# Wilf-classification of mesh patterns of short length


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**Abstract**

This paper starts the Wilf-classification of mesh patterns of length 2. Although there are initially 1024 patterns to consider we introduce automatic methods to reduce the number of potentially different Wilf-classes to at most 65. By enumerating some of the remaining classes we bring that upper-bound further down to 56. Finally, we conjecture that the actual number of Wilf-classes of mesh patterns of length 2 is 46.

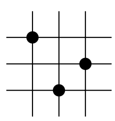


**Keywords:** permutation patterns; Wilf-classification


## 1 Introduction

Let $n$ be a non-negative integer. A *permutation* is a bijection from the set $\{1, 2, \ldots, n\}$ to itself. The permutation that maps $i$ to $\pi_i$ will be written as the word $\pi = \pi_1\pi_2 \cdots \pi_n$. Let $S_n$ be the set of all permutations of length $n$.

A (*classical permutation*) *pattern* is a permutation $p \in S_k$. The pattern $312 \in S_3$ can be drawn as follows, where the horizontal lines represent the values and the vertical lines denote the positions in the pattern.

$312 =$ 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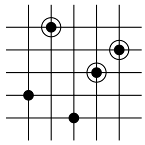
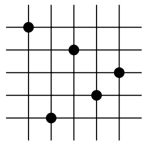
We say that a pattern $p$ *occurs* in a permutation $\pi \in S_n$ if there is a subsequence of $\pi$ whose letters are in the same relative order of size as the letters of $p$. This sequence is called an *occurrence* of the pattern $p$ in the permutation $\pi$. If a pattern occurs in a permutation we say that the permutation *contains* the pattern. For example, the permutation 25134 contains the pattern 312 as the subsequence 534. The diagram below shows the permutation where points corresponding to the occurrence of the pattern have been circled.

$25134 =$ 

A permutation that does not contain a pattern is said to *avoid* the pattern. For example, a permutation $\pi \in S_n$ avoids the pattern 231 if there do not exist $1 \leqslant i < j < k \leqslant n$ with $\pi(k) < \pi(i) < \pi(j)$. An example of a permutation that avoids the pattern 231 is the permutation 51423.

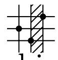
$51423 =$ 

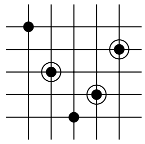
Given a pattern $p$ we let $S_n(p)$ be the set of permutations of length $n$ that avoid $p$.

One of the primary questions in the theory of permutation patterns is that of Wilf-equivalence: Given two patterns $p$ and $q$, are the sizes of the sets $S_n(p)$ and $S_n(q)$ equal for all $n$? Patterns for which the answer is "yes" are called *Wilf-equivalent.* A *Wilf-class* is a maximal set of patterns (necessarily of the same length) that are all Wilf-equivalent. The process of sorting patterns into classes by Wilf-equivalence is called *Wilf-classification.* Classical patterns of length 3 were Wilf-classified by Knuth [7], who showed that the number of permutations avoiding each classical pattern of length 3 is given by the Catalan numbers. Permutations avoiding more than one pattern have also been studied. Simion and Schmidt [9] Wilf-classified all sets of classical patterns of length 3.

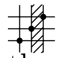
The purpose of this paper is to start the Wilf-classification of mesh patterns. Mesh patterns, whose definition we review below, provide a common extension of several previous generalizations of classical patterns. We show that the 16 mesh patterns of length 1 belong to 4 different Wilf-classes. The classification of the 1024 mesh patterns of length 2 would be very tedious to to by hand without resorting to the usual $D_8$ symmetries of patterns. This however only brings the number of patterns down to 186. In Lemma 11 we give a sufficient condition for when a shading can be added to a mesh pattern. This cuts the number of patterns in half, down to 87. Two other operations allow us to bring that number down to 65. We then use conventional tools of combinatorics, like generating functions, bijective maps, etc., to achieve the upper bound of 56 on the number of Wilf-classes. We conjecture that the actual number of classes is 46.

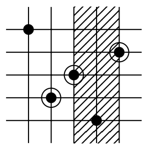
# 2 An overview of generalized patterns

Several generalizations of classical patterns have been introduced. The first extension relevant to us are *vincular patterns*, defined by Babson and Steingrímsson [2]. These patterns can require letters in a permutation to be adjacent. For example the pattern  requires the letters corresponding to 1 and 3 in a permutation to be adjacent. Graphically this means that no points can be in the shaded area. This pattern occurs in the permutation

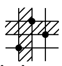
53124 = 

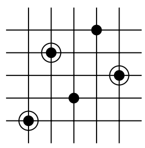
because 53124 contains the classical pattern 213 as the subsequence 324 and because the letters in the permutation that correspond to 1 and 3 in the pattern, i.e., 2 and 4, are adjacent in the permutation.

The permutation 52314 avoids the pattern , since the only occurrence of the classical pattern 123 is the subsequence 234 and the letters 3 and 4 are not adjacent in the permutation.

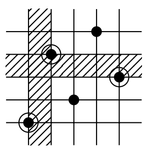
52314 = 

Vincular patterns of length 3 with one shaded column were Wilf-classified by Claesson [5]. He showed that the number of permutations avoiding eight of the vincular patterns is given by the Bell numbers and the remaining four give the Catalan numbers. Several of his results are a consequence of Lemma 11 below.

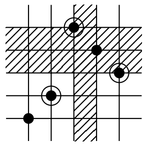
*Bivincular patterns* are a natural extension of vincular patterns where we may also put constraints on the values in a permutation. Bivincular patterns where first introduced by Bousquet-Mélou et al. [3]. The pattern $p =$  requires the letters corresponding to 1 and 3 in a permutation to be adjacent in position and the letters corresponding to 2 and 3 in a permutation to be adjacent in size. The permutation

14253 = 

contains the pattern $p$ because it contains the classical pattern 132 as the subsequence 143 and letters 1 and 4 are adjacent in position and letters 3 and 4 are adjacent in size. On a diagram, we observe that there are no points in the shaded areas.

14253 = 

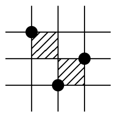
However the permutation 12543 contains one occurrence of the classical pattern 132 as the subsequence 253 but 5 and 3 are not adjacent in size, so that is not an occurrence of the bivincular pattern.

12543 = 

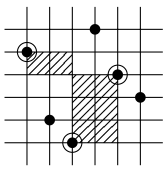
Bivincular patterns of length 2 and 3 were Wilf-classified by Parviainen in [8]. Some of his results are a consequence of Lemma 11 below.

Mesh patterns where first introduced by Brändén and Claesson [4], as a further extension of bivincular patterns, that also subsumes barred patterns with a single bar [13], and interval patterns [14]. A pair $(\tau, R)$, where $\tau$ is a permutation in $S_k$ and $R$ is a subset of $[\![0, k]\!] \times [\![0, k]\!]$, where $[\![0, k]\!]$ denotes the interval of the integers from 0 to $k$, is a *mesh pattern* of length $k$.

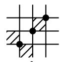
Let $(i, j)$ denote the box whose corners have coordinates $(i, j), (i, j + 1), (i + 1, j + 1)$ and $(i + 1, j)$. An example of a mesh pattern is the classical pattern 312 along with $R = \{(1, 2), (2, 1)\}$. We draw this by shading the boxes in $R$

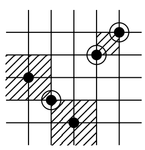


The permutation 521643 contains this pattern, see below

521643 = 

The permutation has an occurrence of the mesh pattern as the subsequence 514, since it forms the classical pattern 312 and there are no points in the shaded areas.

Let's now look at the permutation $\pi = 32145$. This permutation avoids the pattern $(123, \{(0, 1), (1, 0), (2, 2)\})$ = , because for all occurrences of the classical pattern 123 there is at least one point in at least one of the shaded boxes. For example, the subsequence 245 in $\pi$ is an occurrence of the classical pattern 123 but not of the mesh pattern since the point representing 1 is in one of the shaded areas. This can be seen on the following diagram.

32145 = 

## 3 Operations preserving Wilf-equivalence

The number of mesh patterns of length $n$ is $n! \cdot 2^{(n+1)^2}$ so already for $n = 2$ we have 1024 patterns. This makes the Wilf-classification very tedious to do by hand. We therefore review known operations (or symmetries) that preserve Wilf-equivalence as well as introducing new ones. This will allow us to automatically bring the number of patterns to consider down to 65.

The first known operations are the symmetries reverse, complement and inverse. For a given mesh pattern $(\tau, R)$ of length $n$, we define

$$(\tau, R)^r = (\tau^r, R^r), \quad (\tau, R)^c = (\tau^c, R^c), \quad (\tau, R)^i = (\tau^i, R^i),$$

where $\tau^r$ is the usual reverse of the permutation $\tau$, $\tau^c$ the usual complement, $\tau^i$ the usual inverse, and

$$\begin{aligned} R^r &= \{(n - x, y) \colon (x, y) \in R\}, \\ R^c &= \{(x, n - y) \colon (x, y) \in R\}, \\ R^i &= \{(y, x) \colon (x, y) \in R\}. \end{aligned}$$

Hence, reverse is a reflection around the vertical center line, complement is a reflection around the horizontal center line and inverse is the reflection around the southwest to northeast diagonal. Figure 1 is an example of the use of these symmetries on the pattern $p = (312, \{(0, 1), (1, 3), (2, 2)\})$. It is well-known that a permutation $\pi$ avoids a mesh

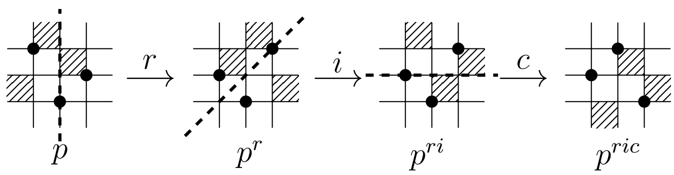



Figure 1: Several symmetries of a mesh pattern

pattern $p$ if and only if the permutation $\pi^r$ avoids $p^r$. That is, the reverse operation preserves Wilf-equivalence. The same applies for complement and inverse or any composition of these three operations.

We now define the first of the new operations.

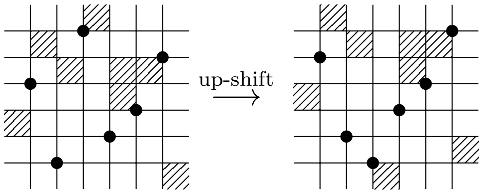
**Definition 1.** Let $\pi$ be a permutation (or a classical pattern) of length $n$. We define the *up-shift* of $\pi$, as $\pi^{\uparrow}$, where

$$\pi^{\uparrow}_i = (\pi_i \bmod n) + 1, \quad \text{for } 1 \leqslant i \leqslant n.$$

Let $p = (\tau, R)$ be a mesh pattern of length $n$. We define the *up-shift* of $p$ as the pattern $p^{\uparrow} = (\tau^{\uparrow}, R^{\uparrow})$, where

$$R^{\uparrow} = \{(a, (b + 1) \bmod (n + 1)) \colon (a, b) \in R\}.$$

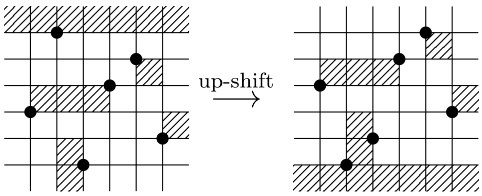
**Example 2.** This example shows the effect of the up-shift operation on a mesh pattern of length 6.




**Proposition 3.** *Let $p = (\tau, R)$ be a pattern of length $n$ where the top line is shaded, that is $\{(0, n), (1, n), \dots, (n, n)\} \subseteq R$. Then a permutation $\pi$ avoids $p$ if and only if $\pi^{\uparrow}$ avoids $p^{\uparrow}$. That is, up-shift preserves Wilf-equivalence for this kind of pattern.*

*Proof.* Let $\pi$ be a permutation with an occurrence $\pi_{i_1}, \dots, \pi_{i_n}$ of a pattern $p$ as described in the proposition. It is easy to see that $\pi^{\uparrow}_{i_1}, \dots, \pi^{\uparrow}_{i_n}$ is an occurrence of $p^{\uparrow}$ in $\pi^{\uparrow}$. Since the map $S_n \to S_n$, $\pi \mapsto \pi^{\uparrow}$ is a bijection the claim follows. $\square$

**Example 4.** Here is another example that shows the effect of up-shift on a mesh pattern of length 6.




According to Proposition 3 these two patterns are Wilf-equivalent.

Given a permutation $\pi \in S_n$, $0\pi$ is the permutation $\pi$ with 0 prepended. Then define $\pi \boxplus 1 = (\pi_1 + 1 \bmod (n+1))(\pi_2 + 1 \bmod (n+1)) \cdots (\pi_n + 1 \bmod (n+1))$. For a given permutation $\lambda$ of $[\![0, n]\!]$, $\lambda_0$ is the permutation in $S_n$ obtained by appending the part of $\lambda$ to the left of 0 to the other part, e.g., $14032_0 = 3214$.

**Definition 5** (Ulfarsson [12])**.** Let $\pi$ be a permutation (or a classical pattern) of length $n$. We define the *toric-shift* of $\pi$ as

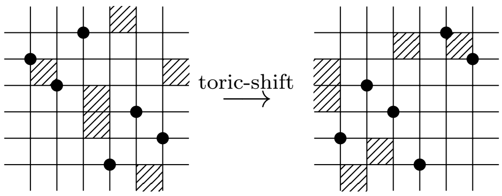
$$\pi^t = (\pi^0 \boxplus 1)_0.$$

Let $p = (\tau, R)$ be a mesh pattern of length $n$. We define the *toric-shift* of $p$ as the pattern $p^t = (\tau^t, R^t)$, where

$$R^t = \{(a + (n + 1 - \ell) \bmod (n+1), b + 1 \bmod (n+1)) \colon (a, b) \in R\}.$$

Here $\ell$ is the position of the letter $n$ in the classical pattern $\tau$.

**Example 6.** This example shows the effect of toric-shift on a mesh pattern of length 6.



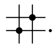
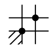
**Observation 7.** *Let $p = (\tau, R)$ be a pattern of length $n$ where the top line is shaded, that is $\{(0, n), (1, n), \ldots, (n, n)\} \subseteq R$. Recall from Ulfarsson [12] that a permutation $\pi$ avoids $p$ if and only if $\pi^t$ avoids $p^t$. That is, toric-shift preserves Wilf-equivalence for this kind of pattern.*

**Definition 8.** Two mesh patterns $p$ and $q$ are said to be *coincident*, denoted by $p \asymp q$, if for any permutation $\pi$, $\pi$ avoids $p$ if and only if $\pi$ avoids $q$.

Coincident patterns are obviously Wilf-equivalent.

**Observation 9.** *Let $R' \subseteq R$. Then any occurrence of $(\tau, R)$ in a permutation is an occurrence of $(\tau, R')$.*

The next example will be generalized below into a powerful lemma that allows adding more shaded boxes to a mesh pattern, while maintaining the coincidence with the original pattern.

**Example 10.** Consider the mesh pattern $p = (12, \varnothing) =$ . Let $u$ be the point $(1, 1)$ and $v$ the point $(2, 2)$, in the pattern. We claim that the mesh pattern $q = (12, \{(0, 0)\}) =$  is coincident to $p$. Because of Observation 9 it suffices to show that if a permutation contains $p$ it also contains $q$. Let $\pi$ be a permutation that contains $p$ and consider a particular occurrence of it. Let $k$ be the number of points in the box $(0, 0)$ in $\pi$. If $k = 0$, it is clear that $\pi$ contains $q$ as well. If $k \geqslant 1$, then we can choose the leftmost (or the lowest point), call it $d$, in the box $(0, 0)$ and replace $u$ with $d$. It is clear that the subsequence $ud$ satisfies the requirements of the mesh in $q$. This can be interpreted as shading the box $(0, 0)$. Figure 2 shows an example of this coincidence. In the left image in the figure, the pattern $p$ can be found with the points $(7, 5)$ as $u$ and $(8, 8)$ as $v$. There are points in the area at the lower left of $u$ and the circled point, $(2, 2)$, is the leftmost point in that area, and thus we denote that point as $d$. In the right image, point $v$ is still $(8, 8)$ but now $u$ has been replaced with $d$.

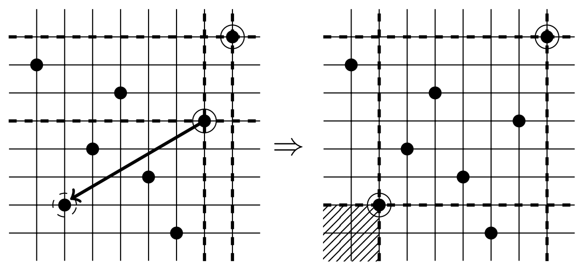


Figure 2: This picture shows two coincident patterns in the permutation 72463158

This example generalizes to a new operation, introduced in Lemma 11, which preserves coincidence of mesh patterns.

**Lemma 11** (Shading Lemma)**.** *Let $(\tau, R)$ be a mesh pattern of length $n$ such that $\tau(i) = j$ and the box $(i, j) \notin R$. If all of the following conditions are satisfied:*

1. *The box $(i-1, j-1)$ is not in $R$;*

2. *At most one of the boxes $(i, j-1)$, $(i-1, j)$ is in $R$;*

3. *If the box $(\ell, j-1)$ is in $R$ (with $\ell \neq i-1, i$) then the box $(\ell, j)$ is also in $R$;*

4. *If the box $(i-1, \ell)$ is in $R$ (with $\ell \neq j-1, j$) then the box $(i, \ell)$ is also in $R$;*

*then the patterns $(\tau, R)$ and $(\tau, R \cup \{(i, j)\})$ are coincident. Analogous conditions determine if other boxes neighboring the point $(i, j)$ can be added to $R$ while preserving the coincidence of the corresponding patterns.*

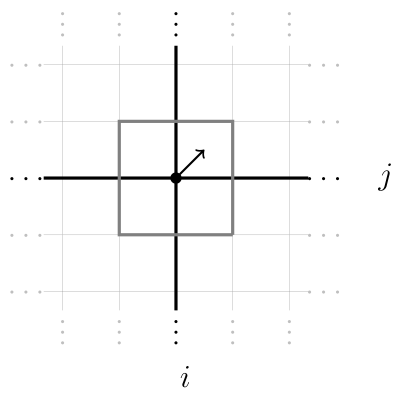


Figure 3: If the condition of the lemma are satisfied the box $(i, j)$ can be shaded

*Proof.* In this proof we assume that the box $(i-1, j)$ is not in $R$. According to Observation 9 we know that if a permutation $\pi$ contains the pattern $q = (\tau, R \cup \{(i, j)\})$, it also contains the pattern $p = (\tau, R)$.

Now we show that if a permutation $\pi$ contains the pattern $p$ it also contains the pattern $q$. Assume we have a particular occurrence of $p$ in $\pi$ so the pattern point $(i, j)$ corresponds to a particular point $(i', j')$ in $\pi$ and the box $(i, j)$ corresponds to a certain region $\mathcal{K}$ in $\pi$ containing $k$ points. If $k = 0$ then we have an occurrence of $q$. If $k \geqslant 1$ then let $(i'', j'')$ be the rightmost[1] point in the region $\mathcal{K}$. This amounts to shifting the line $x = i'$ to the line $x = i''$ and the line $y = j'$ to the line $y = j''$. While we have now an empty region in the permutation corresponding to the box $(i, j)$ in the pattern we need to make sure we have not violated any of the requirements of the mesh $R$. The shifting of the lines has the following effect on the other neighboring boxes of the point $(i, j)$:

- The box $(i-1, j)$ is shrunk from below, but extended to the right. It might contain extra points after the shifting, but that does not matter since it is not shaded.

[1]If we had assumed that the box $(i, j-1)$ was not in $R$ then we would have chosen the topmost point in the region $\mathcal{K}$.

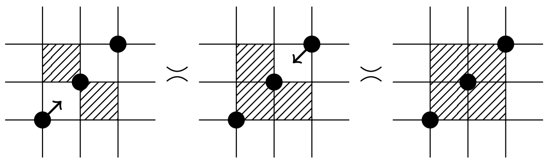
- The box $(i-1, j-1)$ is extended up and to the right. It now contains the point $(i', j')$ (and possibly others) but this box is not a part of the shading, because of requirement (1).
- The box $(i, j-1)$ is shrunk from the left and extended up, but since we choose $(i'', j'')$ as the rightmost point in $\mathcal{K}$ we can be sure that this box is empty.

The effect on the rest of the mesh is as follows:

- Every box $(\ell, j)$ $(\ell \neq i-1, i)$ is shrunk (from below) and any empty box is still empty. In particular every shaded box can remain shaded.
- Every box $(\ell, j-1)$ $(\ell \neq i-1, i)$ is extended into the previous $(\ell, j)$ box. Requirement (3) in the lemma ensures that a shaded box remains empty.
- Every box $(i, \ell)$ $(\ell \neq j, j-1)$ is shrunk (from the left) and any empty box is still empty. In particular every shaded box can remain shaded.
- Every box $(i-1, \ell)$ $(\ell \neq j, j-1)$ is extended into the previous $(i, \ell)$ box. Requirement (4) in the lemma ensures that a shaded box remains empty. □

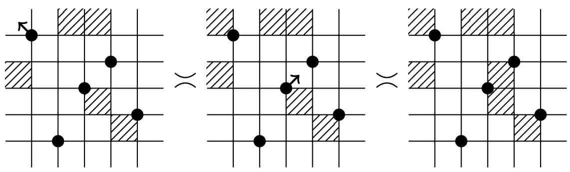
We note that a related result was independently proved by Tenner [11, Theorem 3.5'] which determines when a mesh pattern is coincident to the underlying classical pattern.

**Example 12.** By using Lemma 11 the following coincidence can be found. The point that the arrow is pointing from is the point $(i, j)$ in the lemma.



First we can shade the box $(1, 1)$ because when choosing the rightmost (or the topmost point) in that box, neither of the shaded boxes, $(1, 2)$ and $(2, 1)$, will be extended. When shading the box $(2, 2)$ we choose the leftmost (or the lowest point) and the same applies as before. Notice that the point $(1, 1)$ and the box $(1, 0)$ do not fulfill the condition of the lemma since the box $(2, 1)$ is shaded but the box $(2, 0)$ is not.

**Example 13.**



We can shade the box $(0, 5)$ by choosing the leftmost (or the topmost point) in the box, because none of the shaded boxes touching the lines $x = 0$ or $y = 5$ will be extended by this. We can also shade the box $(3, 3)$ by choosing the rightmost (*not* the topmost point!).

This is because the shaded boxes touching the lines $x = 3$ and $y = 3$ will either not be extended, boxes $(0, 3)$ and $(3, 5)$, or are extended into a shaded area, box $(2, 5)$. However when trying to shade box $(1, 0)$, box $(4, 1)$ would have to be extended into a non-shaded area. Therefore box $(1, 0)$ cannot be shaded.

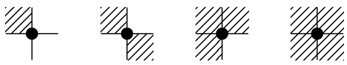
We record one more new operation that preserves Wilf-equivalence in Appendix A since it is only useful for patterns of length three or more.

## 4 Wilf-classes

**Definition 14.** A *Wilf-subclass* is a set containing patterns of the same length which are Wilf-equivalent. A *Wilf-class* is a maximal Wilf-subclass.

In this section we look at the Wilf-classification of mesh patterns of length 1 and 2.

The number of mesh patterns of length 1 is $2 \cdot 2^3 = 16$. The operations from above suffice to sort these 16 patterns into 4 Wilf-classes. Below is one representative from each class.



Occurrences of these patterns in a permutation are well-known. Each occurrence of the first pattern in a permutation $\pi$ is a *left-to-right maximum*. An occurrence of the second pattern in a permutation $\pi$ is a *strong fixed point* in $\pi$.

**Observation 15** (Stanley [10]). *For reference we record the generating function for the number of permutations without strong fixed points is*

$$G(x) = \frac{F(x)}{1 + xF(x)} \quad \textit{where } F(x) = \sum_{n \geqslant 0} n!x^n.$$

Both left-to-right maxima and strong fixed points in connection to mesh patterns were introduced by Brändén and Claesson [4]. All permutations of length $n$ that contain the third pattern begin with $n$. There is only one permutation that contains the last pattern, namely 1.

The number of mesh patterns of length 2 is $2^{10} = 1024$. We start by creating 1024 Wilf-subclasses, one for each of the patterns. By using the operations above many of these subclasses merge. More precisely, the operations reverse, complement and inverse preserve Wilf-equivalence so after merging we arrive at 168 Wilf-subclasses. When further using the shading lemma this number decreases to 87 and finally when taking into account the toric shift and the up-shift we arrive at 65 Wilf-subclasses.

Below we further reduce this number to 56 by combining some of the above classes. We conjecture that the actual number of Wilf-classes is 46.

### 4.1 The Wilf-class containing classical patterns

Here we will bring the number of Wilf-subclasses down to 62 by merging four Wilf-subclasses, represented by the patterns shown in Table 1.

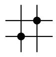
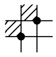
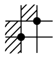
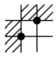
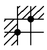
| Nr. | Repr. $p$ | $\lvert S_n(p)\rvert$ for $n = 1, \ldots, 9$ | size of subclass |
|---|---|---|---|
| 1 |  | $1, 1, 1, 1, 1, 1, 1, 1, 1$ | 98 |
| 2 |  | $1, 1, 1, 1, 1, 1, 1, 1, 1$ | 8 |
| 3 |  | $1, 1, 1, 1, 1, 1, 1, 1, 1$ | 16 |
| 4 |  | $1, 1, 1, 1, 1, 1, 1, 1, 1$ | 4 |

Table 1: Patterns Wilf-equivalent to classical patterns

Because of Observation 9 it suffices to show that a permutation containing the classical pattern 12 also contains the last pattern in Table 1. Consider a permutation containing 12. Out of all the occurrences of 12 we can choose the occurrence $\pi_i\pi_j$ such that there are no other occurrence $\pi_{i'}\pi_{j'}$ with $i' < i$. If there are multiple occurrences $\pi_i\pi_{j_1}, \pi_i\pi_{j_2}, \ldots, \pi_i\pi_{j_k}$, we can choose the one where the value $\pi_{j_\ell}$ is maximized. This particular occurrence, $\pi_i\pi_{j_\ell}$, is an occurrence of .

There are no other mesh patterns of length 2 that have the same number of avoiding permutations of length $n = 1, \ldots, 9$ as the patterns in Table 1, so this Wilf-class is comprised of exactly the Wilf-subclasses in the table. This result is shown in Table 2. It is worth noticing that over 1/10th of length 2 mesh patterns lie in this class.

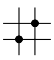
| Repr. $p$ | $\lvert S_n(p)\rvert$ | size of class | OEIS seq. |
|---|---|---|---|
|  | 1 | 126 | A000012 |

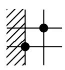
Table 2: The Wilf-class containing classical patterns

### 4.2 Wilf-classes containing vincular patterns

In this section we will deal with subclasses that contain vincular patterns and subclasses that can be merged with them. These subclasses are shown in Table 3. We will bring the number of subclasses down to 58 by combining the first five subclasses into one Wilf-class.

**Proposition 16.** *Here we consider the first three subclasses from Table 3.*

1. *A permutation of length $n$ avoids the pattern*



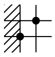
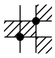
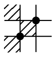
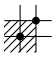
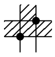
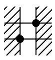
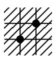
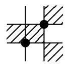
| Nr. | Repr. $p$ | $\lvert S_n(p)\rvert$ for $n = 1, \ldots, 9$ | size of subclass |
|---|---|---|---|
| 5 |  | $1, 1, 2, 6, 24, 120, 720, 5040, 40320$ | 184 |
| 6 |  | $1, 1, 2, 6, 24, 120, 720, 5040, 40320$ | 8 |
| 7 |  | $1, 1, 2, 6, 24, 120, 720, 5040, 40320$ | 16 |
| 8 |  | $1, 1, 2, 6, 24, 120, 720, 5040, 40320$ | 8 |
| 9 |  | $1, 1, 2, 6, 24, 120, 720, 5040, 40320$ | 16 |
| 10 |  | $1, 1, 3, 12, 60, 360, 2520, 20160, 181440$ | 80 |
| 11 |  | $1, 1, 6, 24, 120, 720, 5040, 40320, 362880$ | 2 |

Table 3: Patterns Wilf-equivalent to vincular patterns

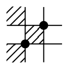
*if and only if it starts with $n$. Therefore, the number of permutations of length $n$ that avoid this pattern is $(n-1)!$.*

2. *A permutation avoids the pattern*



*if and only if it ends with 1. Therefore, the number of permutations of length $n$ that avoid this pattern is $(n-1)!$.*

3. *A permutation of length $n$ avoids the pattern*



*if and only if it starts with $n$. Therefore, the number of permutations of length $n$ that avoid this pattern is $(n-1)!$.*

*Proof.* We only prove part (2) as the others are similar. Let $\pi$ be a permutation that does not end with 1. Let $a$ be the last element of $\pi$ and let $b = a - 1$, which is somewhere to the left of $a$ in $\pi$. It is easy to see that the letters $ab$ form an occurrence of the pattern. Clearly if $\pi$ ends with 1 it can not contain the pattern. □

Recall that $[x^n]f(x)$ denotes the coefficient of $x^n$ in the power series expansion of the function $f(x)$.

**Definition 17.** The $n$-th Eulerian polynomial $E_n(x)$ can be defined by

$$\sum_{k=0}^{\infty}(k+1)^n x^k = \frac{E_n(x)}{(1-x)^{n+1}},$$

see for example [6]. It is well-known that the Eulerian numbers, $T_{n,k} = [x^k]E_n$, can also be defined recursively by

$$T_{n,k} = k \cdot T_{n-1,k} + (n-k+1) \cdot T_{n-1,k-1}, \quad T_{1,1} = 1.$$

Instead of just enumerating avoiders of the fourth and fifth pattern in Table 5 we find the distribution of descents and use it to find the enumeration.

**Proposition 18.** *The distribution for the number of descents in permutations of length $n$ avoiding the pattern $p =$ is*

$$\sum_{\pi \in S_n(\ )} x^{\operatorname{des}(\pi)} = xE_{n-1}(x),$$

*where* $\operatorname{des}(\pi)$ *is the number of descents in the permutation $\pi$. This implies*

$$\left| S_n \left( \ \right) \right| = (n-1)!$$

*for all $n \geqslant 1$.*

*Proof.* In order to construct all permutations of length $n$ we can take each permutation in $S_{n-1}$ and place $n$ in every possible position. Let $\pi'$ be a permutation obtained by adding the letter $n$ to $\pi$, where $\pi$ is a permutation in $S_{n-1}$ which contains the pattern $p$. Then there exist $i < j$ where the only letter lower than $\pi(j)$ in positions $1, 2, \dots j-1$ is $\pi(i)$. From this it follows that we can add the letter $n$ to any position in the permutation $\pi$ for $\pi'$ to still satisfy these conditions and therefore contain the pattern $p$.

Now let $\pi$ be a permutation in $S_{n-1}$ that avoids $p$. Then for all $i < j$ with $\pi(i) < \pi(j)$ there are at least two letters in places $1, 2, \dots j-1$ lower than $\pi(j)$. In order to get a permutation of length $n$ that also avoids $p$ we can add the letter $n$ in all positions except between the letters $\pi(1)$ and $\pi(2)$. By adding the letter $n$ between the letters $\pi(1)$ and $\pi(2)$ we produce an ascent where the letter $n$ has only one letter to the left of it and therefore $\pi'$ would contain the pattern $p$. However by adding the letter $n$ in all other positions does not produce an ascent or it produces an ascent that has more than one letter to the left of it and therefore $\pi'$ avoids the pattern $p$. Since we are not allowed to add the letter $n$ between the letters $\pi(1)$ and $\pi(2)$, it follows that all permutations in $S_n(\ )$ start with a descent.

We let $B_n(x)$ be the distribution for the number of descents in permutations of length $n$ avoiding $p$, i.e.,

$$B_n(x) = \sum_{\pi \in S_n(\ )} x^{\operatorname{des}(\pi)}, \quad \text{and } R_{n,k} = [x^k]B_n(x).$$

In order to construct a permutation $\pi$ of length $n$ with $k$ descents, we have two choices. We can either take a permutation of length $n-1$ with $k$ descents and add $n$ to it without changing the number of descents or take a permutation of length $n-1$ with $k-1$ descents and increase the number of descents by one by adding $n$ to it. By adding $n$ between two letters making up a descent we do not change the number of descents. If we add $n$ in any of the other position we increase the number of descents by one. Above we have shown that we can add the letter $n$ to a permutation of length $n-1$ that avoids $p$ in every position except between the first two letters of the permutation. We have also shown that every permutation avoiding $p$ begins with a descent. From this it follows that

$$R_{n,k} = (k-1)R_{n-1,k} + (n-k+1)R_{n-1,k-1}.$$

We claim that $B_n(x) = xE_{n-1}(x)$, or equivalently $R_{n,k} = T_{n-1,k-1}$. We will prove this by induction. For $n = 2$ we have $B_2(x) = x^1$ and $E_1(x) = x^0$, so $R_{2,1} = 1$ and $T_{1,0} = 1$. Hence, the claim holds for the base case. Now we assume that $R_{N,k} = T_{N-1,k-1}$ holds for all $N < n$ and all $k$. By Definition 17, we have

$$\begin{aligned} R_{n,k} &= (k-1)R_{n-1,k} + (n-k+1)R_{n-1,k-1} \\ &= (k-1)T_{n-2,k-1} + (n-k+1)T_{n-2,k-2} && \text{(by induction hypothesis)} \\ &= T_{n-1,k-1}. \end{aligned}$$

Thus, the claim holds for all $n$. It is well-known that $\sum_{k=0}^{n-1} T_{n-1,k} = (n-1)!$ and therefore the number of permutations of length $n$ avoiding p is $(n-1)!$. □

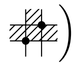
The proof of the next proposition is analogous to the previous one, but instead of considering where $n$ can be added to a permutation we consider where 1 can be added (and the rest of the values raised by 1).

**Proposition 19.** *The distribution for the number of descents in permutations of length $n$ avoiding the pattern p =*  *is*

$$\sum_{\pi \in S_n(\ \cdot\ )} x^{\mathrm{des}(\pi)} = xE_{n-1}(x).$$

*This implies*

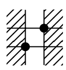
$$\left| S_n\left( \ \cdot\ \right) \right| = (n-1)!$$



*for all $n \geqslant 0$.*

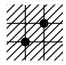
This takes care of merging the first five Wilf-subclasses into one Wilf-class. The remaining propositions in this section give the enumeration of the remaining patterns. Their proofs are left for the reader.

**Proposition 20.** *A permutation contains the pattern*



*if and only if its first letter is smaller than its last. This happens for precisely half of the permutations. Thus the the number of permutations of length $n$ avoiding the pattern is $n!/2$ for $n \geqslant 2$.*

**Proposition 21.** *The only permutation that contains the pattern*



*is* 12*. Therefore the number of permutations of length $n$ that avoid the pattern is $n!$ for $n \geqslant 3$.*

Propositions 16, 18 and 19 show that the first five subclasses of Table 3 have the same enumeration.

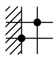
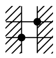
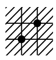
| Repr. $p$ | $\lvert S_n(p)\rvert$ | size of class | OEIS seq. |
|---|---|---|---|
|  | $(n-1)!$ | 232 | A000142 |
|  | $n!/2$, $n \geqslant 2$ | 80 | A001710 |
|  | $n!$, $n \geqslant 3$ | 2 | A000142 ($n \geqslant 3$) |

Table 4: Wilf-classes containing vincular patterns

### 4.3 Wilf-classes containing bivincular patterns

Here we will deal with Wilf-subclasses that contain bivincular patterns and other subclasses that can be merged with them. The subclasses are shown in Table 5. We will bring the number of subclasses down to 57 by combining the last two subclasses.

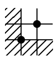
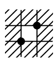
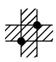
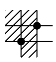
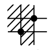
| Nr. | Repr. $p$ | $\lvert S_n(p)\rvert$ for $n = 1, \dots, 9$ | size of subclass |
|---|---|---|---|
| 12 |  | $1, 1, 4, 18, 96, 600, 4320, 35280, 322560$ | 56 |
| 13 |  | $1, 1, 5, 22, 114, 696, 4920, 39600, 357840$ | 18 |
| 14 |  | $1, 1, 3, 11, 53, 309, 2119, 16687, 148329$ | 2 |
| 15 |  | $1, 1, 3, 11, 53, 309, 2119, 16687, 148329$ | 32 |

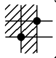
Table 5: Patterns Wilf-equivalent to bivincular patterns

Formulas for the number of permutations avoiding patterns in the first three subclasses were found by Parviainen [8]. Computer experiments show that these subclasses can not be enlarged further. He also shows that the number of permutations of length $n$ that avoid patterns in the third subclass is

$$\sum_{k=0}^{n} (-1)^k (n-k+1) \frac{n!}{k!}.$$

The following proposition shows that the last subclass in the table can be merged with the third one.

**Proposition 22.** *A permutation* $\pi$ *avoids the pattern* $p =$  *if and only if each ascent in* $\pi$ *is the* 12 *of a* 312 *pattern or the* 13 *of a* 213 *pattern, or both. If*

$$\left| S_n \left( \right. \right.$$  $$\left. \left. \right) \right| = a_n$$

*then* $a_n = (n-1)a_{n-1} + (n-2)a_{n-2}$ *and* $a_0 = a_1 = 1$*. Furthermore*

$$a_n = \sum_{k=0}^{n} (-1)^k (n-k+1)\frac{n!}{k!}.$$

*Proof.* We leave the characterization in terms of ascents to the reader, as well as the last formula for $a_n$. Let $A_n$ be the set of all permutations that avoid $p$ and $a_n$ be the size of $A_n$. Now, let $A_{n,k} = \{\pi \in A(n)\colon \pi(n) = k\}$. For $k \neq n$ we define a mapping $\varphi_k : A_{n,k} \to A_{n-1}$ such that for a permutation $\pi$, the mapping removes $k$ from $\pi$ and then subtracts 1 from all letters larger than $k$. Then we can also define $\varphi_k^{-1} : A_{n-1} \to A_{n,k}$ to be the mapping that appends $k$ to the end of a permutation $\pi$ and then adds 1 to all letters that are equal to $k$ or larger. For $k = n$ we also have a mapping $\varphi_n : A_{n,n} \to A_{n-1} \setminus A_{n-1,n-1}$ such that for a permutation $\pi$ where the last letter is $n$, the mapping removes $n$. The range of the mapping is $A_{n-1} \setminus A_{n-1,n-1}$ because if $\varphi(\pi)(n) = n-1$ then $\pi$ would end with $(n-1)n$ which is an occurrence of the pattern $p$. It is easy to see that the inverse is $\varphi_n^{-1} = A_{n-1} \setminus A_{n-1,n-1} \to A_{n,n}$; the mapping that appends $n$ to the end of a permutation.

As explained above, in order to construct all permutations of length $n$ avoiding the pattern $A_n$, we can append $n$ at the end of all permutations of length $n-1$ except for those ending with the letter $n-1$. Hence,

$$a_n = na_{n-1} - a_{n-1,n-1}.$$

From the mappings we get $a_{n,n} = a_{n-1} - a_{n-1,n-1}$ which gives $a_{n-1,n-1} = a_{n-1} - a_{n,n}$. We also have $a_{n,n} = (n-2)a_{n-2}$ since for producing a permutation from $A_{n,n}$ we take a permutation in $A_{n-2}$ and append a letter from the set $\{1, 2, \ldots, n-2\}$ to it. Lastly we append $n$ at the end of the permutation. Therefore,

$$\begin{aligned} a_n &= na_{n-1} - (a_{n-1} - a_{n,n}) \\ &= na_{n-1} - a_{n-1} + (n-2)a_{n-2} \\ &= (n-1)a_{n-1} + (n-2)a_{n-2}, \end{aligned}$$

which is what we wanted to prove. □

The results for the bivincular Wilf-classes are shown in Table 6.

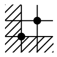
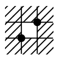
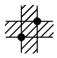
| Repr. $p$ | $\lvert S_n(p)\rvert$ | size of class | OEIS seq. |
|---|---|---|---|
|  | $n! - (n-1)!\ (n \geqslant 2)$ | 56 | A094258 |
|  | $n! - (n-2)!\ (n \geqslant 2)$ | 18 | A213167 $(n \geqslant 2)$ |
|  | $\sum_{k=0}^{n}(-1)^k(n-k+1)\frac{n!}{k!}$ | 34 | A000255 |

Table 6: Wilf-classes containing bivincular patterns

### 4.4 Enumerated Wilf-classes, not containing bivincular patterns

In this section we provide formulas for the enumeration of the Wilf-subclasses shown in Table 7. The first seven subclasses have unique enumeration sequences for $n \leqslant 9$ and can therefore not be enlarged further. The last two subclasses are shown to have the same enumeration in Propositions 30 and 31. They therefore merge into one Wilf-class. Hence, the number of Wilf-classes is decreased from 57 to 56.

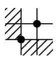
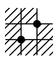
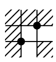
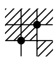
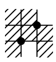
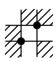
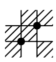
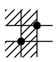
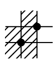
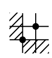
| Nr. | Repr. $p$ | $\lvert S_n(p)\rvert$ for $n = 1, \ldots, 9$ | size of subclass |
|---|---|---|---|
| 16 |  | $1, 1, 4, 17, 91, 574, 4173, 34353, 316012$ | 60 |
| 17 |  | $1, 1, 5, 21, 110, 677, 4817, 38956, 353237$ | 8 |
| 18 |  | $1, 1, 3, 13, 70, 446, 3276, 27252, 253296$ | 84 |
| 19 |  | $1, 1, 3, 14, 80, 528, 3948, 33072, 307584$ | 16 |
| 20 |  | $1, 1, 4, 19, 104, 656, 4728, 38508, 350592$ | 24 |
| 21 |  | $1, 1, 2, 8, 47, 332, 2644, 23296, 225336$ | 4 |
| 22 |  | $1, 1, 3, 15, 89, 594, 4434, 36892, 340308$ | 4 |
| 23 |  | $1, 1, 2, 7, 33, 191, 1304, 10241, 90865$ | 16 |
| 24 |  | $1, 1, 2, 7, 33, 191, 1304, 10241, 90865$ | 16 |

Table 7: Wilf-subclasses not containing bivincular patterns, with a known counting sequence

**Proposition 23.** *The number of permutations of length $n$ that avoid the pattern $p =$* 

*is*

$$n\,[x^n]\log\left(1+\sum_{n\geqslant 1}(n-1)!x^n\right).$$

*Proof.* We define two sequences,

$$a_n = \left|S_n\left(\;\right)\right| \quad \text{and} \quad b_n = |S_n(\;)|,$$

and let us show that

$$a_n = b_n + b_{n-1}. \tag{1}$$

We define a map,

$$\varphi : S_n(\;) \cup S_{n-1}(\;) \longrightarrow S_n(p).$$

The mapping $\varphi$ maps $\pi \in S_n(\;)$ to itself in $S_n(p)$, we know that $\pi$ avoids , and therefore it also avoids $p$. For $\pi \in S_{n-1}(\;)$ the mapping $\varphi$ appends the letter $n$ after the last letter in $\pi$, and we obtain a permutation of length $n$ with the letter $n$ in the $n$-th position.

The inverse map

$$\varphi^{-1} : S_n(p) \longrightarrow S_n(\;) \cup S_{n-1}(\;)$$

can be defined as follows: For $\pi \in S_n(p)$ that ends with the letter $n$, we remove $n$ and then $\varphi^{-1}(\pi)$ is in $S_{n-1}(\;)$. For $\pi \in S_n(p)$ that does not end with the letter $n$, $\pi$ maps to itself in $S_n(\;)$. The permutation $\pi$ avoids  as well for the following reasons. For a letter $u$ in $\pi$ that appears to the left of the letter $n$ there must be points in at least one of the shaded areas for $\pi$ to avoid $p$. Also for $v$ in $\pi$ that appears to the right of $n$ then there is at least one point, $n$, in the upper shaded box. Therefore, $\pi$ avoids , and hence $\varphi^{-1}(\pi) \in S_n(\;)$. This proves equation 1.

We define the generating function for $a_n$ to be

$$D(x) = \sum_{n\geqslant 1} a_n x^n.$$

The generating function for $b_n$ is given in Observation 15. Since $a_n = b_n + b_{n-1}$ we obtain

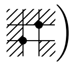
$$\begin{aligned} D(x) &= G(x) + xG(x) - 1 \\ &= \frac{\sum_{n\geqslant 0} n!x^n}{1 + x\sum_{n\geqslant 0} n!x^n} + x\left(\frac{\sum_{n\geqslant 0} n!x^n}{1 + x\sum_{n\geqslant 0} n!x^n}\right) - 1 \\ &= \frac{\sum_{n\geqslant 0} n!x^n - 1}{1 + x\sum_{n\geqslant 0} n!x^n} = \frac{\sum_{n\geqslant 1} n!x^n}{1 + \sum_{n\geqslant 1}(n-1)!x^n}. \end{aligned}$$

We now have $D(x) = xA'(x)$ where $A(x) = \log(1 + \sum_{n\geqslant 1}(n-1)!x^n)$ which implies that $A(x)$ is the logarithmic generating function of $a_n$, i.e.,

$$A(x) = \sum_{n\geqslant 1} \frac{a_n}{n} x^n = \log(1 + \sum_{n\geqslant 1}(n-1)!x^n). \qquad \square$$

**Proposition 24.** *The number of permutations of length $n$ which have $n$ in position $i$ counted from the right and contain the pattern $p =$*  *is $\frac{(n-1)!}{i}$ and therefore*

$$\left| S_n \left( \quad \right) \right| = n! - \sum_{i=1}^{n-1} \frac{(n-1)!}{i}.$$



*Proof.* To find an occurrence of the pattern $p$ in a permutation of length $n$, we must use the first letter in the permutation and the letter $n$. Then it must also hold that all the letters to the right of $n$ must be greater than the first letter. Thus, the number of permutations containing the pattern $p$ depends on the position of the letter $n$.

Recall that $i$ is the position of the letter $n$ counted from the right and let $k$ be the size of the first letter. Obviously, the letter $n$ cannot be in the last position counted from the right, and thus, $1 \leqslant k \leqslant n-1$ and $1 \leqslant i \leqslant n-1$.

Now, we choose $i-1$ letters greater than $k$ to fill the positions to the right of $n$, which can be done in $\binom{n-(k+1)}{i-1}$ ways. Then these letters can be arranged in $(i-1)!$ ways. The remaining $n-i-1$ letters will be placed between $n$ and $k$, which can be done in $(n-i-1)!$ ways. This must hold for each $1 \leqslant k \leqslant n-1$. Therefore, the number of permutations containing the pattern $p$ is

$$\sum_{k=1}^{n-1} \binom{n-k-1}{i-1} (i-1)!(n-i-1)!.$$

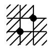
We have

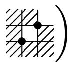
$$\begin{aligned} i \sum_{k=1}^{n-1} \binom{n-k-1}{i-1} (i-1)!(n-i-1)! \\ &= i!(n-i-1)! \sum_{k=1}^{n-1} \binom{n-k-1}{i-1} \\ &= i!(n-i-1)! \binom{n-1}{i} \\ &= i!(n-i-1)! \frac{(n-1)!}{i!(n-1-i)!} \\ &= (n-1)!, \end{aligned}$$

which implies

$$\sum_{k=1}^{n-1} \binom{n-k-1}{i-1} (i-1)!(n-i-1)! = \frac{(n-1)!}{i}.$$ □

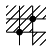
We note that a permutation contains the pattern in the next proposition if and only if it starts with 1 and the remaining letters form a permutation with a strong fixed point. The rest of the proof is left to the reader

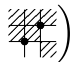
**Proposition 25.** *A permutation $\pi$ of length $n$ contains the pattern $p =$*  *if and only if $\pi$ begins with 1 and ends with a permutation of length $n-1$ that contains a strong fixed point. Thus,*

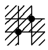
$$\left| S_n \left( \right. \right.$$  $$\left. \left. \right) \right| = n! - (n-1)! + [x^{n-1}] \frac{F(x)}{1 + xF(x)},$$

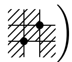
*where $F(x) = \sum_{n \geqslant 0} n!x^n$, as in Observation 15.*

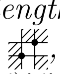
The proof of the next four propositions are similar to the proof of Proposition 24 and are therefore omitted.

**Proposition 26.** *The number of permutations of length $n$ which have $n$ in position $i$, where $i = 0$ is the rightmost position and contain the pattern $p =$*  *is $i!(n-1-i)!$. Therefore*

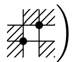
$$\left| S_n \left( \right. \right.$$  $$\left. \left. \right) \right| = n! - \sum_{i=0}^{n-2} i!(n-i-1)!.$$

**Proposition 27.** *The number of permutations of length $n$ that start with $k$ and contain the pattern $p =$*  *is $(k-1)!(n-k-1)!$ and therefore*

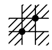
$$\left| S_n \left( \right. \right.$$  $$\left. \left. \right) \right| = n! - \sum_{k=1}^{n-1} (k-1)!(n-k-1)!.$$

**Proposition 28.** *The number of permutations of length $n$ containing the pattern $p$, with $i$ as the height of the first point of the pattern $p =$* *, counted from above, and $\ell$ as the distance between the two points, is $(i-\ell)!(n-i-\ell)!\ell!$.*

*Hence,*

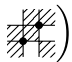
$$\left| S_n \left( \right. \right.$$  $$\left. \left. \right) \right| = n! - \sum_{i=1}^{n-1} \sum_{\ell=1}^{i} (i-\ell)!(n-i-\ell)!\ell!,$$

*for all $n \geqslant 0$.*

**Proposition 29.** *The number of permutations of length $n$ containing the pattern $p =$*  *with $k$ as the height of the first point of the pattern, counted from above, and $j$ the distance between the two points is*

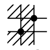
$$j!(k-j)!(n-k)!.$$

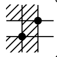
*Thus,*

$$\left| S_n \left( \right. \right.$$  $$\left. \left. \right) \right| = n! - \sum_{k=0}^{n-2} \sum_{j=0}^{k} j!(k-j)!(n-2-k)!$$

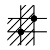
*for all $n \geqslant 0$.*

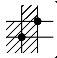
The proofs of the next two proposition follow similar arguments as the proof of Proposition 22 and are therefore omitted.

**Proposition 30.** *A permutation $\pi$ avoids the pattern $p =$*  *if and only if each ascent in $\pi$ is the 23 of a 123 pattern or the 12 of a 312 pattern, or both. If,*

$$\left| S_n \left( \right. \right.$$  $$\left. \left. \right) \right| = a_n$$

*then $a_n = n \cdot a_{n-1} - a_{n-2}$ and $a_{-1} = 0, a_0 = 1$.*

**Proposition 31.** *A permutation $\pi$ avoids the pattern $p =$*  *if and only if each ascent in $\pi$ is the 13 of a 213 pattern or the 23 of a 123 pattern, or both. If*

$$\left| S_n \left( \right. \right.$$  $$\left. \left. \right) \right| = a_n$$

*then $a_n = n \cdot a_{n-1} - a_{n-2}$ and $a_{-1} = 0, a_0 = 1$.*

Table 8 collects together the results of this section.

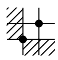
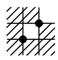
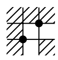
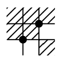
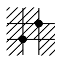
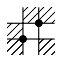
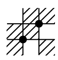
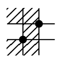
| Repr. $p$ | $\lvert S_n(p) \rvert$ | size of class | OEIS seq. |
|---|---|---|---|
|  | $n[x^n]\left(1 + \sum_{i=1}^{n}(i-1)! \cdot x^i\right)$ | 60 | A141154 |
|  | $n! - (n-1)! + [x^n]\frac{F(x)}{1+xF(x)}$ | 8 | |
|  | $n! - \sum_{i=1}^{n-1} \frac{(n-1)!}{i}$ | 84 | A121586 ($n \geqslant 2$) |
|  | $n! - \sum_{i=0}^{n-2} i!(n-1-i)!$ | 16 | |
|  | $n! - \sum_{k=1}^{n-1}(k-1)!(n-k-1)!$ | 24 | |
|  | $n! - \sum_{i=1}^{n-1}\sum_{\ell=1}^{i}(i-\ell)!(n-i-\ell)!\ell!$ | 4 | |
|  | $n! - \sum_{k=0}^{n-2}\sum_{j=0}^{k} j!(k-j)!(n-2-k)!$ | 4 | |
|  | $a_n = n \cdot a_{n-1} - a_{n-2}$, $a_{-1} = 0, a_0 = 1$ | 32 | A058797 |

Table 8: Wilf-classes not containing bivincular patterns, with a known counting sequence

### 4.5 Unenumerated Wilf-classes, not containing bivincular patterns

Tables 9 and 10 show the Wilf-subclasses that we have not mentioned before and have a unique enumeration sequence up to and including $S_9$. This shows that these subclasses cannot be merged with any other Wilf-subclasses and are therefore Wilf-classes.

Although we are unable to provide a formula for the enumeration of these classes, we do have a conjecture for one of them.

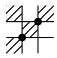
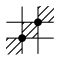
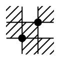
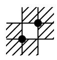
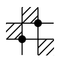
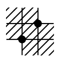
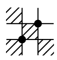
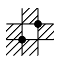
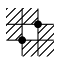
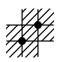
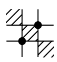
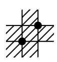
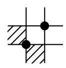
| Nr. | Repr. $p$ | $\lvert S_n(p)\rvert$ for $n = 1, \dots, 9$ | size of subclass |
|---|---|---|---|
| 25 |  | $1, 1, 3, 11, 55, 337, 2437, 20211, 188537$ | 4 |
| 26 |  | $1, 1, 2, 7, 35, 218, 1598, 13398, 126157$ | 2 |
| 27 |  | $1, 1, 4, 18, 99, 631, 4592, 37675, 344809$ | 4 |
| 28 |  | $1, 1, 3, 16, 94, 613, 4507, 37203, 341817$ | 2 |
| 29 |  | $1, 1, 2, 8, 41, 251, 1809, 14986, 139963$ | 4 |
| 30 |  | $1, 1, 5, 21, 109, 673, 4797, 38845, 352541$ | 2 |
| 31 |  | $1, 1, 3, 11, 56, 349, 2560, 21453, 201545$ | 4 |
| 32 |  | $1, 1, 3, 13, 70, 448, 3307, 27618, 257363$ | 4 |
| 33 |  | $1, 1, 5, 20, 106, 657, 4707, 38267, 348341$ | 2 |
| 34 |  | $1, 1, 4, 20, 107, 664, 4755, 38621, 351151$ | 2 |
| 35 |  | $1, 1, 3, 11, 53, 315, 2217, 17990, 165057$ | 2 |
| 36 |  | $1, 1, 3, 14, 76, 480, 3491, 28792, 265708$ | 4 |

Table 9: Wilf-classes not containing bivincular patterns, without a known counting sequence – Part I.

**Conjecture 32.** The following pattern is a representative pattern for Wilf-subclass 39 in Table 10.

$p =$ 

The number of permutations of length $n$ avoiding the pattern is the same as the absolute value of the $n$-th line in column 0 of a triangular matrix given by the formula

$$T(n,k) = \sum_{j=0}^{n-k} T(n-k,j) \cdot T(j+k-1,k-1)^2$$

for $n \geqslant k > 0$ with $T(0,0) = 1$ and $T(n,0) = -\sum_{j=1}^{n} T(n,j)$ for $n > 0$.

[2]http://oeis.org/A101900

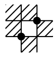
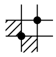
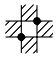
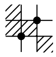
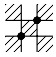
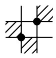
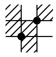
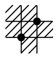
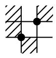
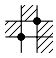
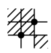
| Nr. | Repr. $p$ | $\lvert S_n(p)\rvert$ for $n = 1, \dots, 9$ | size of subclass |
|---|---|---|---|
| 38 |  | $1, 1, 4, 15, 80, 493, 3565, 29279, 269621$ | 4 |
| 39 |  | $1, 1, 2, 5, 17, 71, 357, 2101, 14203$ | 8 |
| 40 |  | $1, 1, 3, 10, 50, 290, 2018, 16023, 143601$ | 2 |
| 41 |  | $1, 1, 3, 12, 61, 376, 2715, 22416, 207950$ | 12 |
| 42 |  | $1, 1, 4, 15, 80, 501, 3666, 30467, 283196$ | 2 |
| 43 |  | $1, 1, 2, 7, 35, 217, 1586, 13287, 125237$ | 4 |
| 44 |  | $1, 1, 3, 13, 69, 437, 3209, 26751, 249329$ | 8 |
| 45 |  | $1, 1, 4, 16, 83, 512, 3671, 29983, 274757$ | 4 |
| 46 |  | $1, 1, 3, 12, 62, 387, 2819, 23409, 217949$ | 24 |
| 47 |  | $1, 1, 3, 12, 62, 385, 2789, 23040, 213566$ | 16 |

Table 10: Wilf-classes not containing bivincular patterns, without a known counting sequence – Part II. Only subclass 39 is recognized by the OEIS, as A101900.

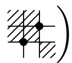
### 4.6 The remaining subclasses

In this section we will list all the remaining subclasses. As can be seen in Tables 11 and 12 some of these subclasses have the same enumeration sequence for $n \leqslant 9$. We believe those subclasses can be merged so that the final number of Wilf-classes will be 46.

We have only been able to find formula for the enumeration of one subclass.

**Proposition 33.** *The permutations avoiding the pattern* $p =$  *are the connected permutations, thus*

$$\left| S_n \left( \; \square \; \right) \right| = [x^n] \left( 1 - \frac{1}{\sum_n n! x^n} \right).$$



To be able to provide that proof we have to consider invariant sets and connected permutations.

**Definition 34.** If a permutation sends letters $1, 2, \dots, j$ to $1, 2, \dots, j$, where $0 < j < n$, we say that $\{1, 2, \dots, j\}$ is an *invariant set*. Permutations that do not contain an invariant set are called *connected* permutations.

According to Aguiar and Sottile in [1] the number of permutations of length $n$ with

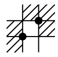
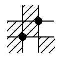
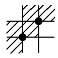
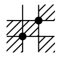
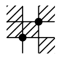
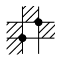
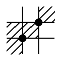
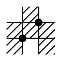
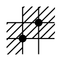
| Nr. | Repr. $p$ | $\lvert S_n(p)\rvert$ for $n = 1, \ldots, 9$ | size of subclass |
|---|---|---|---|
| 48 |  | $1, 1, 2, 9, 54, 370, 2849, 24483, 232913$ | 8 |
| 49 |  | $1, 1, 2, 9, 54, 370, 2849, 24483, 232913$ | 8 |
| 50 |  | $1, 1, 2, 9, 54, 370, 2849, 24483, 232913$ | 4 |
| 51 |  | $1, 1, 3, 12, 64, 412, 3074, 25946, 243996$ | 8 |
| 52 |  | $1, 1, 3, 12, 64, 412, 3074, 25946, 243996$ | 8 |
| 53 |  | $1, 1, 2, 8, 43, 277, 2070, 17567, 166648$ | 8 |
| 54 |  | $1, 1, 2, 8, 43, 277, 2070, 17567, 166648$ | 8 |
| 55 |  | $1, 1, 3, 15, 85, 549, 4043, 33559, 310429$ | 8 |
| 56 |  | $1, 1, 3, 15, 85, 549, 4043, 33559, 310429$ | 8 |

Table 11: Remaining subclasses – Part I

no global descents[3] is given by

$$1 - \frac{1}{\sum_n n!x^n},$$

which is the same as the number of connected permutations of length $n$.

*Proof of Proposition 33.* We will show the contrapositive, i.e., that the pattern $p$ occurs in a permutation $\pi$ if and only if $\pi$ has an invariant set.

In the pattern $p =$ , let us call the points in the pattern $v$ and $w$, respectively. If $p$ occurs in a permutation $\pi$, then there is no letter to the left of $w$ that is greater than $v$. Also, there can be no smaller letter than $v$ to the right of $w$. Hence, if there exist letters $a_1, a_2, \ldots, a_k$ to the left of $w$, then $a_1, a_2, \ldots, a_k$ are all smaller than $v$.

On one hand, if $\{a_1, a_2, \ldots, a_k\} = \varnothing$, then $v$ is both the leftmost and the smallest letter in $\pi$. Then $\{v\}$ is an invariant set. On the other hand, if $\{a_1, a_2, \ldots, a_k\} \neq \varnothing$ then the points $a_1, a_2, \ldots, a_k$ are in the first $k$ positions in $\pi$ and hence $\{a_1, a_2, \ldots, a_k, v\}$ is an invariant set.

If $\pi$ has an invariant set, we know that the lowest $k$ letters in the permutation are in the first $k$ positions. Let us choose the highest letter in the invariant set, and call it $v$. Then we choose $w$ in position $k + 1$. Now, $v$ and $w$ form the pattern $p$. □

[3] A global descent in a permutation $\pi$ of length $n$ is an index $k < n$ such that $\pi_i > \pi_j$ for all $i \leqslant k$ and $j \geqslant i + 1$.

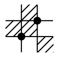
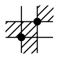
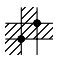
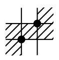
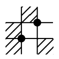
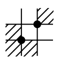
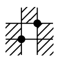
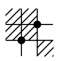
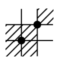
| Nr. | Repr. $p$ | $\lvert S_n(p)\rvert$ for $n = 1, \ldots, 9$ | size of subclass |
|---|---|---|---|
| 57 |  | $1, 1, 2, 8, 42, 265, 1956, 16482, 155739$ | 4 |
| 58 |  | $1, 1, 2, 8, 42, 265, 1956, 16482, 155739$ | 4 |
| 59 |  | $1, 1, 2, 9, 52, 341, 2540, 21360, 200536$ | 8 |
| 60 |  | $1, 1, 2, 9, 52, 341, 2540, 21360, 200536$ | 4 |
| 61 |  | $1, 1, 2, 8, 44, 290, 2204, 18930, 181120$ | 8 |
| 62 |  | $1, 1, 2, 8, 44, 290, 2204, 18930, 181120$ | 4 |
| 63 |  | $1, 1, 3, 13, 71, 461, 3447, 29093, 273343$ | 8 |
| 64 |  | $1, 1, 3, 13, 71, 461, 3447, 29093, 273343$ | 4 |
| 65 |  | $1, 1, 3, 13, 71, 461, 3447, 29093, 273343$ | 4 |

Table 12: Remaining subclasses – Part II

## 5 Open questions

We end with three open questions:

1. What is the actual number of Wilf-classes for mesh patterns of length 2? As mentioned above it is at most 56 but based on computer experiments we conjecture it to be 46.

2. Suppose $p$ and $q$ are mesh patterns such that $|S_n(p)| = |S_n(q)|$ for all $1 \leqslant n \leqslant N$; how large does $N$ have to be (as a function of the length of the patterns $p$ and $q$) to guarantee that $p$ and $q$ are Wilf-equivalent?

3. Is there a stronger version of the Shading Lemma (Lemma 11) that explains more pattern coincidences, perhaps strong enough to give all the coincidences between the patterns in Table 1.

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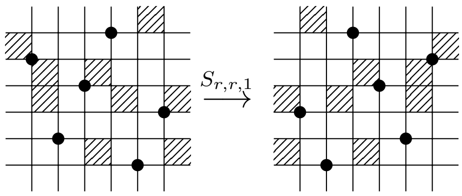
# A Switch operations

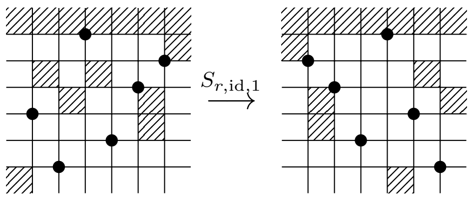
**Definition 35.** We define a set of operations called *switch operations*. Each operation takes a pattern $p$ and breaks it down into two parts by the largest letter $n$. The first part contains the part of the pattern $p$ that appears before $n$. The second part contains the part of $p$ that appears after $n$. Then one of the two operations, $\mathrm{id}, r$, is used on each part, where $(\tau, R)^{\mathrm{id}} = (\tau, R)$. After that theses parts can be switched, i.e., part two will then appear before $n$ and part one after $n$. An operation will be denoted as $S_{a,b,d}$, Where $a$ is the operation used on part one and $b$ the operation used on part two. The letter $d$ is 1 if part one and two are switched and 0 otherwise. These switch operations can also be used on permutations.

**Example 36.** This example shows the effect of the switch operation $S_{r,r,1}$ on a mesh pattern of length 6.




**Observation 37.** *Let $p = (\tau, R)$ be a pattern of length $n$ where the top line is shaded, that is $\{(0, n), (1, n), \ldots, (n, n)\} \subseteq R$. Then a permutation $\pi$ avoids $p$ if and only if $\pi^{S_{a,b,d}}$ avoids $p^{S_{a,b,d}}$. That is, the switch operations preserve Wilf-equivalence for this kind of pattern.*

**Example 38.** Here is another example showing the effect of the switch operation $S_{r,\mathrm{id},1}$ on a mesh pattern of length 6.




According to Observation 37 theses two patterns are Wilf-equivalent.

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