# An exact Turán result for tripartite 3-graphs

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**Abstract**

Mantel's theorem says that among all triangle-free graphs of a given order the balanced complete bipartite graph is the unique graph of maximum size. We prove an analogue of this result for 3-graphs. Let $K_4^- = \{123, 124, 134\}$, $F_6 = \{123, 124, 345, 156\}$ and $\mathcal{F} = \{K_4^-, F_6\}$: for $n \neq 5$ the unique $\mathcal{F}$-free 3-graph of order $n$ and maximum size is the balanced complete tripartite 3-graph $S_3(n)$ (for $n = 5$ it is $C_5^{(3)} = \{123, 234, 345, 145, 125\}$). This extends an old result of Bollobás that $S_3(n)$ is the unique 3-graph of maximum size with no copy of $K_4^- = \{123, 124, 134\}$ or $F_5 = \{123, 124, 345\}$.

## 1 Introduction

If $r \geqslant 2$ then an *$r$-graph* $G$ is a pair $G = (V(G), E(G))$, where $E(G)$ is a collection of $r$-sets from $V(G)$. The elements of $V(G)$ are called *vertices* and the $r$-sets in $E(G)$ are called *edges*. The number of vertices is the *order* of $G$, while the number of edges, denoted by $e(G)$, is the *size* of $G$.

Given a family of $r$-graphs $\mathcal{F}$, an $r$-graph $G$ is *$\mathcal{F}$-free* if it does not contain a subgraph isomorphic to any member of $\mathcal{F}$. For an integer $n \geqslant r$ we define the *Turán number of $\mathcal{F}$* to be

$$\mathrm{ex}(n, \mathcal{F}) = \max\{e(G) : G \text{ an } \mathcal{F}\text{-free } r\text{-graph of order } n\}.$$

The related asymptotic *Turán density* is the following limit (an averaging argument due to Katona, Nemetz and Simonovits [7] shows that it always exists)

$$\pi(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{r}}.$$

The problem of determining the Turán density is essentially solved for all 2-graphs by the Erdős–Stone–Simonovits Theorem.

**Theorem 1 (Erdős and Stone [5], Erdős and Simonovits [4])** *Let $\mathcal{F}$ be a family of 2-graphs. If $t = \min\{\chi(F) : F \in \mathcal{F}\} \geqslant 2$, then*

$$\pi(\mathcal{F}) = \frac{t-2}{t-1}.$$

It follows that the set of all Turán densities for 2-graphs is $\{0, 1/2, 2/3, 3/4, \ldots\}$.

There is no analogous result for $r \geqslant 3$ and most progress has been made through determining the Turán densities of individual graphs or families of graphs. A central problem, originally posed by Turán, is to determine $\mathrm{ex}(n, K_4^{(3)})$, where $K_4^{(3)} = \{123, 124, 134, 234\}$ is the complete 3-graph of order 4. This is a natural extension of determining the Turán number of the triangle for 2-graphs, a question answered by Mantel's theorem [9]. Turán gave a construction that he conjectured to be optimal that has density 5/9 but this question remains unanswered despite a great deal of work. The current best upper bound for $\pi(K_4^{(3)})$ is 0.561666, given by Razborov [11].

A related problem due to Katona is given by considering cancellative hypergraphs. A hypergraph $H$ is *cancellative* if for any distinct edges $a, b \in H$, there is no edge $c \in H$ such that $a \triangle b \subseteq c$ (where $\triangle$ denotes the symmetric difference). For 2-graphs, this is equivalent to forbidding all triangles. For a 3-graph, it is equivalent to forbidding the two non-isomorphic configurations $K_4^- = \{123, 124, 134\}$ and $F_5 = \{123, 124, 345\}$.

An $r$-graph $G$ is *$k$-partite* if there is a partition of its vertices into $k$ classes so that all edges of $G$ contain at most one vertex from each class. It is *complete $k$-partite* if there is a partition into $k$ classes such that all edges meeting each class at most once are present. If the partition of the vertices of a complete $k$-partite graph is into classes that are as equal as possible in size then we say that $G$ is *balanced*.

Let $S_3(n)$ be the complete balanced tripartite 3-graph of order $n$.

**Theorem 2 (Bollobás [3])** *For $n \geqslant 3$, $S_3(n)$ is the unique cancellative 3-graph of order $n$ and maximum size.*

This result was refined by Frankl and Füredi [6] and Keevash and Mubayi [8], who proved that $S_3(n)$ is the unique $F_5$-free 3-graph of order $n$ and maximum size, for $n$ sufficiently large.

The *blow-up* of an $r$-graph $H$ is the $r$-graph $H(t)$ obtained from $H$ by replacing each vertex $a \in V(H)$ with a set of $t$ vertices $V_a$ in $H(t)$ and inserting a complete $r$-partite $r$-graph between any $r$ vertex classes corresponding to an edge in $H$. The following result is an invaluable tool in determining the Turán density of an $r$-graph that is contained in the blow-ups of other $r$-graphs:

**Theorem 3 (Brown and Simonovits [1], [2])** *If $F$ is a $k$-graph that is contained in a blow-up of every member of a family of $k$-graphs $\mathcal{G}$, then $\pi(F) = \pi(F \cup \mathcal{G})$.*

Since $F_5$ is contained in $K_4^-(2)$, Theorems 2 and 3 imply that $\pi(F_5) = 2/9$.

A natural question to ask is which 3-graphs (that are not subgraphs of blow-ups of $F_5$) also have Turán density 2/9? Baber and Talbot [2] considered the 3-graph $F_6 = \{123, 124, 345, 156\}$, which is not contained in any blow-up of $F_5$. Using Razborov's flag algebra framework [10], they gave a computational proof that $\pi(F_6) = 2/9$. In this paper, we obtain a new (non-computer) proof of this result. In fact we go further and determine the exact Turán number of $\mathcal{F} = \{F_6, K_4^-\}$.

**Theorem 4** *If $n \geqslant 3$ then the unique $\mathcal{F}$-free 3-graph with $ex(n, \mathcal{F})$ edges and $n$ vertices is $S_3(n)$ unless $n = 5$ in which case it is $C_5^{(3)} = \{123, 234, 345, 145, 125\}$.*

As $F_6$ is contained in $K_4^-(2)$, we have the following corollary to Theorem 4.

**Corollary 5** $\pi(F_6) = 2/9$.

## 2 Turán number

*Proof of Theorem 4:* The underlying proof structure is the same as that employed by Keevash and Mubayi [8] in their proof of Bollobás's theorem (Theorem 2).

We use induction on $n$. Note that the result holds trivially for $n = 3, 4$. For $n = 5$ it is straightforward to check that the only $\mathcal{F}$-free 3-graphs with 4 edges are $S_3(5)$, $\{123, 124, 125, 345\}$ and $\{123, 234, 345, 451\}$. Of these the first two are edge maximal while the third can be extended by a single edge to give $C_5^{(3)}$. Thus we may suppose that $n \geqslant 6$ and the theorem is true for $n - 3$.

For $k \geqslant 2$ let $T_k(n)$ be the $k$-partite Turán graph of order $n$: this is the complete balanced $k$-partite graph. We denote the number of edges in $S_3(n)$ and $T_k(n)$ by $s_3(n)$ and $t_k(n)$ respectively. Let $G$ be $\mathcal{F}$-free with $n \geqslant 6$ vertices and $\mathrm{ex}(n, \mathcal{F})$ edges. Since $S_3(n)$ is $\mathcal{F}$-free we have $e(G) \geqslant s_3(n)$.

The inductive step proceeds as follows: select a special edge $abc \in E(G)$ (precisely how we choose this edge will be explained in Lemma 6 below). For $0 \leqslant i \leqslant 3$ let $f_i$ be the number of edges in $G$ meeting $abc$ in exactly $i$ vertices. By our inductive hypothesis we have

$$e(G) = f_0 + f_1 + f_2 + f_3 \leqslant \mathrm{ex}(n - 3, \mathcal{F}) + f_1 + f_2 + 1. \tag{1}$$

Note that unless $n - 3 = 5$ our inductive hypothesis says that $\mathrm{ex}(n - 3, \mathcal{F}) = s_3(n - 3)$ with equality iff $G - \{a, b, c\} = S_3(n - 3)$. For the moment we will assume that $n \neq 8$ and so we have the following bound

$$e(G) \leqslant s_3(n - 3) + f_1 + f_2 + 1, \tag{2}$$

with equality iff $G - \{a, b, c\} = S_3(n - 3)$.

Let $V^- = V(G) - \{a, b, c\}$. For each pair $xy \in \{ab, ac, bc\}$ define $\Gamma_{xy} = \{z \in V^- : xyz \in E(G)\}$ and let $\Gamma_{abc} = \Gamma_{ab} \cup \Gamma_{ac} \cup \Gamma_{bc}$ be the *link-neighbourhood* of $abc$. Note that since $G$ is $K_4^-$-free this is a disjoint union, so

$$f_2 = |\Gamma_{ab}| + |\Gamma_{ac}| + |\Gamma_{bc}| = |\Gamma_{abc}|.$$

For $x \in \{a, b, c\}$ define $L(x)$ to be the *link-graph of* $x$, so $V(L(x)) = V^-$ and $E(L(x)) = \{yz \subset V^- : xyz \in E(G)\}$. The *link-graph of the edge abc* is the edge labelled graph $L_{abc}$

with vertex set $V^-$ and edge set $L(a) \cup L(b) \cup L(c)$. The label of an edge $yz \in E(L_{abc})$ is $l(yz) = \{x \in \{a, b, c\} : xyz \in E(G)\}$. The *weight* of an edge $yz \in L_{abc}$ is $|l(yz)|$ and the weight of $L_{abc}$ is $w(L_{abc}) = \sum_{yz \in L_{abc}} |l(yz)|$. Note that $f_1 = w(L_{abc})$. To ease our presentation we will express the label of an edge as, for example, $ab$ rather than $\{a, b\}$.

By a subgraph of $L_{abc}$ we mean an ordinary subgraph of the underlying graph where the labels of edges are non-empty subsets of the labels of the edges in $L_{abc}$. For example if $xy \in E(L_{abc})$ has $l(xy) = ab$ then in any subgraph of $L_{abc}$ containing the edge $xy$ it must have label $a, b$ or $ab$.

A triangle in $L_{abc}$ is said to be *rainbow* iff all its edges have weight one and are labelled $a, b, c$. Given an edge labelled subgraph $H$ of $L_{abc}$ and an (unlabelled) graph $G$ we say that $H$ is a *rainbow* $G$ if all of the edges in $H$ have weight 1 and all the triangles in $H$ are rainbow.

The following lemma provides our choice of edge $abc$.

**Lemma 6** *If $G$ is an $\mathcal{F}$-free 3-graph with $n \geqslant 6$ vertices and $ex(n, \mathcal{F})$ edges then there is an edge $abc \in E(G)$ such that*

$$f_1 + f_2 = w(L_{abc}) + |\Gamma_{abc}| \leqslant t_3(n-3) + n - 3,$$

*with equality iff $L_{abc}$ is a rainbow $T_3(n-3)$ and $\Gamma_{abc} = V^-$.*

Underlying all our analysis are some simple facts regarding $\mathcal{F}$-free 3-graphs that are contained in Lemmas 7 and 8.

**Lemma 7** *If $G$ is $\mathcal{F}$-free and $abc \in E(G)$ then the following configurations cannot appear as subgraphs of $L_{abc}$. Moreover any configuration that can be obtained from one described below by applying a permutation to the labels $\{a, b, c\}$ must also be absent.*

*($F_6$-1) The triangle $xy, xz, yz$ with $l(xy) = l(xz) = a$ and $l(yz) = b$.*

*($F_6$-2) The pair of edges $xy, xz$ with $l(xy) = ab$ and $l(xz) = c$.*

*($F_6$-3) A vertex $x \in \Gamma_{ab}$ and edges $xy, yz$ with labels $l(xy) = c$ and $l(yz) = a$.*

*($F_6$-4) A vertex $x \in \Gamma_{ab}$ and edges $xy, yz, zw$ with labels $l(xy) = l(zw) = a$ and $l(yz) = b$.*

*($F_6$-5) Vertices $x \in \Gamma_{ac}, y \in \Gamma_{bc}, z \in \Gamma_{ab}$ and the edge $xy$ with label $l(xy) = b$.*

*($K_4^-$-1) The triangle $xy, xz, yz$ with $l(xy) = l(xz) = l(yz) = a$.*

*($K_4^-$-2) The vertex $x \in \Gamma_{ab}$ and edge $xy$ with label $l(xy) = ab$.*

*($K_4^-$-3) The vertices $x, y \in \Gamma_{ab}$ and edge $xy$ with label $l(xy) = a$.*

**Lemma 8** *If $G$ is $\mathcal{F}$-free and $abc \in E(G)$ then the link-graph and link-neighbourhood satisfy:*

*(i) The only triangles in $L_{abc}$ are rainbow.*

*(ii)* *The only* $K_4$*s in* $L_{abc}$ *are rainbow.*

*(iii)* $L_{abc}$ *is* $K_5$*-free.*

*(iv)* *If* $xy \in E(L_{abc})$ *has* $l(xy) = abc$ *then* $x$ *and* $y$ *meet no other edges in* $L_{abc}$ *and* $x, y \notin \Gamma_{abc}$.

*(v)* *If* $V^4_{abc} = \{x \in V^- :$ *there is a* $K_4$ *containing* $x\}$ *then* $\Gamma_{abc} \cap V^4_{abc} = \varnothing$.

*(vi)* *There are no edges in* $L_{abc}$ *between* $\Gamma_{abc}$ *and* $V^4_{abc}$.

*(vii)* *If* $x \in V^4_{abc}$ *then* $|l(xy)| \leqslant 1$ *for all* $y \in V^-$.

*(viii)* *If* $x \in \Gamma_{ac}$, $y \in \Gamma_{bc}$ *and* $l(xy) = ab$, *then* $\Gamma_{bc} = \varnothing$. *Moreover, if* $xz \in E(L_{abc})$ *with* $z \neq y$ *then* $z \notin \Gamma_{abc}$ *and* $l(xz) = a$, *while if* $yz \in E(L_{abc})$ *with* $z \neq x$ *then* $z \notin \Gamma_{abc}$ *and* $l(yz) = b$.

*(ix)* *If* $xy, xz \in E(L_{abc})$, $l(xy) = ab$ *and* $z \in \Gamma_{abc}$ *then* $|l(xz)| \leqslant 1$.

We also require the following identities, that are easy to verify.

**Lemma 9** *If* $n \geqslant k \geqslant 3$ *then*

*(i)* $s_3(n) = s_3(n-3) + t_3(n-3) + n - 2$.

*(ii)* $t_3(n) = t_3(n-3) + 2n - 3$.

*(iii)* $t_3(n) = t_3(n-2) + n - 1 + \lfloor n/3 \rfloor$.

*(iv)* $t_k(n) = t_k(n-1) + n - \lceil n/k \rceil$.

Let $abc \in E(G)$ be a fixed edge given by Lemma 6.

By assumption $e(G) \geqslant s_3(n)$ so Lemma 9 (i) and Lemma 6 together with the bound on $e(G)$ given by (2) imply that $e(G) = s_3(n)$ and hence $G - \{a,b,c\} = S_3(n-3)$, $L_{abc}$ is a rainbow $T_3(n-3)$ and $\Gamma_{abc} = V^-$. To complete the proof we need to show that $G = S_3(n)$. First note that since $L_{abc}$ is a rainbow $T_3(n-3)$ and $\Gamma_{abc} = V^-$, Lemma 8 (i) and Lemma 7($F_6$-3) imply that no vertex in $\Gamma_{ab}$ is in an edge with label $c$ and similarly for $\Gamma_{ac}, \Gamma_{bc}$. Hence $L_{abc}$ is the complete tripartite graph with vertex classes $\Gamma_{ab}$, $\Gamma_{ac}$ and $\Gamma_{bc}$ and the edges between any two parts are labelled with the common label of the parts (e.g. all edges from $\Gamma_{ab}$ to $\Gamma_{ac}$ receive label $a$). So $L_{abc}$ is precisely the link graph of an edge $abc \in S_3(n)$.

In order to deduce that $G = S_3(n)$ we need to show that $G - \{a,b,c\} = S_3(n-3)$ has the same tripartition as $L_{abc}$. This is straightforward: any edge $xyz \in E(G - \{a,b,c\})$ not respecting the tripartition of $L_{abc}$ meets one of the parts at least twice. But if $x, y, z \in \Gamma_{ab}$ then $|\Gamma_{ac}| \geqslant 2$ so let $u \in \Gamma_{ac}$. Setting $a = 1, b = 2, x = 3, y = 4, z = 5, u = 6$ gives a copy of $F_6$. If $x, y \in \Gamma_{ab}$ and $z \in \Gamma_{ac}$ then $a = 1, x = 3, y = 4, z = 2$ gives a copy of $K_4^-$.

Hence $G = S_3(n)$ and the proof is complete in the case $n \neq 8$.

For $n = 8$ we note that if $G - \{a, b, c\}$ is $F_5$-free then Theorem 2 implies that the result follows as above, so we may assume that $G - \{a, b, c\}$ contains a copy of $F_5$. In this case it is sufficent to show that $e(G) \leqslant 17 < 18 = s_3(8)$.

If $V(G - \{a, b, c\}) = \{s, t, u, v, w\}$ then we may suppose that $stu, stv, uvw, abc \in G$. Since $G$ is $K_4^-$-free it does not contain $suv$ or $tuv$. Moreover it contains at most 3 edges from $\{u, v, w\}^{(2)} \times \{a, b, c\}$ and at most 5 edges from $\{s, t, u, v, w\} \times \{a, b, c\}^{(2)}$. Since $G$ is $F_6$-free it contains no edges from $\{s, t\} \times \{w\} \times \{a, b, c\}$.

The only potential edges we have yet to consider are those in $\{st, su, tu, sv, tv\} \times \{w, a, b, c\}$. Since $G$ is $K_4^-$-free it contains at most 2 edges from $std, sud, tud, svd, tvd$, for any $d \in \{w, a, b, c\}$. Moreover, since $G$ is $F_6$-free, if it contains 2 such edges for a fixed $d$ then it can contain at most 3 such edges in total for the other choices of $d$. Hence at most 5 such edges are present.

Thus in total $e(G) \leqslant 4 + 3 + 5 + 5 = 17$, as required. □

In order to prove Lemma 6 we first need an edge with large link-neighbourhood.

**Lemma 10** *If $G$ is $K_4^-$-free 3-graph of order $n$ with $s_3(n)$ edges, then there is an edge $abc \in E(G)$ with $|\Gamma_{abc}| \geqslant n - \lfloor n/3 \rfloor - 3$.*

*Proof of Lemma 10:* Let $G$ be $K_4^-$-free with $n$ vertices and $s_3(n)$ edges. For $x, y \in V(G)$ let $d_{xy} = |\{x : xyz \in E(G)\}|$. If $uvw \in E(G)$ then $\Gamma_{uvw} = \Gamma_{uv} \cup \Gamma_{uw} \cup \Gamma_{vw}$ is a union of pairwise disjoint sets and $|\Gamma_{uvw}| = d_{uv} + d_{uw} + d_{vw} - 3$. Thus if the lemma fails to hold then for every edge $uvw \in E(G)$ we have $d_{uv} + d_{uw} + d_{vw} \leqslant n - \lfloor n/3 \rfloor - 1$. Note that since $\sum_{xy \in \binom{V}{2}} d_{xy} = 3e(G)$, convexity implies that

$$e(G)(n - \left\lfloor \frac{n}{3} \right\rfloor - 1) \geqslant \sum_{uvw \in E(G)} d_{uv} + d_{uw} + d_{vw} = \sum_{xy \in \binom{V}{2}} d_{xy}^2 \geqslant \frac{9e^2(G)}{\binom{n}{2}}.$$

Thus

$$e(G) \leqslant \frac{1}{18} n(n-1)(n - \lfloor n/3 \rfloor - 1).$$

But it is easy to check that this is less than $s_3(n)$. □

Our next objective is to describe various properties of the link-graph $L_{abc}$ and link-neighbourhood $\Gamma_{abc}$.

Lemma 8 (v) allows us to partition the vertices of $L_{abc}$ as $V^- = \Gamma_{abc} \cup V_{abc}^4 \cup R_{abc}$, where $V_{abc}^4 = \{x \in V^- : \text{there is a } K_4 \text{ containing } x\}$ and $R_{abc} = V^- - (\Gamma_{abc} \cup V_{abc}^4)$. To prove Lemma 6 we require the following result to deal with the part of $L_{abc}$ not meeting any copies of $K_4$.

**Lemma 11** *Let $H$ be a subgraph of $L_{abc}$ with $s \geqslant 3$ vertices satisfying $V(H) \cap V_{abc}^4 = \varnothing$. If $H_\Gamma = V(H) \cap \Gamma_{abc}$ and $|H_\Gamma| \geqslant s - \lfloor s/3 \rfloor - 1$ then*

$$w(H) + |H_\Gamma| \leqslant t_3(s) + s,$$

*with equality iff $H_\Gamma = V(H)$ and $H$ is a rainbow $T_3(s)$.*

*Proof of Lemma 6:* Let $G$ be $\mathcal{F}$-free with $n \geqslant 6$ vertices and $\mathrm{ex}(n, \mathcal{F})$ edges. By Lemma 10 we can choose an edge $abc \in E(G)$ such that $|\Gamma_{abc}| \geqslant n - \lfloor n/3 \rfloor - 3$. Let $V^- = \Gamma_{abc} \cup R_{abc} \cup V^4_{abc}$ be the partition of $V^-$ given by Lemma 8 (v). If $s = |V^-|$, $j = |\Gamma_{abc}|$, $k = |R_{abc}|$ and $l = |V^4_{abc}|$ then $n - 3 = s = j + k + l$ and $j \geqslant s - \lfloor s/3 \rfloor - 1 \geqslant j + k - \lfloor (j + k)/3 \rfloor - 1$. We can apply Lemma 11 to $H = L_{abc}[\Gamma_{abc} \cup R_{abc}]$, to deduce that

$$w(L_{abc}[\Gamma_{abc} \cup R_{abc}]) + |\Gamma_{abc}| \leqslant t_3(j + k) + j + k,$$

with equality iff $R_{abc} = \varnothing$ and $L_{abc}[\Gamma_{abc}]$ is a rainbow $T_3(j + k)$. Now if $L_{abc}$ is $K_4$-free then $V^4_{abc} = \varnothing$ and the proof is complete, so suppose there is a $K_4$ in $L_{abc}$. In this case $4 \leqslant |V^4_{abc}| \leqslant n - 3 - |\Gamma_{abc}| \leqslant \lfloor n/3 \rfloor$, so $n \geqslant 12$.

We now need to consider the edges in $L_{abc}$ meeting $V^4_{abc}$. By Lemma 8 (iii) we know that $L_{abc}$ is $K_5$-free, while Lemma 8 (vii) says that $V^4_{abc}$ meets no edges of weight 2 or 3, so by Turán's theorem $w(L_{abc}[V^4_{abc}]) \leqslant t_4(l)$.

Lemma 8 (vi) implies that there are no edges from $\Gamma_{abc}$ to $V^4_{abc}$ so the total weight of edges between $\Gamma_{abc} \cup R_{abc}$ and $V^4_{abc}$ is at most $kl$. Thus

$$w(L_{abc}) + |\Gamma_{abc}| \leqslant t_3(j + k) + j + k + t_4(l) + kl.$$

Finally Lemma 12 with $s = n - 3$ implies that

$$w(L_{abc}) + |\Gamma_{abc}| \leqslant t_3(n - 3) + n - 3,$$

with equality iff $R_{abc} = V^4_{abc} = \varnothing$ and $L_{abc}$ is a rainbow $T_3(n - 3)$ as required. □

**Lemma 12** *If $j, k, l \geqslant 0$ are integers satisfying $j + k + l = s \geqslant 5$ and $j \geqslant s - \lfloor s/3 \rfloor - 1$ then*

$$t_3(j + k) + t_4(l) + j + k + kl \leqslant t_3(s) + s, \tag{3}$$

*with equality iff $l = 0$.*

*Proof of Lemma 12:* If $l = 0$ then the result clearly holds, so suppose that $l \geqslant 1$, $j + k + l = s \geqslant 5$ and $j \geqslant s - \lfloor s/3 \rfloor - 1$. Let $f(j, k, l)$ be the LHS of (3). We need to check that $\Delta(j, k, l) = f(j, k + 1, l - 1) - f(j, k, l) > 0$. Since if this holds then we have

$$f(j, k, l) < f(j, k + 1, l - 1) < \cdots < f(j, k + l, 0) = t_3(s) + s.$$

Using Lemma 9 (iv) we have

$$\begin{aligned} \Delta(j, k, l) &= j - \lceil (j + k + 1)/3 \rceil + \lceil l/4 \rceil + 1 \\ &= j + \lceil l/4 \rceil - \lfloor (j + k)/3 \rfloor. \end{aligned}$$

So it is sufficient to check that $j + l/4 > (j + k)/3$. This follows easily from $j \geqslant s - \lfloor s/3 \rfloor - 1$, $k \leqslant \lfloor s/3 \rfloor + 1$, $l \geqslant 1$ and $s \geqslant 5$. □

*Proof of Lemma 11:* We prove this by induction on $s \geqslant 3$. The result holds for $s = 3, 4$ (see the end of this proof for the tedious details) so suppose that $s \geqslant 5$ and the result holds for $s - 2$.

Let $H$ be a subgraph of $L_{abc}$ with $s \geqslant 5$ vertices satisfying $V(H) \cap V^4_{abc} = \varnothing$. Let $H_\Gamma = V(H) \cap \Gamma_{abc}$ and suppose that $|H_\Gamma| \geqslant s - \lfloor s/3 \rfloor - 1$.

Note that if $H$ contains no edges of weight 2 or 3 then the result follows directly from Turán's theorem and Lemma 8 (i), so we may suppose there are edges of weight 2 or 3. With this assumption it is sufficient to show that

$$w(H) + |H_\Gamma| \leqslant t_3(s) + s - 1.$$

By Lemma 9 (iii) this is equivalent to showing that the following inequality holds:

$$w(H) + |H_\Gamma| \leqslant t_3(s-2) + 2s - 2 + \lfloor s/3 \rfloor \tag{4}$$

**Case (i):** There exists an edge of weight 3, $l(xy) = abc$.

Lemma 8 (iv) implies that $x, y \notin H_\Gamma$ and $x, y$ meet no other edges in $H$, so we can apply the inductive hypothesis to $H' = H - \{x, y\}$ to obtain

$$w(H) + |H_\Gamma| \leqslant w(H') + |H'_\Gamma| + 3 \leqslant t_3(s-2) + s - 2 + 3.$$

Hence (4) holds as required. So we may suppose that $H$ contains no edges of weight 3.

**Case (ii):** The only edges of weight 2 are contained in $H_\Gamma$

Let $xy \in E(H)$ have weight 2, say $l(xy) = ab$. Now Lemma 7 ($K_4^-$-2) implies that $x, y \notin \Gamma_{ab}$, while Lemma 7 ($K_4^-$-3) implies that $x, y$ cannot both belong to $\Gamma_{ac}$ or $\Gamma_{bc}$ so we may suppose that $x \in \Gamma_{ac}$ and $y \in \Gamma_{bc}$. Lemma 8 (viii) implies that $x, y$ have no more neighbours in $H_\Gamma$. If $H_\Gamma = V(H)$ then we can apply the inductive hypothesis to $H' = H - \{x, y\}$ to obtain

$$w(H) + |H_\Gamma| \leqslant t_3(s-2) + s - 2 + 2 + 2,$$

in which case (4) holds, so suppose $V(H) \neq H_\Gamma$.

Let $z \in V(H) - H_\Gamma$ be a neighbour of $x$ in $H$ if one exists otherwise let $z$ be any vertex in $V(H) - H_\Gamma$. By our assumption that all edges of weight 2 are contained in $H_\Gamma$, $z$ meets no edges of weight 2. Moreover, by Lemma 8 (viii), all edges containing $x$ (except $xy$) have label $b$, so $x$ is not in any triangles in $H$. Hence $x$ and $z$ have no common neighbours in $H$ and so the total weight of edges meeting $\{x, z\}$ is at most $2 + 1 + s - 3$ (if $xz$ is an edge) and at most $2 + s - 2$ otherwise. Applying our inductive hypothesis to $H' = H - \{x, z\}$ we have

$$w(H) + |H_\Gamma| \leqslant t_3(s-2) + s - 2 + 1 + s,$$

and (4) holds.

**Case (iii):** There is an edge of weight 2 meeting $V(H) - H_\Gamma$.

So suppose that $xy \in E(H)$, $l(xy) = ab$ and $y \notin H_\Gamma$. Lemma 8 (ix) implies that for any $z \in H_\Gamma$ we have $|l(xz)|, |l(yz)| \leqslant 1$. Let $\gamma_{xy} = |\{x, y\} \cap H_\Gamma| \leqslant 1$. Thus, since $xy$ is not in any triangles, the total weight of edges meeting $\{x, y\}$ is at most

$$2 + s - 2 + |V(H) - H_\Gamma| - (2 - \gamma_{xy}).$$

Applying the inductive hypothesis to $H' = H - \{x, y\}$ we have

$$w(H) + |H_\Gamma| \leqslant t_3(s-2) + s - 2 + s + s - |H_\Gamma| - 2 + 2\gamma_{xy},$$

with equality holding only if $|H'_\Gamma| = s - 2$. Now $|H_\Gamma| \geqslant s - \lfloor s/3 \rfloor - 1$ implies that

$$w(H) + |H_\Gamma| \leqslant t_3(s-2) + 2s - 3 + \lfloor s/3 \rfloor + 2\gamma_{xy}, \tag{5}$$

with equality only if $|H'_\Gamma| = s - 2$ and $|H_\Gamma| = s - \lfloor s/3 \rfloor - 1$. If $\gamma_{xy} = 0$ then (4) holds as required, so suppose $\gamma_{xy} = 1$. In this case (4) holds, unless (5) holds with equality. But if (5) is an equality then $|H_\Gamma| = |H'_\Gamma| + 1 = s - 1$, while $|H_\Gamma| = s - \lfloor s/3 \rfloor - 1$, which is impossible for $s \geqslant 3$.

We finally need to verify the cases $s = 3, 4$. It is again sufficient to prove that if $H$ contains edges of weight 2 or 3 then $w(H) + |H_\Gamma| \leqslant t_3(s) + s - 1$, thus we need to show that $w(H) + |H_\Gamma|$ is at most 5 if $s = 3$ and at most 8 if $s = 4$.

We note that argument in Case (i) above implies that if $H$ contains an edge of weight 3 then $|H_\Gamma| \leqslant s - 2$ and $w(H) \leqslant 3 + 3\binom{s-2}{2}$, so if $s = 3$ then $w(H) + |H_\Gamma| \leqslant 4$ and if $s = 4$ then $w(H) + |H_\Gamma| \leqslant 8$ so the result holds. So we may suppose there are no edges of weight 3.

Now let $xy$ be an edge of weight 2. Using the fact that $xy$ is not in any triangles and Lemma 8 (viii) and (ix) we find that for $s = 3$ we have $w(H) + |H_\Gamma| \leqslant 2 + 3 - |H_\Gamma|$, while for $s = 4$ we have $w(H) + |H_\Gamma| \leqslant 2 + 6 - |H_\Gamma|$, so the result holds. □

Finally we need to establish our two stuctural lemmas.

*Proof of Lemma 7:* In each case we describe a labelling of the vertices of the given configuration to show that if it is present then $G$ is not $\mathcal{F}$-free.

($F_6$-1) $a = 1$, $b = 5$, $c = 6$, $x = 2$, $y = 3$, $z = 4$.

($F_6$-2) $a = 3$, $b = 4$, $c = 5$, $x = 1$, $y = 2$, $z = 6$.

($F_6$-3) $a = 1$, $b = 2$, $c = 3$, $x = 4$, $y = 5$, $z = 6$.

($F_6$-4) $a = 1$, $b = 3$, $x = 2$, $y = 4$, $z = 5$, $w = 6$.

($F_6$-5) $a = 5$, $b = 1$, $c = 3$, $x = 4$, $y = 2$, $z = 6$.

($K_4^-$-1) $a = 1$, $x = 2$, $y = 3$, $z = 4$.

($K_4^-$-2) $a = 3$, $b = 4$, $x = 1$, $y = 2$.

($K_4^-$-3) $a = 1$, $b = 2$, $x = 3$, $y = 4$. □

*Proof of Lemma 8:* We will make repeated use of Lemma 7.

(i) This follows immediately from ($F_6$-1) and ($K_4^-$-1).

(ii) This follows immediately from (i): if $uvwx$ is a copy of $K_4$ then we may suppose $l(uv) = a, l(uw) = b, l(vw) = c$, thus $l(ux) = c$ (otherwise (i) would be violated) continuing we see that $uvwx$ must be rainbow.

(iii) This follows immediately from (ii): if $xyzuv$ is a copy of $K_5$ then by (ii) we may suppose that $l(xy), l(xz), l(xu), l(xv)$ are all distinct single labels from $\{a, b, c\}$ but this is impossible since there are only 3 labels in total.

(iv) This follows immediately from ($F_6$-2) and ($K_4^-$-2).

(v) If $x$ is in a $K_4$ then by (ii) it lies in edges with labels $a, b, c$, so ($F_6$-3) implies that $x \notin \Gamma_{abc}$.

(vi) If $x \in \Gamma_{abc}$, say $x \in \Gamma_{ab}$, and $y \in V_{abc}^4$ with $xy \in E(L_{abc})$ then ($F_6$-3) implies that $l(xy) \neq c$, while ($F_6$-4) implies that $l(xy) \neq a, b$ (since there are $t, u, v, w$ such that $l(yt) = b, l(tu) = a$ and $l(yv) = a, l(vw) = b$).

(vii) This follows immediately from the fact that all $v \in V_{abc}^4$ meet edges with labels $a, b, c$ and ($F_6$-2).

(viii) ($F_6$-5) implies that $\Gamma_{bc} = \varnothing$. If $xz \in E(L_{abc})$ then ($F_6$-3) implies that $l(xz) = a$. Now ($K_4$-3) implies that $z \notin \Gamma_{ac}$ while ($F_6$-3) implies that $z \notin \Gamma_{bc}$. Hence $z \notin \Gamma_{abc}$. Similarly if $yz \in E(L_{abc})$ then $l(yz) = b$ and $z \notin \Gamma_{abc}$.

(ix) If $x \in \Gamma_{abc}$ or $y \in \Gamma_{abc}$ then this follows directly from (viii) so suppose that $x, y \notin \Gamma_{abc}$, $l(xy) = ab$ and $|l(xz)| = 2$. In this case, ($F_6$-2) implies that $l(xz) = ab$ so ($K_4$-2) implies that $z \in \Gamma_{ac} \cup \Gamma_{bc}$. But then ($F_6$-3) is violated. Hence $|l(xz)| \leqslant 1$. □

## 3 Conclusion

Many Turán-type results have associated "stability" versions, and we were able to obtain such a result. For reasons of length we state it without proof.

**Theorem 13** *For any $\epsilon > 0$ there exist $\delta > 0$ and $n_0$ such that the following holds: if $H$ is an $\mathcal{F}$-free 3-graph of order $n \geqslant n_0$ with at least $(1 - \delta)\, s_3(n)$ edges, then there is a partition of the vertex set of $H$ as $V(H) = U_1 \cup U_2 \cup U_3$ so that all but at most $\epsilon n^3$ edges of $H$ have one vertex in each $U_i$.*


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