c(k,r) > 0$ such that $e(G) \geqslant t_{n,r} - cn$. Thus, by Corollary 3.2, G can be made r-partite by removing a set F of $O_{k,r}(1)$ vertices. Let $V_1, \ldots, V_r$ be the partition classes of the remaining subgraph; each of them must be of size $(1 + o(1))n/r$, or else we would have $e(G) \leqslant t_{n,r} - \omega(n^2)$, a contradiction.

Note that G is $(r+1)$-saturated. Let $T_i \subset V_i$ denote the subset of vertices with twins. Since $e(G)$ exceeds the bound of Corollary 3.10, each T_i has size $t_i := |T_i| = (1+o(1))n/r$.

Claim 4.1. *We may assume that each T_i forms a single twin class, that each pair (T_i, T_j) induces a complete bipartite graph, and that there exists $c > 0$ such that for all i and j,*

$$\sum_{i=1}^{r} |V_i \setminus T_i| \geqslant c|t_i - t_j|. \tag{4.3}$$

Proof of Claim 4.1. First, it is easy to see that if u, $v \in T_i$, then $\deg(u) = \deg(v)$: if not, then either $Z_{u,v}(G)$ or $Z_{v,u}(G)$ is K_{r+1}-free, is $(\geqslant k)$-chromatic and has strictly more edges than G, a contradiction. So, by applying Zykov symmetrization within each V_i, we may assume that each T_i is a single twin class. (Note that because we only symmetrize vertices that have twins, this process will not decrease $\chi(G)$.)

Therefore, for all i and j, $G[T_i \cup T_j]$ is either empty or complete bipartite. Because $t_i = (1 + o(1))n/r$ for all i, we must have $E(T_i, T_j) \neq \emptyset$ for all i and j: otherwise, $e(G) \leqslant t_{n,r} - \omega(n^2)$ a contradiction. Hence, for all i and j, $G[T_i \cup T_j]$ must be complete bipartite.

It remains to show that (4.3) holds. Since, by assumption, the number of twin classes in G is unbounded, so must be the left hand side of (4.3). Hence, if (4.3) does not hold, then there exist i and j such that $t_i - t_j = f(n)$, where $f(n)$ tends to infinity with n. In this case we have $\sum_{i=1}^r |V_i \backslash T_i| = o(f(n))$. Then the fact that all edges are present between different T_i implies that if $v_i \in T_i$ and $v_j \in T_j$, then $\deg(v_j) - \deg(v_i) \geqslant (1 - o(1))f(n)$. Therefore, if we replace v_i with a twin of v_j, then we obtain a graph with strictly more edges than G that is K_{r+1}-free and (because we have symmetrized vertices that have twins) is still $(\geqslant k)$-chromatic, which is a contradiction. This proves the claim. $\square$

Let

$$T = \max_i t_i \qquad \text{and} \qquad t = \min_i t_i.$$

Let G' be the graph obtained by identifying t of the vertices of each T_i. Then G' has $n' := n - r(t-1)$ vertices; moreover, n' is at least the number of twin classes in G, so by assumption n' tends to infinity with n.

We want to show that G' has a large twin-free independent set. Indeed, observe that $n' \leqslant r(T - t) + \sum_{i=1}^r |V_i \setminus T_i|$. It follows from Claim 4.1 that there exists $c' > 0$ such that for some i, we have $|V_i \setminus T_i| \geqslant c'n'$. The set $V_i \setminus T_i$ is twin-free by definition, so, if n (and hence n') is large enough, Corollary 3.10 implies that

$$e(G') \leqslant t_{n',r} - Cn'$$

for some large constant C.

Let H'' be a K_{r+1}-free, k-chromatic graph on $\ell = C_{k,r}$ vertices such that $\Lambda_r(H'') = \Lambda_r(k)$ (recall that $C_{k,r}$ was defined as the smallest order of such a graph). Let $K \subset V(H'')$ be a clique that achieves the value of $\Lambda_r(H'')$ and let H' be the graph on n' vertices obtained from H'' by blowing up each vertex of K by a factor of $(n' - \ell)/r + 1$. Observe that

$$e(H') \geqslant t_{n',r} - cn'$$

for some small constant c and that

$$\Lambda_r(H') = \Lambda_r(H'') = \Lambda_r(k),$$

where the value of $\Lambda_r(H') = \Lambda_r(k)$ is realised in H' by the same clique K. In particular, we have

$$e(H') > e(G') \tag{4.4}$$

and

$$\Lambda_r(H') \leqslant \Lambda_r(G'). \tag{4.5}$$

For each i, let $v_i \in V(G')$ denote the vertex obtained by identifying the t vertices of T_i. Thus, we obtain G from G' by blowing up each v_i by a factor of t. Let H be the graph on n vertices obtained by blowing up each $v \in K \subset H'$ by a factor of t. Denote $C = \{v_1, \ldots, v_r\} \subset V(G')$. It follows from (4.1), (4.4) and (4.5) that

$$\begin{aligned} e(G) &= e(G') + (t-1) \sum_{v \in V(G')} \deg_C(v) + (t-1)^2 \binom{r}{2} \\ &\leqslant e(G') + (t-1)\big((r-1)n' - \Lambda_r(G')\big) + (t-1)^2 \binom{r}{2} \end{aligned}$$

$$
\begin{aligned}
&< e(H') + (t-1)\big((r-1)n' - \Lambda_r(H')\big) + (t-1)^2\binom{r}{2} \\
&= e(H),
\end{aligned}
$$

contradicting the extremality of G. $\square$

Remark 4.2. As mentioned in the Introduction, Theorem 1.2 is only a special case of Simonovits's result in [17]. Theorem 1.2 was first proved in the more general setting of 'chromatic conditions'—properties that are natural generalizations of statements such as 'G has chromatic number at least k'. (For a precise statement, see [17, Definition 1.5].) It is not hard to verify that our proof of Theorem 1.2 extends to K_{r+1}-free graphs that satisfy these more general conditions. Simonovits's result also extends to a larger class of forbidden subgraphs than the class of complete graphs.

With Theorem 1.2 at our disposal, it is straightforward to derive a formula for the asymptotics of the extremal numbers for K_{r+1}-free, $(\geqslant k)$-chromatic graphs. In fact, the coefficient in the linear term can be conveniently described using the quantity $\Lambda_r(k)$.

Theorem 4.3. *Let $r \geqslant 2$ and let $k \geqslant r+1$. If G is a K_{r+1}-free, $(\geqslant k)$-chromatic graph maximising $e(G)$ over all such graphs of order n, then*

$$
e(G) = t_{n,r} - \frac{\Lambda_r(k)}{r} \cdot n + O_{k,r}(1). \tag{4.6}
$$

Given a graph H of order ℓ with $\omega(H) = r$, the following lemma tells us which of its blow-ups to order n maximises $e(G)$.

Lemma 4.4. *Let H be a graph on ℓ vertices with $\omega(H) = r$ and let G be a blow-up of H with $|V(G)| = n$. For large n, $e(G)$ is maximised up to $O(1)$ by letting C be an r-clique in H for which the quantity*

$$
\sum_{v \in C} \deg(v)
$$

is maximised and by blowing up each $v \in C$ by a factor of $(n-\ell)/r + 1$.

The proof of the lemma is a variant of the proof Turán's theorem due to Motzkin and Straus [13] (see also [1]). Nevertheless, we include the details of the argument.

Proof. Observe that up to a $O(1)$ error term, maximising $e(G)$ is equivalent to determining

$$
\max\left\{2\sum_{v_iv_j \in E(H)} x_ix_j \;:\; \sum_{i=1}^{\ell} x_i = n, x_i \geqslant 1 \text{ for all } i\right\}.
$$

Letting $y_i = x_i - 1$, this is equivalent to determining

$$
\max\left\{2\left(\sum_{v_iv_j \in E(H)} y_iy_j + \sum_{i=1}^{\ell} y_i \deg(v_i) + e(H)\right) \;:\; \sum_{i=1}^{\ell} y_i = n - \ell, y_i \geqslant 0 \text{ for all } i\right\}. \tag{4.7}
$$

Thus, given $\mathbf{y} \in \mathbb{R}^\ell$, we define

$$f(\mathbf{y}) = 2 \sum_{v_i v_j \in E(H)} y_i y_j + 2 \sum_{i=1}^{\ell} y_i \deg(v_i) + 2e(H).$$

By compactness, the maximum in (4.7) is achieved. Suppose that $\mathbf{y}$ is a vector that achieves the maximum in (4.7) with the minimum number of non-zero entries and let $C = \{i : y_i > 0\}$. We claim that C corresponds to a clique in H. Suppose to the contrary that $i_1, i_2 \in C$ but $v_1 v_2 \notin E(H)$ and, for t with $-y_1 \leqslant t \leqslant y_2$, let $\mathbf{y}_t = (y_1 + t, y_2 - t, y_3, \ldots, y_\ell)$. If $\deg(v_1) \neq \deg(v_2)$, then it is easy to see that there exists $t \neq 0$ such that $f(\mathbf{y}_t) > f(\mathbf{y})$, a contradiction. Otherwise, by assumption, $f(\mathbf{y}_t)$ achieves its maximum value on the interval $-y_1 \leqslant t \leqslant y_2$ at $t = 0$. Because $v_1 v_2 \notin E(H)$, $f(\mathbf{y}_t)$ is linear in t, and because it achieves its maximum on the interior of the interval, it is a constant function of t. Thus, if we let $t = y_2$, then we obtain a vector with fewer non-zero entries than $\mathbf{y}$ that also achieves the maximum in (4.7), a contradiction.

We may thus assume that C corresponds to a clique. Now we observe that

$$(n-\ell)^2 = (y_1 + \cdots + y_\ell)^2 = 2 \sum_{i,j \in C} y_i y_j + \sum_{i \in C} y_i^2 = f(\mathbf{y}) - 2 \sum_{i \in C} y_i \deg(v_i) - 2e(H) + \sum_{i \in C} y_i^2.$$

It follows that $f(\mathbf{y})$ is maximised when the quantity

$$\sum_{i \in C} \big(y_i^2 - 2 y_i \deg(v_i)\big) \tag{4.8}$$

is minimised. Recalling the assumption that $\sum_{i \in C} y_i = n - \ell$ and computing the Lagrange multipliers, we find that the quantity in (4.8) is minimised when

$$y_i - \deg(i) = y_j - \deg(j)$$

for all i and j, which shows that the y_i must differ by constants (with respect to n). Observe that if we shift constant weights in order to make all weights in C equal to $(n - \ell)/|C|$, then we change f, and hence $e(G)$, only by a constant. For this choice of $\mathbf{y}$, we have

$$f(\mathbf{y}) = (n-\ell)^2 \left(1 - \frac{1}{|C|}\right) + 2(n-\ell) \sum_{i \in C} \deg(v_i),$$

which is maximised when $|C| = r$ and $\sum_{i \in C} \deg(v_i)$ is maximal. This completes the proof. $\square$

The next result follows from Lemma 4.4 by straightforward calculations.

Corollary 4.5. *Let H be a graph on ℓ vertices with $\omega(H) = r$. If a graph G of order n is a blow-up of H with the maximum number of edges, then*

$$e(G) = t_{n,r} - \frac{\Lambda_r(H)}{r} \cdot n + O(1).$$

It is now a short step to complete the proof of Theorem 4.3.

Proof of Theorem 4.3. Corollary 3.4 (for $r = 2$) or Theorem 1.2 (in general) implies that G is a blow-up of a fixed size graph. The result then follows from Corollary 4.5. $\square$

Remark 4.6. With additional work, one can show that there exists a finite time algorithm that, for each r and k, determines the extremal size of a K_{r+1}-free, $(\geqslant k)$-chromatic graph exactly. Indeed, by Theorem 1.2, it suffices to consider blow-ups of finitely many graphs. The extremal value is determined by analysing the quadratic programming argument in the proof of Lemma 4.4 more carefully. We omit further details. We also note that Simonovits [16] observed that such an algorithm exists for $r = 2$.

Remark 4.7. Theorem 4.3 also holds for graphs that satisfy the chromatic conditions discussed in Remark 4.2. (If $\mathcal{A}$ is a chromatic condition, then (4.6) holds with $\Lambda_r(k)$ replaced by $\Lambda_r(\mathcal{A})$, which we define to be the minimum value of $\Lambda_r(H)$ over all K_{r+1}-free graphs H satisfying $\mathcal{A}$.)

Recall that Theorem 1.1 gives the largest number of edges in a K_{r+1}-free, $(\geqslant k)$-chromatic graph where $k = r + 1$. As an application of Theorem 4.3, we shall establish the analogous result for $k = r + 2$, up to a $O(1)$ error term.

Theorem 4.8. *Let G be a K_{r+1}-free, $(\geqslant r+2)$-chromatic graph that maximises $e(G)$ over all such graphs of order n. If $r = 2$, then*

$$e(G) = t_{n,2} - \frac{3n}{2} + O(1),$$

and if $r \geqslant 3$, then

$$e(G) = t_{n,r} - \frac{2n}{r} + O_r(1).$$

By Theorem 4.3, in order to prove Theorem 4.8, it is enough to determine $\Lambda_r(r + 2)$ for all $r \geqslant 2$.

Lemma 4.9. *We have $\Lambda_2(4) = 3$.*

Let us note that Lemma 4.9 was also proved in [16].

Proof. Let H be a triangle-free graph and let v and w be adjacent vertices of H such that $|V(H)| - \deg(v) - \deg(w) = \Lambda_2(H)$ (cf. (4.1)). Let S denote the set of common non-neighbours of v and w. Because v and w are adjacent and H is triangle-free, $|V(H)| - \deg(v) - \deg(w)$ is exactly $|S|$. We claim that if $|S| \leqslant 2$, then H is 3-colourable.

First, suppose that $|S| = 1$ and let $S = \{x\}$. Then, because $N(v)$ and $N(w)$ are independent sets, we may give colour 1 to each vertex in $N(w)$ (including v), colour 2 to each vertex in $N(v)$ (including w), and colour 3 to x.

Second, suppose that $|S| = 2$ and let $S = \{x, y\}$. If x and y are not adjacent, then we may give colour 3 to both of them. Otherwise, we modify the colouring above: we give colour 3 to x, colour 1 to y and colour 3 to all vertices of $N(y) \cap N(w)$. Because x and y are adjacent, they have no common neighbours, and so we have properly 3-coloured H.

It follows that $\Lambda_2(4) \geqslant 3$. Finally, $\Lambda_2(4) = 3$ is realised when H is the Grötzsch graph and v and w are adjacent vertices whose degree sum is 8. $\square$

Now we establish a relation between extremal numbers for different values of r and k.

Lemma 4.10. *We have* $\Lambda_r(k) \leqslant \Lambda_{r-1}(k-1)$.

Proof. Take a graph H that realises $\Lambda_{r-1}(k-1)$ and add a new vertex u adjacent to every vertex of H. Then $\Lambda_r(H \cup \{u\}) = \Lambda_{r-1}(H) = \Lambda_{r-1}(k-1)$, and the result follows. ☐

Next, we give a lower bound on $\Lambda_r(k)$.

Lemma 4.11. *We have* $\Lambda_r(k) \geqslant k - r$.

Proof. Let H be a graph with $\omega(H) = r$ and $\chi(H) \geqslant k$ and let $C \subset V(H)$ be an r-clique that achieves the value of $\Lambda_r(H)$. Let $S = \{v \notin C : \deg_C(v) = r-1\}$. We observe that $H[C \cup S]$ is r-colourable: after properly colouring C, give each $v \in S$ the colour of its non-neighbour in C, and observe that because $\omega(H) = r$, if u, $v \in S$ have the same non-neighbour in C, then they are independent. Because $\chi(H) \geqslant k$, it follows that H must contain at least $k-r$ vertices not in $C \cup S$, and our assumption that H is K_{r+1}-free means that each such vertex is adjacent to at most $r-2$ vertices of C. It follows from (4.2) that each such vertex contributes at least 1 to $\Lambda_r(H)$, which proves the lemma. ☐

In order to prove Theorem 4.8, it remains to compute $\Lambda_r(r+2)$ for all $r \geqslant 3$. However, it turns out to be enough to determine $\Lambda_3(5)$, which we now do.

Lemma 4.12. *We have* $\Lambda_3(5) = 2$.

Proof. Lemma 4.11 implies that $\Lambda_3(5) \geqslant 2$. To show that equality holds, we define a graph H as follows. Let v_1, v_2 and v_3 be the vertices of a triangle. Let a_{12} and b_{12} be adjacent to v_1 and v_2, let b_{23} and c_{23} be adjacent to v_2 and v_3, and let a_{13}, b_{13} and c_{13} be adjacent to v_1 and v_3. Let x be adjacent to v_1 and to both of the a_{ij}, let y be adjacent to v_3 and to both of the c_{ij}, and let x and y be adjacent to each other and to all of the b_{ij}. Finally, if $(i,j) \neq (k,\ell)$, let a_{ij} be adjacent to $c_{k\ell}$.

By inspection, H is K_4-free and $\Lambda_3(H) = 2$. We shall show that H is not 4-colourable. Let $f : V(H) \to \mathbb{N}$ be a proper colouring of H and, for each i, let $f(v_i) = i$. We shall show that no matter what colours we give to x and y, some vertex must receive colour 5. If we let $f(x) = 3$ and $f(y) = 1$, then neither a_{12} nor c_{23} can receive colours 1, 2 or 3, which means that one of them must receive colour 5. In the same way, if we let $f(x) = 3$ and $f(y) = 2$, then either a_{12} or c_{13} must receive colour 5, and if we let $f(x) = 2$ and $f(y) = 1$, then either a_{13} or c_{23} must receive colour 5. Finally, if we give colour 4 to either x or y, then no matter what colour we give to the other, some b_{ij} must receive colour 5.

It follows that $\Lambda_3(5) = \Lambda_3(H) = 2$, as claimed. ☐

Proof of Theorem 4.8. The result for $r = 2$ follows from Lemma 4.9 and Theorem 4.3. If $r \geqslant 3$, then Lemmas 4.10, 4.11 and 4.12 imply that $\Lambda_r(r+2) = 2$. The result then follows from Theorem 4.3. ☐

It is possible to determine $\Lambda_2(k)$ for other small values of k, as in the following case.

Proposition 4.13. *We have* $\Lambda_2(5) = 6$.

We omit the proof; the argument is similar to the proofs of Lemmas 4.9 and 4.12.

Remark 4.14. Using results from Ramsey theory, it is possible to give good asymptotic bounds on $\Lambda_2(k)$. (We note that this connection was also observed in [16].)

First, letting $f_2(k)$ denote the minimum order of a triangle-free graph with chromatic number at least k, it is not hard to show that there exist constants C_1, $C_2 > 0$ such that

$$C_1 k^2 \log k \leqslant f_2(k) \leqslant C_2 k^2 \log k. \tag{4.9}$$

Indeed, Ajtai, Komlós and Szemerédi [2, 3] and Kim [12] proved that there exist constants c_1, $c_2 > 0$ such that for every $t \geqslant 2$, the Ramsey number $R(3,t)$ satisfies

$$c_1 \frac{t^2}{\log t} \leqslant R(3,t) \leqslant c_2 \frac{t^2}{\log t}.$$

In other words, every triangle-free graph on n vertices has an independent set of size at least $c_3\sqrt{n \log n}$, while there exist triangle-free graphs on n vertices with no independent set of size more than $c_4\sqrt{n \log n}$. The upper bound in (4.9) then follows from the fact that any graph G satisfies $|V(G)| \leqslant \alpha(G)\chi(G)$. The lower bound can be derived using a greedy algorithm: we repeatedly colour a largest independent set with a single colour and then remove it from the graph. As the resulting graph is still triangle-free, we may recursively apply the lower bound on $\alpha(G)$ to obtain the claimed upper bound on $\chi(G)$. (See [10, pp. 124–125] for details.)

It follows from (4.9) that there exist constants c_5, $c_6 > 0$ such that

$$c_5 k^2 \log k \leqslant \Lambda_2(k) \leqslant c_6 k^2 \log k.$$

To see this, let H be a triangle-free, k-chromatic graph on at most $c_6 k^2 \log k$ vertices. We have

$$\Lambda_2(k) \leqslant \Lambda_2(H) \leqslant |V(H)| \leqslant c_6 k^2 \log k.$$

On the other hand, given a triangle-free, k-chromatic graph H with vertices v and w realising $\Lambda_2(H)$, put $F = H \setminus (N(v) \cup N(w))$. Since $\chi(F) \geqslant k-2$, (4.1) and (4.9) imply that

$$\Lambda_2(H) = |V(F)| \geqslant C_1(k-2)^2 \log(k-2) \geqslant c_5 k^2 \log k,$$

for a suitably chosen constant c_5. Since this holds for every H, we conclude that $\Lambda_2(k) \geqslant c_5 k^2 \log k$, as claimed.

It is possible to derive asymptotic bounds on $\Lambda_r(k)$ for fixed $r \geqslant 3$ in a similar fashion. However, the existing bounds on $R(r+1,t)$ for fixed $r \geqslant 3$ are too far apart to give matching upper and lower bounds on $\Lambda_r(k)$.

5 Open Problems

For the convenience of the reader, in this section we state precisely the open questions mentioned at various points in the paper. The first two relate to Theorem 1.4 and its consequences. The latter two concern generalizations of Theorem 1.5.

Problem 5.1. For $r \geqslant 2$, determine the supremum of all values c such that if n is sufficiently large, then every $(r+1)$-saturated graph G on n vertices with $e(G) \geqslant t_{n,r} - cn \log_2 n$ has a pair of twins.

Remark 5.2. Let c_r be the supremum defined in Problem 5.1. Observe that Proposition 3.8 and Example 3.9 imply that $1/20 \leqslant c_2 \leqslant 1$. For $r \geqslant 3$, Example 3.11 implies that $c_r \leqslant 1$. On the other hand, following the proof of Theorem 1.4 while being a little more careful about the calculations gives the lower bound $c_r \geqslant 1/r(r-2)$.

The next question asks for the best possible result along the lines of Corollary 3.10.

Question 5.3. Let $r \geqslant 2$ and let $c \leqslant c_r$. For n sufficiently large, if G is an $(r+1)$-saturated graph on n vertices with at least $t_{n,r} - cn \log_2 n$ edges, what is the smallest number of vertices with twins that G may contain?

The proof Theorem 1.5 shows that a 3-saturated graph of order n with enough edges must contain a complete tripartite subgraph of order $n - O(\sqrt{n})$, but we do not know whether this is best possible.

Question 5.4. What is the smallest function f such that given $c > 0$, every 4-saturated graph G of order n with $e(G) > t_{n,3} - cn$ contains a complete tripartite graph on at least $n - f(c, n)$ vertices?

The final question asks whether Theorem 1.5 extends to larger values of r.

Question 5.5. Given $r \geqslant 4$ and $c \geqslant 0$, does every $(r+1)$-saturated graph of order n with at least $t_{n,r} - cn$ edges contain a complete r-partite subgraph on $(1 - o(1))n$ vertices?

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Appendix

Proof of Theorem 2.2. Let G be as in the statement of the theorem. Because $e(G)$ is maximal, we may assume that no edges were added to G during Step 3 of the algorithm described in the proof of Theorem 1.1. Let us therefore consider Step 2 of the construction. Because we did not add any edges to G in Step 3, we may assume that the graph G' obtained from having performed the switches in Step 2 is isomorphic to $G_{n,r}$. Now, if G was transformed into G' by switches, then we can transform G' into G by a series of inverse switches. To perform an inverse switch, we need a pair (v, w) with $v \in V_i$ and $w \in V_j$, $i \neq j$, such that $uv, vw \in E(G')$ and $uw \notin E(G')$.

By the definition of $G_{n,r}$, we must have $w \in V_1 \setminus \{v_1\}$ or $w \in V_2 \setminus \{v_2\}$. If we let $v = v_2$ and repeatedly choose $w \in V_1 \setminus \{v_1\}$, then after each inverse switch, we obtain the graph $G_{n,r}^{(\ell)}$ for some ℓ. Similarly, if we let $v = v_1$ and repeatedly choose $w \in V_2 \setminus \{v_2\}$, then we obtain either $G_{n,r}^{(\ell)}$ or (if $n = kr + 2$ for some $k \geqslant 2$) $G_{n,r}'^{(\ell)}$ for some ℓ. This means that throughout this process, the graph remains K_{r+1}-free and is not r-colourable.

Now we show that performing any other inverse switch creates a graph that does not satisfy the hypotheses of the theorem. First, we may not join u to vertices in both $V_1 \setminus \{v_1\}$ and $V_2 \setminus \{v_2\}$: if, for some $w_1 \in V_1 \setminus \{v_1\}$ and $w_2 \in V_2 \setminus \{v_2\}$, we added the edges uw_1 and uw_2 and removed the edges v_1w_2 and v_2w_1, then for $j = 3, \ldots, r$, there exist $v_j \in V_j$ such that $\{u, w_1, w_2, v_3, \ldots, v_r\}$ would induce a copy of K_{r+1}. Second, we may not join u to all of V_1 (respectively, to all of V_2), because the resulting graph would be r-colourable: we could give colour 2 to u and give colour 1 to v_2 (respectively, to v_1).

Finally, we may not have $v \in V_j$ for any $j \geqslant 3$. Indeed, suppose that for some $w \in V_1 \setminus \{v_1\}$ and $v \in V_3$, say, we added the edge uw and removed the edge wv. If V_3 contains a vertex x besides v (as must be the case if $n \geqslant 2r + 1$), then there exist vertices $v_i \in V_i$, $i = 4, \ldots, r$, such that $\{u, w, v_2, x, v_4, \ldots, v_r\}$ would induce a copy of K_{r+1}. If $|V_3| = 1$, then $n \leqslant 2r$, and in particular, $|V_1| = |V_2| = 2$. In this case, the resulting graph would be r-colourable: letting $y \in V_2$ with $y \neq v_2$, we could give colour 1 to u and y, colour 2 to v and w, colour 3 to v_1 and v_2, and, for $i = 4, \ldots, r$, colour i to all vertices of V_i.

Hence we may assume that G can be transformed into some $G_{n,r}^{(\ell)}$ or some $G_{n,r}'^{(\ell)}$ (where $1 \leqslant \ell \leqslant s - 1$) by a series of IZS's in Step 1 of the algorithm. In what follows, we assume that G can be transformed into $G'' \cong G_{n,r}^{(\ell)}$ for some ℓ; the other case is nearly identical. Observe that because $e(G)$ is maximal, each IZS leaves the number of edges unchanged, meaning that at each step we symmetrize two vertices of equal degrees. Again, reversing the procedure, G'' can be transformed into G by a series of inverse symmetrizations: take two twins x and y, remove x and add a new vertex x' such that $\deg(x') = \deg(y)$, $x' \not\sim y$ and $N(x') \neq N(y)$. Letting $W = N_{G''}(u) \cap V_1$, it is easy to see that the only twins in G'' are pairs of vertices from W, pairs of vertices from $V_1 \setminus W$, pairs of vertices from $V_2 \setminus \{v_2\}$, and pairs of vertices from some class V_i with $i \geqslant 3$. (If $|V_1| = |V_2| = 1$, then v_1 and v_2 are twins, but this contradicts our assumption that $n \geqslant r + 3$.)

If x, $y \in V_1 \setminus W$ and x' is a twin of some $w \in W$, then the resulting graph G is isomorphic to $G_{n,r}^{(\ell+1)}$. Similarly, if x, $y \in W$ and x' is a twin of some $w \in V_1 \setminus W$, then $G \cong G_{n,r}^{(\ell-1)}$. (In this case, if $\ell = 2$ and x' is a twin of v_2, then G is isomorphic either to $G_{n,r}^{(2)}$ or to $G_{n,r}'^{(2)}$.) Any other inverse symmetrization would create a copy of K_{r+1}: in all other cases, either x' has pairwise adjacent neighbours in all of the V_j, or x' is adjacent to u and to vertices in all but one of the V_j, all of which are adjacent to one another and to u.

Thus, our extremal graph G must be either of the form $G_{n,r}^{(\ell)}$ or of the form $G_{n,r}'^{(\ell)}$ for some $1 \leqslant \ell \leqslant s - 1$. This completes the proof. $\square$