Fractional coloring of triangle-free planar graphs*

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Abstract

We prove that every planar triangle-free graph on n vertices has fractional chromatic number at most $3 - \frac{3}{3n+1}$.

1 Introduction

The interest in the chromatic properties of triangle-free planar graphs originated with Grötzsch's theorem [6], stating that such graphs are 3-colorable. Since then, several simpler proofs have been given, e.g., by Thomassen [13, 14]. Algorithmic questions have also been addressed: while most proofs readily yield quadratic algorithms to 3-color such graphs, it takes considerably more effort to obtain asymptotically faster algorithms. Kowalik [10] proposed an algorithm running in time $O(n \log n)$, which relies on the design of an

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advanced data structure. More recently, Dvořák, Kawarabayashi and Thomas [2] managed to obtain a linear-time algorithm, yielding at the same time a yet simpler proof of Grötzsch's theorem.

The fact that all triangle-free planar graphs admit a 3-coloring implies that all such graphs have an independent set containing at least one third of the vertices. Albertson, Bollobás and Tucker [1] had conjectured that there is always a larger independent set, which was confirmed by Steinberg and Tovey [12] even in a stronger sense: all triangle-free planar n-vertex graphs admit a 3-coloring where not all color classes have the same size, and thus at least one of them forms an independent set of size at least $\frac{n+1}{3}$. This bound turns out to be tight for infinitely many triangle-free graphs, as Jones [8] showed. As an aside, let us mention that the graphs built by Jones have maximum degree 4: this is no coincidence as Heckman and Thomas later established that all triangle-free planar n-vertex graphs with maximum degree at most 3 have an independent set of order at least $\frac{3n}{8}$, which again is a tight bound—actually attained by planar graphs of girth 5.

All these considerations naturally lead us to investigate the fractional chromatic number χ_f of triangle-free planar graphs. Indeed, this invariant is known to correspond to a weighted version of the independence ratio. In addition, since $\chi_f(G) \leqslant \chi(G)$ for every graph G, Grötzsch's theorem implies that $\chi_f(G) \leqslant 3$ whenever G is triangle-free and planar. On the other hand, Jones's construction shows the existence of triangle-free planar graphs with fractional chromatic number arbitrarily close to 3. Thus one wonders whether there exists a triangle-free planar graph with fractional chromatic number exactly 3. Let us note that this happens for the circular chromatic number χ_c, which is a different relaxation of the ordinary chromatic number such that $\chi_f(G) \leqslant \chi_c(G) \leqslant \chi(G)$ for every graph G.

The purpose of this work is to answer this question. We do so by establishing the following upper bound on the fractional chromatic number of triangle-free planar n-vertex graphs, which depends on n.

Theorem 1. *Every planar triangle-free graph on n vertices has fractional chromatic number at most $\frac{9n}{3n+1} = 3 - \frac{3}{3n+1}$.*

A consequence of Theorem 1 is that no (finite) triangle-free planar graph has fractional chromatic number equal to 3. How much is it possible to improve the bound of Theorem 1? The aforementioned construction of Jones [8] yields, for each $n \geqslant 2$ such that $n \equiv 2 \pmod 3$, a triangle-free planar graph G_n with $\alpha(G_n) = \frac{n+1}{3}$. Consequently, $\chi_f(G_n) \geqslant \frac{3n}{n+1} = 3 - \frac{3}{n+1}$. Therefore, the bound of form $3 - \frac{c}{n}$ for some c in Theorem 1 is qualitatively the best possible.

The bound can be improved for triangle-free planar graphs with maximum degree at most four, giving an exact result for such graphs.

Theorem 2. *Every planar triangle-free n-vertex graph of maximum degree at most four has fractional chromatic number at most $\frac{3n}{n+1}$.*

Furthermore, the graphs of Jones's construction contain a large number of separating 4-cycles (actually, all their faces have length five). We show that planar triangle-free graphs of *maximum degree* 4 and *without separating* 4*-cycles* cannot have fractional chromatic number arbitrarily close to 3.

Theorem 3. *There exists $\delta > 0$ such that every planar triangle-free graph of maximum degree at most four and without separating* 4*-cycles has fractional chromatic number at most* $3 - \delta$*.*

Dvořák and Mnich [5] proved that there exists $\beta > 0$ such that all planar triangle-free n-vertex graphs without separating 4-cycles contain an independent set of size at least $n/(3-\beta)$. This gives evidence that the restriction on the maximum degree in Theorem 3 might not be necessary.

Conjecture 4. There exists $\delta > 0$ such that every planar triangle-free graph without separating 4-cycles has fractional chromatic number at most $3 - \delta$.

Faces of length four are usually easy to deal with in the proofs by collapsing; thus the following seemingly simpler variant of Conjecture 4 is likely to be equivalent to it.

Conjecture 5 (Dvořák and Mnich [5])**.** There exists $\delta > 0$ such that every planar graph of girth at least five has fractional chromatic number at most $3 - \delta$.

2 Notation and auxiliary results

Consider a graph G. For an integer $a \geqslant 1$, let $[a] = \{1, \ldots, a\}$. An *a-fractional coloring* of G is a function φ assigning to each vertex of G a subset of $[a]$, such that $\varphi(u) \cap \varphi(v) = \emptyset$ for all edges uv of G. Let $f\colon V(G) \to [a]$ be any function. If the a-fractional coloring φ satisfies $|\varphi(v)| \geqslant f(v)$ for every $v \in V(G)$, then φ is an *(a, f)-coloring* of G. If $|\varphi(v)| = f(v)$ for every $v \in V(G)$, then the (a, f)-coloring φ is *tight*. Note that if G has an (a, f)-coloring, then it also has a tight one. If f is the constant function assigning to each vertex of G the value $b \in [a]$, then an (a, f)-coloring is said to be an *$(a : b)$-coloring*. An *a-coloring* is an $(a : 1)$-coloring.

Let $f_1\colon V(G) \to [a_1]$ and $f_2\colon V(G) \to [a_2]$ be arbitrary functions, and let $f\colon V(G) \to [a_1+a_2]$ be defined by $f(v) = f_1(v)+f_2(v)$ for all $v \in V(G)$. Suppose that φ_i is an (a_i, f_i)-coloring of G for $i \in \{1, 2\}$. Let φ be defined by setting $\varphi(v) = \varphi_1(v) \cup \{a_1 + c : c \in \varphi_2(v)\}$ for every $v \in V(G)$. Then φ is an $(a_1 + a_2, f)$-coloring of G, and we write $\varphi = \varphi_1 + \varphi_2$. For an integer $k \geqslant 1$, we define $k\varphi$ to be $\underbrace{\varphi + \cdots + \varphi}_{k \text{ times}}$.

The fractional chromatic number of a graph can be expressed in various equivalent ways, see [11] for details. In this paper, we use the following definition. The *fractional chromatic number* of G is

$$\chi_f(G) = \inf\left\{\frac{a}{b} : G \text{ has an } (a : b)\text{-coloring}\right\}.$$

We need several results related to Grötzsch's theorem. The following lemma was proved for vertices of degree at most three by Steinberg and Tovey [12]. The proof for vertices of degree four follows from the results of Dvořák and Lidický [4], as observed by Dvořák, Král' and Thomas [3].

Lemma 6. *If G is a triangle-free planar graph and v is a vertex of G of degree at most four, then there exists a 3-coloring of G such that all neighbors of v have the same color.*

In fact, Dvořák, Král' and Thomas [3] proved the following stronger statement.

Lemma 7. *There exists an integer $D \geqslant 4$ with the following property. Let G be a triangle-free planar graph without separating 4-cycles and let X be a set of vertices of G of degree at most four. If the distance between every two vertices in X is at least D, then there exists a 3-coloring of G such that all neighbors of vertices of X have the same color.*

Let G be a triangle-free plane graph. A 5-face $f = v_1v_2v_3v_4v_5$ of G is *safe* if v_1, v_2, v_3 and v_4 have degree exactly three, their neighbors $x_1, \dots, x_4$ (respectively) not incident with f are pairwise distinct and non-adjacent, and

- the distance between x_2 and v_5 in $G - \{v_1, v_2, v_3, v_4\}$ is at least four, and
- $G - \{v_1, v_2, v_3, v_4\}$ contains no path of length exactly three between x_3 and x_4.

Lemma 8 (Dvořák, Kawarabayashi and Thomas [2, Lemma 2.2])**.** *If G is a plane triangle-free graph of minimum degree at least three and all faces of G have length five, then G has a safe face.*

Finally, let us recall the folding lemma, which is frequently used in the coloring theory of planar graphs.

Lemma 9 (Klostermeyer and Zhang [9])**.** *Let G be a planar graph with odd-girth at least $g > 3$. If $C = v_0v_1 \dots v_{r-1}$ is a facial circuit of G with $r \neq g$, then there is an integer $i \in \{0, \dots, r-1\}$ such that the graph G' obtained from G by identifying v_{i-1} and v_{i+1} (where indices are taken modulo r) is also of odd-girth at least g.*

3 Proofs

First, let us show a lemma based on the idea of Hilton *et al.* [7].

Lemma 10. *Let G be a planar triangle-free graph. For a vertex $v \in V(G)$, let $f_v \colon V(G) \to [3]$ be defined by $f_v(v) = 2$ and $f_v(w) = 1$ for $w \in V(G) \setminus \{v\}$. If v has degree at most 4, then G has a $(3, f_v)$-coloring.*

Proof. Lemma 6 implies that there exists a 3-coloring of G such that all neighbors of v have the same color, without loss of generality the color $\{1\}$. Hence, we can color v by the set $\{2, 3\}$. □

Theorem 2 now readily follows.

Proof of Theorem 2. Let $V(G) = \{v_1, \dots, v_n\}$. For $i \in \{1, \dots, n\}$, let $f_{v_i} \colon V(G) \to [3]$ be defined as in Lemma 10, and let φ_i be a $(3, f_{v_i})$-coloring of G. Then $\varphi_1 + \dots + \varphi_n$ is a $(3n : n+1)$-coloring of G. □

Similarly, Lemma 7 implies Theorem 3.

Proof of Theorem 3. Let D be the constant of Lemma 7, let $m = 4^D$ and let $\delta = \frac{3}{m+1}$. We show that every planar triangle-free graph G of maximum degree at most four and without separating 4-cycles has a $(3m : m+1)$-coloring, and thus $\chi_f(G) \leqslant \frac{3m}{m+1} = 3 - \delta$.

Let G' be the graph obtained from G by adding edges between all pairs of vertices at distance at most $D-1$. The maximum degree of G' is less than $4^D = m$, and thus G' has a coloring by at most m colors. Let $C_1, \ldots, C_m$ be the color classes of this coloring (some may be empty). For $i \in [m]$, let f_i be the function defined by $f_i(v) = 2$ for $v \in C_i$ and $f_i(v) = 1$ for $v \in V(G) \setminus C_i$. Note that the distance in G between any distinct vertices in C_i is at least D, and thus Lemma 7 ensures that G has a $(3, f_i)$-coloring φ_i. Then $\varphi_1 + \cdots + \varphi_m$ is a $(3m : m+1)$-coloring of G. □

The proof of Theorem 1 is somewhat more involved. Let G be a plane triangle-free graph. We say that G is a *counterexample* if there exists an integer $n \geqslant |V(G)|$ such that G does not have a $(9n : 3n+1)$-coloring. We say that G is a *minimal counterexample* if G is a counterexample and no plane triangle-free graph with fewer than $|V(G)|$ vertices is a counterexample. Observe that every minimal counterexample is connected.

Lemma 11. *If G is a minimal counterexample, then G is 2-connected. Consequently, the minimum degree of G is at least two.*

Proof. Let $n \geqslant |V(G)|$ be an integer such that G does not have a $(9n : 3n+1)$-coloring. Since $9n > 2(3n+1)$, it follows that G has at least three vertices. Hence, it suffices to prove that G is 2-connected, and the bound on the minimum degree will follow.

Suppose for a contradiction that G is not 2-connected, and let G_1 and G_2 be subgraphs of G such that $G = G_1 \cup G_2$, the graph G_1 intersects G_2 in exactly one vertex v, and $|V(G_1)|, |V(G_2)| < |V(G)|$. By the minimality of G, neither G_1 nor G_2 is a counterexample, and thus for $i \in \{1, 2\}$, there exists a $(9n : 3n+1)$-coloring φ_i of G_i. By permuting the colors, we can assume that $\varphi_1(v) = \varphi_2(v)$. Hence, $\varphi_1 \cup \varphi_2$ is a $(9n : 3n+1)$-coloring of G, which is a contradiction. □

Lemma 12. *If G is a minimal counterexample, then every face of G has length exactly 5.*

Proof. Let $n \geqslant |V(G)|$ be an integer such that G does not have a $(9n : 3n+1)$-coloring. Suppose for a contradiction that G has a face f of length other than 5. Since G is triangle-free, it has odd girth at least five, and by Lemma 9, there exists a path $v_1v_2v_3$ in the boundary of f such that the graph G' obtained by identifying v_1 with v_3 to a single vertex z has odd girth at least five as well. It follows that G' is triangle-free. Since G is a minimal counterexample, G' has a $(9n : 3n+1)$-coloring, and by giving both v_1 and v_3 the color of z, we obtain a $(9n : 3n+1)$-coloring of G. This is a contradiction. □

Lemma 13. *If G is a minimal counterexample, then G has minimum degree at least three.*

Proof. Let $n \geqslant |V(G)|$ be an integer such that G does not have a $(9n : 3n+1)$-coloring. By Lemma 11, the graph G has minimum degree at least two. Suppose for a contradiction that $v \in V(G)$ has degree two. Let f_v be defined as in Lemma 10 and let φ_1 be a $(3, f_v)$-coloring of G.

Since G is a minimal counterexample and $|V(G-v)| \leqslant n-1$, there exists a tight $(9n-9 : 3n-2)$-coloring φ_2 of $G-v$. Let $f(x) = 3n-2$ for $x \in V(G-v)$ and $f(v) = 3n-5$. Since both neighbors of v are assigned sets of $3n-2$ colors, there are at least $(9n-9) - 2(3n-2) = 3n-5$ colors not appearing at any neighbor of v, and thus φ_2 can be extended to a $(9n : f)$-coloring of G.

However, $3\varphi_1 + \varphi_2$ is a $(9n : 3n+1)$-coloring of G, which is a contradiction. □

Lemma 14. *No minimal counterexample contains a safe* 5*-face.*

Proof. Let G be a minimal counterexample. Let $n \geqslant |V(G)|$ be an integer such that G does not have a $(9n : 3n+1)$-coloring. Suppose for a contradiction that f contains a safe 5-face $f = v_1v_2v_3v_4v_5$, and let $x_1, \ldots, x_4$ be the neighbors of $v_1, \ldots, v_4$ that are not incident with f, respectively. For $i \in \{1, \ldots, 4\}$, let f_{v_i} be defined as in Lemma 10 and let φ_i be a $(3, f_{v_i})$-coloring of G.

Let G' be the plane graph obtained from $G - \{v_1, v_2, v_3, v_4\}$ by identifying x_2 with v_5 into a new vertex u_1, and x_3 with x_4 into a new vertex u_2. Since f is safe, G' is triangle-free. Let $N = 9n - 54$. Since G is a minimal counterexample and $|V(G')| \leqslant n-6$, we conclude that G' has a tight $(N : 3n-17)$-coloring φ_5. Let $f(x) = 3n-17$ for $x \in V(G - \{v_1, v_2, v_3, v_4\})$ and $f(v_i) = 3n-20$ for $i \in \{1, \ldots, 4\}$. We extend φ_5 to an (N, f)-coloring of G as follows.

Let $\varphi_5(x_2) = \varphi_5(v_5) = \varphi_5(u_1)$ and $\varphi_5(x_3) = \varphi_5(x_4) = \varphi_5(u_2)$. Note that $|\varphi_5(x_1) \cup \varphi_5(v_5)| \leqslant 2(3n-17)$, and thus we can choose $\varphi_5(v_1)$ as a subset of $[N] \setminus (\varphi_5(x_1) \cup \varphi_5(v_5))$ of size $3n-20$. Similarly, choose $\varphi_5(v_2)$ as a subset of $[N] \setminus (\varphi_5(x_2) \cup \varphi_5(v_1))$ of size $3n-20$. Let $M_3 = [N] \setminus (\varphi_5(v_2) \cup \varphi_5(x_3))$ and $M_4 = [N] \setminus (\varphi_5(v_5) \cup \varphi_5(x_4))$. Note that $|M_3| \geqslant 3n-20$ and $|M_4| \geqslant 3n-20$. Furthermore, since $\varphi_5(x_3) = \varphi_5(x_4)$ and $\varphi_5(v_2) \cap \varphi_5(v_5) = \varphi_5(v_2) \cap \varphi_5(x_2) = \emptyset$, we have $|M_3 \cup M_4| = N - |\varphi_5(x_3)| = N - (3n-17) > 2(3n-20)$. Let $\varphi_5(v_3) \subseteq M_3$ be a set of size $3n-20$ chosen so that $|\varphi_5(v_3) \cap M_4|$ is minimum. Observe that $|M_4 \setminus \varphi_5(v_3)| \geqslant 3n-20$, and thus we can choose a set $\varphi_5(v_4) \subseteq M_4 \setminus \varphi_5(v_3)$ of size $3n-20$. This gives an (N, f)-coloring of G.

Also, by Grötzsch's theorem, G has a $(3 : 1)$-coloring φ_6. However, $3(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) + \varphi_5 + 6\varphi_6$ is a $(9n : 3n+1)$-coloring of G, which is a contradiction. □

We can now establish Theorem 1.

Proof of Theorem 1. Suppose for a contradiction that there exists a planar triangle-free graph G on n vertices with fractional chromatic number greater than $3 - \frac{3}{3n+1}$. Then G has no $(9n : 3n+1)$-coloring, and thus G is a counterexample. Therefore, there exists a minimal counterexample G_0. Lemmas 13, 12 and 8 imply that G_0 has a safe 5-face. However, that contradicts Lemma 14. □

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