Inversion Formulae on Permutations Avoiding 321

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Abstract

We will study the inversion statistic of 321-avoiding permutations, and obtain that the number of 321-avoiding permutations on $[n]$ with m inversions is given by

$$|\mathcal{S}_{n,m}(321)| = \sum_{b \vdash m} \binom{n - \frac{\Delta(b)}{2}}{l(b)}.$$

where the sum runs over all compositions $b = (b_1, b_2, \ldots, b_k)$ of m, i.e.,

$$m = b_1 + b_2 + \cdots + b_k \quad \text{and} \quad b_i \geqslant 1,$$

$l(b) = k$ is the length of b, and $\Delta(b) := |b_1| + |b_2 - b_1| + \cdots + |b_k - b_{k-1}| + |b_k|$. We obtain a new bijection from 321-avoiding permutations to Dyck paths which establishes a relation on inversion number of 321-avoiding permutations and valley height of Dyck paths.

Keywords: pattern avoidance; Catalan number; Dyck path; generating function

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1 Introduction

Let $\mathcal{S}_n$ denote the permutation group on $[n] = \{1, 2, \ldots, n\}$. Write $\sigma \in \mathcal{S}_n$ in the form $\sigma = \sigma_1\sigma_2\cdots\sigma_n$. For $m \leqslant n$, if $\sigma \in \mathcal{S}_n$ and $\pi = \pi_1\cdots\pi_m \in \mathcal{S}_m$, we say that σ *contains the pattern* π if there is an index subsequence $1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n$ such that $\sigma_{i_j} < \sigma_{i_k}$ iff $\pi_j < \pi_k$ for $1 \leqslant j, k \leqslant m$, that is, σ has a subsequence which is order isomorphic to π. Otherwise, σ *avoids the pattern* π, or say, σ is π-*avoiding*. We denote by $\mathcal{S}_n(\pi)$ the set of all permutations $\sigma \in \mathcal{S}_n$ that are π-avoiding, i.e.,

$$\mathcal{S}_n(\pi) = \{\sigma \in \mathcal{S}_n \mid \sigma \text{ avoids the pattern } \pi\}.$$

For example, the permutation 41253 avoids the pattern 321, but contains the pattern 132 since its subsequence 153 is order isomorphic to 132, hence $41253 \in \mathcal{S}_5(321)$ and $41253 \notin \mathcal{S}_5(132)$.

In 1970's, Knuth [12, 13] obtained a well known result on permutations avoiding patterns, that is for any $\pi \in \mathcal{S}_3$,

$$|\mathcal{S}_n(\pi)| = C_n = \frac{1}{n+1}\binom{2n}{n},$$

where C_n is the n-th Catalan number which counts the number of Dyck paths of length $2n$. In past decades, various articles considered the bijections between 321-avoiding permutations and Dyck paths, see [4, 7, 10, 11, 14, 15, 17, 18, 19, 21, 22].

In this paper, we will study the inversion distribution of 321-avoiding permutations. For $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{S}_n(\pi)$, we define the inversion set $\mathrm{Inv}(\sigma)$ to be

$$\mathrm{Inv}(\sigma) = \{(\sigma_i, \sigma_j) \mid i < j \text{ and } \sigma_i > \sigma_j\},$$

and denote by $\mathrm{inv}(\sigma) = \#\mathrm{Inv}(\sigma)$, called the *inversion number* of σ, where the hash sign denotes cardinality. The generating function $I_n(\pi, q)$ of the inversion numbers is

$$I_n(\pi, q) = \sum_{\sigma \in S_n(\pi)} q^{\mathrm{inv}(\sigma)}.$$

for $\sigma \in \mathcal{S}_n(\pi)$. This generating function was first introduced and explored in [8, 20] and some recurrence formulae have been obtained for $\pi \in S_3$ and $\pi \neq 321$. Conjecture 3.2 of [8] states that, for all $n \geqslant 1$,

$$I_n(321, q) = I_{n-1}(321, q) + \sum_{i=0}^{n-2} q^{i+1} I_i(321, q) I_{n-i-1}(321, q). \quad (1)$$

Soon afterwards a bijective proof of the recursive formula (1) was obtained by Szu-En Cheng et al. [6]. There are some other works on inversions of restricted permutations, see [1, 3, 5, 9, 15, 16]. In 2014, M. Barnabei, F. Bonetti, S. Elizalde and M. Silimbani [2] studied the distribution of descents and major indexes of 321-avoiding involutions.

Motivated by [2, 6], in this paper we will study the inversion distribution of 321-avoiding permutations. As the main result, we give an explicit formula counting the number of 321-avoiding permutations with the fixed inversion number. We also find a bijection between 321-avoiding permutations and Dyck paths, which is new to the best of our knowledge. From this bijection, we show that the inversion number of 321-avoiding permutations and the valley-sum of Dyck paths are equally distributed.

2 Inversions of Permutations Avoiding 321

For $1 \leqslant k \leqslant n$, let $\mathcal{S}_n^k(321)$ be the collection of 321-avoiding permutations of $[n]$ and containing $12\cdots k$ as a subsequence,

$$\mathcal{S}_n^k(321) = \{\sigma \in \mathcal{S}_n(321) \mid \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(k)\}.$$

More precisely, if $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{S}_n^k(321)$ and $\sigma_{i_1} = 1, \sigma_{i_2} = 2, \ldots, \sigma_{i_k} = k$, then $i_1 < i_2 < \cdots < i_k$. Obviously, we have

$$\mathcal{S}_n(321) = \mathcal{S}_n^1(321) \supseteq \mathcal{S}_n^2(321) \supseteq \cdots \supseteq \mathcal{S}_n^n(321) = \{\mathrm{id}\}.$$

For $1 \leqslant k \leqslant n$, let $I_n^k(321, q)$ be the generating function defined by

$$I_n^k(321, q) = \sum_{\sigma \in S_n^k(321)} q^{\mathrm{inv}(\sigma)}.$$

Then we have $I_n(321, q) = I_n^1(321, q)$ and $I_n^n(321, q) = 1$ for all $n \geqslant 1$.

Lemma 1. *For $1 \leqslant k \leqslant n$, we have*

$$I_n^k(321, q) = I_n^{k+1}(321, q) + \sum_{i=1}^{k} q^i I_{n+i-k-1}^i(321, q).$$

Proof. Given $\sigma \in \mathcal{S}_n^k(321)$ with $1 \leqslant k \leqslant n-1$, consider the position of $\sigma^{-1}(k+1)$. Assuming $\sigma^{-1}(0) = 0$, we have either $\sigma^{-1}(k) < \sigma^{-1}(k+1)$, or $\sigma^{-1}(i) < \sigma^{-1}(k+1) < \sigma^{-1}(i+1)$ for some $i \leqslant k-1$. (i): If $\sigma \in \mathcal{S}_n^k(321)$ and $\sigma^{-1}(k) < \sigma^{-1}(k+1)$, it follows that $\sigma \in \mathcal{S}_n^{k+1}(321)$. So this case contributes a term $I_n^{k+1}(321, q)$ to the generating function $I_n^k(321, q)$. (ii): If $\sigma \in \mathcal{S}_n^k(321)$ and $\sigma^{-1}(i) < \sigma^{-1}(k+1) < \sigma^{-1}(i+1)$ for some $i \leqslant k-1$, since σ avoids the pattern 321, it forces that $\sigma^{-1}(j) > \sigma^{-1}(k+1)$ for all $j \geqslant k+2$. Otherwise, we have $\sigma^{-1}(j) < \sigma^{-1}(k+1) < \sigma^{-1}(i+1)$ which is obviously a contradiction. It implies that $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ satisfies $\sigma_1 = 1, \sigma_2 = 2, \ldots, \sigma_i = i, \sigma_{i+1} = k+1$. Denote by $\bar{\sigma} = \sigma_{i+2}\sigma_{i+3}\cdots\sigma_n$. Then $\bar{\sigma}$ is a permutation of $\{i+1, \ldots, k, k+2, \ldots, n\}$ satisfying $\bar{\sigma}^{-1}(i+1) < \bar{\sigma}^{-1}(i+2) < \cdots < \bar{\sigma}^{-1}(k)$ and $\mathrm{inv}(\sigma) = k - i + \mathrm{inv}(\bar{\sigma})$. It implies that case (ii) contributes a term $q^{k-i} I_{n-i-1}^{k-i}(321, q)$ to $I_n^k(321, q)$ for $0 \leqslant i \leqslant k-1$. Changing the index i to $k-i$, the proof will be complete by combining (i) and (ii). □

In the sequel, we always denote by $\bar{\delta} : \mathbb{R}^2 \to \{0,1\}$ a function such that

$$\bar{\delta}(u,v) = \begin{cases} 0, & u = v; \\ 1, & \text{otherwise.} \end{cases}$$

In order to characterize the generating function $I_n(321,q)$ as a counting function of lattice points in a lattice polytope, we introduce the following lemma.

Lemma 2. *Assuming $x_0 = 0$, for all $1 \leqslant t \leqslant n$, we have*

$$I^1_{n+1}(321,q) = \sum_{x_1=0}^{1} \sum_{x_2=x_1}^{2} \cdots \sum_{x_t=x_{t-1}}^{t} I^{t+1-x_t}_{n+1-x_t}(321,q) \prod_{i=1}^{t} q^{\bar{\delta}(x_i,x_{i-1})(i+1-x_i)}$$

Proof. The statement is true for $t = 1$ by Lemma 1. To use induction on t, suppose the above equality holds for t. From Lemma 1, we have

$$I^{t+1-x_t}_{n+1-x_t}(321,q) = \sum_{x_{t+1}=x_t}^{t+1} I^{t+2-x_{t+1}}_{n+1-x_{t+1}}(321,q)\, q^{\bar{\delta}(x_{t+1},x_t)(t+2-x_{t+1})}.$$

Using above formula to substitute the term $I^{t+1-x_t}_{n+1-x_t}(321,q)$ in the formula of this Lemma, we can easily conclude that the equality holds for the case $t+1$. ☐

Let Ω_n be a convex lattice polytope defined by

$$\Omega_n = \{(x_1 \ldots, x_n) \in \mathbb{Z}^n \mid 0 \leqslant x_1 \leqslant \cdots \leqslant x_i \leqslant i \text{ for all } 1 \leqslant i \leqslant n\}.$$

Recall that $I_{n+1}(321,q) = I^1_{n+1}(321,q)$ and $I^{n+1-x_n}_{n+1-x_n}(321,q) = 1$. From above lemma by taking $t = n$ we can easily obtain

Proposition 3. *Assuming $x_0 = 0$, we have*

$$I_{n+1}(321,q) = \sum_{x\in\Omega_n} \prod_{i=1}^{n} q^{\bar{\delta}(x_i,x_{i-1})(i+1-x_i)}.$$

In the following we will give a more explicit interpretation about this formula. Let $\mathrm{inv}_k(\sigma)$ be the number of inversions of σ whose first element is k, i.e,

$$\mathrm{inv}_k(\sigma) = \#\{i \mid (k,i) \in \mathrm{Inv}(\sigma)\}$$

It is obvious that $\mathrm{inv}_k(\sigma) \leqslant k-1$. From the definition of $I_{n+1}(321,q)$ and Proposition 3, we have

$$\sum_{\sigma\in\mathcal{S}_{n+1}(321)} q^{\mathrm{inv}(\sigma)} = \sum_{x\in\Omega_n} q^{\sum_{i=1}^{n} \bar{\delta}(x_i,x_{i-1})(i+1-x_i)}$$

Below we recursively define a map

$$\varphi : \mathcal{S}_{n+1}(321) \to \Omega_n, \qquad \varphi(\sigma) = (x_1,\ldots,x_n) = x, \tag{2}$$

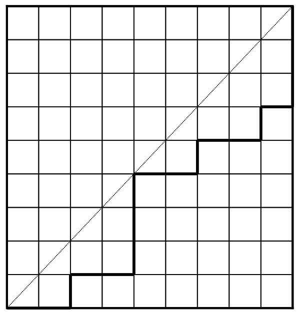

Figure 1: $\varphi : \sigma = 312579468 \mapsto (0,0,2,0,1,0,2,0,3) \mapsto x = (0,1,1,4,4,5,5,6)$

such that $x_1 = \mathrm{inv}_2(\sigma)$ and for $k \geqslant 2$,

$$x_k = \begin{cases} x_{k-1}, & \text{if } \mathrm{inv}_{k+1}(\sigma) = 0; \\ k + 1 - \mathrm{inv}_{k+1}(\sigma), & \text{otherwise.} \end{cases}$$

Figure.1 shows an example, where the second vector is $\big(\mathrm{inv}_1(\sigma), \dots, \mathrm{inv}_9(\sigma)\big)$.

Theorem 4. *The map φ defined above is a bijection. Moreover, if $\varphi(\sigma) = (x_1, \dots, x_n)$, then*

$$\mathrm{inv}(\sigma) = \sum_{i=1}^{n} \bar{\delta}(x_i, x_{i-1})(i + 1 - x_i).$$

Proof. We first show that φ is well defined in the sense that if $x = \varphi(\sigma)$ then $x \in \Omega_n = \{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid 0 \leqslant x_1 \leqslant \cdots \leqslant x_i \leqslant i \ \text{ for all } \ 1 \leqslant i \leqslant n\}$. We use induction on i. For $i = 1$, it is obvious $x_1 = inv_2(\sigma) \leqslant 1$. Suppose $0 \leqslant x_1 \leqslant \cdots \leqslant x_{i-1} \leqslant i - 1$. If $\mathrm{inv}_{i+1}(\sigma) = 0$, then $x_i = x_{i-1} \leqslant i$ by the induction hypothesis. If $\mathrm{inv}_{i+1}(\sigma) \neq 0$, then $x_i = i + 1 - \mathrm{inv}_{i+1}(\sigma) \leqslant i$. It remains to show that if $\mathrm{inv}_{i+1}(\sigma) \neq 0$, then $x_{i-1} \leqslant x_i$. Let $k \leqslant i$ be maximal such that $\mathrm{inv}_k(\sigma) \neq 0$, i.e., $\mathrm{inv}_{k+1}(\sigma) = \cdots = \mathrm{inv}_i(\sigma) = 0$. It follows that there exists an inversion $(k, l) \in \mathrm{Inv}(\sigma)$. Since σ is 321-avoiding, we have $\sigma^{-1}(k) < \sigma^{-1}(i+1)$, otherwise $(i + 1, k, l)$ is a subsequence of σ and of type 321. Hence we obtain $\mathrm{inv}_{i+1}(\sigma) \leqslant \mathrm{inv}_k(\sigma) + i - k$, and

$$x_i = i + 1 - \mathrm{inv}_{i+1}(\sigma) \geqslant k + 1 - \mathrm{inv}_k(\sigma) = x_{k-1} + 1 > x_{k-1}.$$

Note that $\mathrm{inv}_{k+1}(\sigma) = \cdots = \mathrm{inv}_i(\sigma) = 0$. By definitions, we have $x_{k-1} = \cdots = x_{i-1}$ which proves that φ is well defined. To prove the map φ is a bijection, note that each permutation σ can be uniquely recovered from its inversion vector $\big(\mathrm{inv}_1(\sigma), \dots, \mathrm{inv}_{n+1}(\sigma)\big)$. Now we construct an inverse map $\psi : \Omega_n \to \mathcal{S}_{n+1}(321)$ of φ recursively as follows. Given $x = (x_1, \dots, x_n)$, define $\psi(x) = \sigma = \sigma_1 \cdots \sigma_{n+1}$ such that $\mathrm{inv}_1(\sigma) = 0$ and for $2 \leqslant k \leqslant n + 1$,

$$\mathrm{inv}_k(\sigma) = \begin{cases} 0 & \text{if } x_{k-1} = x_{k-2}; \\ k - x_{k-1} & \text{otherwise.} \end{cases}$$

It is not difficult to see that both $\psi \circ \varphi$ and $\varphi \circ \psi$ are identity map, i.e., φ is a bijection. This completes the proof. ∎

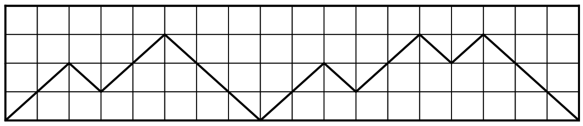

Figure 2: $x = (0,1,1,4,4,5,5,6) \mapsto D = uuduudddduuduududddd$

When $n = 3$, the inversion polynomial of $\mathcal{S}_n(321)$ is $I_3(321,q) = q^4+4q^3+5q^2+3q+1$. Below is the list of the bijection φ,

$$
\begin{aligned}
q^0 &: \{1234\} \xrightarrow{\varphi} \{(0,0,0)\};\\
q^1 &: \{1243, 2134, 1324\} \xrightarrow{\varphi} \{(0,0,3),(1,1,1),(0,2,2)\};\\
q^2 &: \{1342, 1423, 2143, 2314, 3124\} \xrightarrow{\varphi} \{(0,2,3),(0,0,2),(1,1,3),(1,2,2),(0,1,1)\};\\
q^3 &: \{2341, 2413, 3142, 4123\} \xrightarrow{\varphi} \{(1,2,3),(1,1,2),(0,1,3),(0,0,1)\};\\
q^4 &: \{3412\} \xrightarrow{\varphi} \{(0,1,2)\}.
\end{aligned}
$$

A *Dyck path* D is a lattice path from $(0,0)$ to $(2n,0)$ in the (x,y)-plane with up-steps $(1,1)$ (abbreviated as 'u') and down-steps $(1,-1)$ (abbreviated as 'd'), such that D never falls below the x-axis. A valley du of the Dyck path D is a down-step followed by an up-step. The height of a valley is defined to be the y-coordinate of its bottom. Denote by $\mathcal{D}_n$ the set of all Dyck paths of length $2n$. Several bijections between $\mathcal{S}_n(321)$ and $\mathcal{D}_n$ have been established in the literature, see [4, 7, 10, 11, 14, 15, 17, 18, 19, 21, 22]. Here we will give a new bijection obtained easily from the above theorem. Morever, this bijection will allow to read the inversion number of a permutation as the sum of all valley heights and the number of valleys in the corresponding Dyck path.

Indeed, for $x = (x_1, \ldots, x_n) \in \Omega_n$, assuming $x_0 = 0$ and $x_{n+1} = n+1$, we construct a Dyck path D_x as follows. By reading i from 1 to $n+1$, for each i we add an up-step and $x_i - x_{i-1}$ down-steps from left to right. Figure.2 presents an example. It is obvious that this construction gives an bijection from Ω_{n+1} to $\mathcal{D}_{n+1}$.

If all valleys of a Dyck path D have heights $a_1, \ldots, a_k$, denote by

$$v(D) = \sum_{i=1}^{k}(a_i+1).$$

Combining with Theorem 4, we can easily obtain our first main result.

Theorem 5. *The map $\sigma \to D_{\varphi(\sigma)}$ is a bijection from $\mathcal{S}_{n+1}(321)$ to $\mathcal{D}_{n+1}$ such that*

$$\mathrm{inv}(\sigma) = v(D_{\varphi(\sigma)}),$$

where φ is defined in (2).

As an application of Theorem 5, we will give a counting formula on the number of 321-avoiding permutations with a fixed inversion number. For any $D \in \mathcal{D}_n$, we define a *tunnel* of D to be a horizontal segment between two lattice points of D that intersects D only in these two points, and stays always below D. From Theorem 5, for $m \geqslant 0$, there is a bijection

$$\mathcal{S}_{n,m}(321) := \{\sigma \in \mathcal{S}_n(321) : inv(\sigma) = m\} \longrightarrow \mathcal{D}_{n,m} := \{D \in \mathcal{D}_n : v(D) = m\}.$$

Theorem 6. *For every $m \geqslant 0$,*

$$|\mathcal{S}_{n,m}(321)| = \sum_{b \vdash m} \binom{n - \frac{\Delta(b)}{2}}{l(b)}.$$

where the sum runs over all compositions $b = (b_1, b_2, \ldots, b_k)$ of m, denoted $b \vdash m$, i.e.,

$$m = b_1 + b_2 + \cdots + b_k \text{ and } b_i \geqslant 1,$$

$l(b) = k$ is the length of b, and $\Delta(b) := |b_1| + |b_2 - b_1| + \cdots + |b_k - b_{k-1}| + |b_k|$.

Proof. It is sufficient to consider $|\mathcal{D}_{n,m}|$. For any $D \in \mathcal{D}_{n,m}$, suppose that D has k valleys with heights $a_1, a_2, \cdots, a_k$, then $m = v(D) = \sum_{i=1}^{k}(a_i + 1)$. Let l_i be the length of the path D located between the i-th and $(i+1)$-th valley, for $i = 0, 1, 2, \cdots, k$. Then we have

$$l_i = |a_{i+1} - a_i| + 2t_i, \quad \sum_{i=0}^{k} l_i = 2n, \quad t_i \geqslant 1.$$

Where t_i is the number of tunnels between the i-th and $(i+1)$-th valley. Let $a_0 = 0$ and $a_{k+1} = 0$ be the heights of the starting point and the terminal point of the Dyck path D, respectively. Write

$$\Delta(a) = \sum_{i=0}^{k} |a_{i+1} - a_i|.$$

Then

$$\begin{aligned} & \#\{D \in \mathcal{D}_n \mid \text{all valleys of } D \text{ have heights } a_1, a_2, \ldots, a_k\} \\ = \; & \#\{(l_0, l_1, \cdots, l_k) \mid l_i = |a_{i+1} - a_i| + 2t_i, \sum_{i=0}^{k} l_i = 2n, t_i \geqslant 1\}. \\ = \; & \#\{(t_0, t_1, \cdots, t_k) \mid t_0 + t_1 + \cdots + t_k = n - \frac{\Delta(a)}{2}, t_i \geqslant 1\}. \\ = \; & \binom{n - \frac{\Delta(a)}{2} - 1}{k} \end{aligned}$$

So we have

$$|\mathcal{D}_{n,m}| = \sum_{\substack{(a_1+1)+(a_2+1)+\cdots+(a_k+1)=m \\ a_i+1 \geqslant 1}} \binom{n - \frac{\Delta(a)}{2} - 1}{k}$$

Let $b_i = a_i + 1$ for $1 \leqslant i \leqslant k$, $b_0 = b_{k+1} = 0$, obviously $\Delta(b) = \sum_{i=0}^{k} |b_{i+1} - b_i| = \Delta(a) + 2$. Hence

$$|\mathcal{D}_{n,m}| = \sum_{\substack{b_1+b_2+\cdots+b_k=m \\ b_i \geqslant 1}} \binom{n - \frac{\Delta(b)}{2}}{k}.$$

□

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