Digraph representations of 2-closed permutation groups with a normal regular cyclic subgroup

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Abstract

In this paper, we classify 2-closed (in Wielandt's sense) permutation groups which contain a normal regular cyclic subgroup and prove that for each such group G, there exists a circulant Γ such that $\mathrm{Aut}(\Gamma) = G$.

1 Introduction

In 1969, Wielandt [15] introduced the concept of the 2-closure of a permutation group. Let G be a finite permutation group on a set Ω, the *2-closure* $G^{(2)}$ of G on Ω is the largest subgroup of $\mathrm{Sym}(\Omega)$ containing G that has the same orbits as G in the induced action on $\Omega \times \Omega$, and we say G is *2-closed* if $G = G^{(2)}$. It seems impossible to classify all 2-closed transitive permutation groups. However, certain classes of 2-closed transitive groups have been determined. For example, in [16, 17] the author determined all 2-closed odd-order transitive permutation groups of degree pq where p, q are distinct odd primes. In this paper, one of our main purposes is to classify all 2-closed permutation groups with a normal regular cyclic subgroup, see Theorem 1.2. Recall that a permutation group is *regular* if it is transitive and the only element that fixes a point is the identity. And for more information about the 2-closures of permutation groups containing a cyclic regular subgroup, see also [7].

Another research topic of this paper is the study of the automorphism groups of (di)graphs. The full automorphism group of a (di)graph Γ must be 2-closed since any permutation of the vertex set that preserves the orbits of $\mathrm{Aut}(\Gamma)$ on ordered pairs preserves adjacency. However, not every 2-closed permutation group is the full automorphism group

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of some (di)graph. Therefore, the concept of 2-closed groups is more general than the concept of the full automorphism groups of (di)graphs, and the classification of 2-closed groups is closely related to the study of the full automorphism groups of the corresponding digraphs. In this paper, in order to determine 2-closed groups that contain a normal regular cyclic subgroup, we also study circulant digraphs, that is Cayley digraphs of cyclic groups. See Section 2 for a more detailed explanation.

Furthermore, we discuss the following representation problem. A digraph Γ with vertex set Ω is said to *represent* a permutation group $G \leqslant \mathrm{Sym}(\Omega)$ if $\mathrm{Aut}(\Gamma) = G$. In this case, we also say that the permutation group G has a digraph *representation* Γ.

Digraph representation problem: given a 2-closed group G, is there a digraph Γ that represents G?

Suppose the digraph Γ represents a 2-closed group $G \leqslant \mathrm{Sym}(\Omega)$. Then for any $g \in \mathrm{Sym}(\Omega)$, to determine whether g lies in G we only need to test if g preserves the single 2-relation given by the arc set of Γ, instead of checking all G-invariant 2-relations. We say a digraph Γ is *arc-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on the arc set of Γ. This means, the arc set of Γ is actually a minimal $\mathrm{Aut}(\Gamma)$-invariant 2-relation. Suppose further that the 2-closed group G can be represented by an arc-transitive digraph Γ. Then a permutation g lies in G if and only if g leaves invariant the minimal G-invariant 2-relation given by the arc set of Γ. We will show that there are arc-transitive digraph representations for most 2-closed groups that contain a normal regular cyclic subgroup, see the remark after Lemma 3.12.

Replacing digraph with graph, we obtain the graph representation problem which asks for an undirected graph to represent a 2-closed group. These two questions have previously appeared in the literature, see for example [1, 4]. Clearly, the graph version problem is much more complicated than the digraph one. Since we are interested in understanding the concept of 2-closed groups, we concentrate on the digraph representation problem in this paper.

A regular permutation group is 2-closed, and in 1980, Babai [2] proved that with five exceptions, every finite regular permutation group occurs as the automorphism group of a digraph. This is the famous DRR (digraphical regular representations) problem [2]. It is proved in [14] that for any prime power q, the semilinear group $\Gamma\mathrm{L}(1, q)$ can be represented by an arc-transitive circulant digraph. Moreover, it is shown in [16, 17] that every 2-closed odd-order transitive permutation group of degree pq has a tournament digraph representation. As for graphical representation problem, see for example [3, 6, 8, 9, 10, 13].

In this paper, we will prove that every 2-closed permutation group G with a normal regular cyclic subgroup is the full automorphism group of a circulant digraph. We may suppose that $G = Z_n \rtimes G_0$ acting on Z_n naturally where $G_0 \leqslant \mathrm{Aut}(Z_n)$. We first describe the necessary and sufficient condition for G_0 such that G is 2-closed. For the detailed explanation of notation, see Section 2 and Section 3.3.1.

Conditions 1.1. *Let* $n = 2^{d_1} p_2^{d_2} \cdots p_t^{d_t}$, $d_1 \geqslant 0$, $d_2, \ldots, d_t \geqslant 1$, $t \geqslant 1$ *where* $p_2, \ldots, p_t$ *are distinct odd primes (also write* $p_1 = 2$*). And let* $\mathrm{Aut}(Z_n) = \mathrm{Aut}(Z_{2^{d_1}}) \times \cdots \times \mathrm{Aut}(Z_{p_t^{d_t}}) = D_1 D_2 \cdots D_t$, *where* D_i *is the direct factor subgroup of* $\mathrm{Aut}(Z_n)$ *that fixes each component* *of the elements of Z_n except for the i-th component. So $D_i \cong \mathrm{Aut}(Z_{p_i^{d_i}})$ for each i. In fact D_i induces a faithful action on the subgroup $Z_{p_i^{d_i}}$. Note that the induced action D_1 on the subgroup $Z_{2^{d_1}}$ is permutation isomorphic to $\langle(-1)^*\rangle\times\langle 5^*\rangle (d_1 \geqslant 3)$, the multiplicative group of units of the ring $\mathbb{Z}_{2^{d_1}}$ acting on the additive group $\mathbb{Z}_{2^{d_1}}$, let $\phi : \langle(-1)^*\rangle \times \langle(5)^*\rangle \to D_1$ be the corresponding group isomorphism.*

Let $G_0 \leqslant \mathrm{Aut}(Z_n)$.

(i) *if $i \geqslant 2$, $d_i = 1$ and $p_i \geqslant 5$, then $D_i \nleqslant G_0$.*

(ii) *if $i \geqslant 2$ and $d_i \geqslant 2$, then $D_i \cap G_0 \leqslant Z_{p_i-1}$.*

(iii) *if $d_1 = 3$, then $D_1 \nleqslant G_0$.*

(iv) *if $d_1 \geqslant 4$, then either $|D_1 \cap G_0| \leqslant 2$ or $|D_1 \cap G_0| = 4$ and $D_1 \cap G_0 \nleqslant \langle\phi(5^*)\rangle$.*

The main result of this paper is the following theorem.

Theorem 1.2. *Suppose $G = Z_n \rtimes G_0$ acting on Z_n naturally where $G_0 \leqslant \mathrm{Aut}(Z_n)$. Then G is 2-closed if and only if G_0 satisfies Conditions 1.1. Moreover, if G is 2-closed then G can be represented by a circulant digraph.*

2 Preliminary results and notation

First we introduce some concepts and notation concerning Cayley digraphs. Given a finite group H, and a subset $S \subset H\backslash\{1\}$, the Cayley digraph $\Gamma = \mathrm{Cay}(H, S)$ with respect to S is defined as the directed graph with vertex set H and arc set $A\Gamma = \{(g, sg) \mid g \in H, s \in S\}$. Moreover, a Cayley digraph of a cyclic group is called a *circulant*. It is easy to check that the right regular representation $\hat{H}$ is contained in $\mathrm{Aut}(\Gamma)$. In fact, a digraph is a Cayley digraph if and only if its automorphism group contains a regular subgroup. Moreover let $\mathrm{Aut}(H, S) = \{\sigma \in \mathrm{Aut}(H) \mid S^\sigma = S\}$, then each element in $\mathrm{Aut}(H, S)$ induces an automorphism of the Cayley digraph $\Gamma = \mathrm{Cay}(H, S)$. It is proved in [10] that the normalizer of $\hat{H}$ in $\mathrm{Aut}(\Gamma)$ is $\hat{H} \rtimes \mathrm{Aut}(H, S)$. We say a Cayley digraph $\Gamma = \mathrm{Cay}(H, S)$ is *normal* if $\hat{H}$ is normal in $\mathrm{Aut}(\Gamma)$, that is, $\mathrm{Aut}(\Gamma) = \hat{H} \rtimes \mathrm{Aut}(H, S)$, see [10, 18]. So the automorphism group of a normal circulant must be a 2-closed group that contains a normal regular cyclic group. Conversely, we will show that each such 2-closed group is the automorphism group of some normal circulant.

Throughout the rest of this paper, let Z_n be an abstract cyclic group of order n and let $G \leqslant \mathrm{Sym}(Z_n)$ be a transitive permutation group which contains a normal regular cyclic group $\hat{Z}_n$ where

$$\hat{Z}_n = \{\hat{g} : x \to xg \ \forall x \in Z_n \mid g \in Z_n\}. \tag{1}$$

Therefore G is a semidirect product $\hat{Z}_n \rtimes G_0$ for some subgroup $G_0 \leqslant \mathrm{Aut}(Z_n)$ acting naturally on Z_n. Since $\hat{Z}_n \cong Z_n$, we may also write $G = Z_n \rtimes G_0$ directly. Our goal is to determine all such 2-closed groups.

The mail tool used in this paper is the Kovács-Li classification of arc-transitive circulants [11, 12]. Praeger and the author [14] refined the Kovács-Li classification and obtained the following theorem.

Theorem 2.1. [14, Theorem 1.1] *Let $G = Z_n \rtimes G_0 \leqslant Z_n \rtimes \mathrm{Aut}(Z_n)$ acting naturally on Z_n. Then, up to isomorphism, there is a unique connected Z_n-circulant Γ on which G acts arc-transitively. Moreover either $\mathrm{Aut}(\Gamma) = G$ or one of the following holds.*

(a) $n = p \geqslant 5$ is prime, $\Gamma = K_p$, and $G = \mathrm{AGL}(1, p)$;

(b) $n = bm > 4$, where $b \geqslant 2$, p divides m for each prime p dividing b, $\Gamma = \Sigma[\overline{K}_b]$;

(c) $n = pm$, where p is prime, $5 \leqslant p < n$, and $\gcd(m, p) = 1$, $\Gamma = \Sigma[\overline{K}_p] - p.\Sigma$, $G_0 = \mathrm{Aut}(Z_p) \times H \leqslant \mathrm{Aut}(Z_p) \times \mathrm{Aut}(Z_m)$, and Σ is a connected $(Z_m \rtimes H)$-arc-transitive Z_m-circulant.

We point out that up to isomorphism, in the above theorem Γ can be defined as $\mathrm{Cay}(Z_n, z^{G_0})$ where z is a generator of Z_n and z^{G_0} is the orbit of z under G_0. Moreover, if case (b) happens, then the group Z_n has a subgroup Y of order b, and $\Gamma = \mathrm{Cay}(Z_n, S)$ where S is a union of Y-cosets each consisting of generators for Z.

As a simple application of Theorem 2.1, we determine the 2-closed transitive permutation groups of degree p where p is a prime.

Corollary 2.2. *Let p be a prime. Let $G \leqslant \mathrm{Sym}(\Omega)$ be a 2-closed transitive permutation group of degree p. Then there exists a digraph representing G. Moreover, G is one of the following.*

1. *The symmetric group S_p $(p \geqslant 2)$ which is 2-transitive on Ω.*
2. *An affine subgroup $Z_p \rtimes Z_k$ where $p \geqslant 3$, $1 \leqslant k < (p-1)$ and $k|(p-1)$.*

Conversely, each group of the above two types is 2-closed.

Proof. Suppose G is a 2-closed transitive permutation group of degree p. By a classical result of Burnside, G is either 2-transitive or is affine. If G is 2-transitive, then $G = G^{(2)} = S_p$ and $p \geqslant 2$. If G is not 2-transitive, then $G = Z_p \rtimes Z_k$ where $p \geqslant 3$, $1 \leqslant k < (p-1)$ and $k|(p-1)$.

For the converse, note that S_p is the full automorphism group of the complete graph K_p and so S_p is indeed 2-closed. Next, let $G = Z_p \rtimes Z_k$ where $p \geqslant 3$, $1 \leqslant k < (p-1)$ and $k|(p-1)$. By Theorem 2.1, there is a connected arc-transitive circulant Γ of order p such that $\mathrm{Aut}(\Gamma) = G$, and so G is 2-closed. □

Remark: If $p = 2, 3$ then $S_p = Z_p \rtimes \mathrm{Aut}(Z_p)$ is 2-closed; and if $p \geqslant 5$ then $Z_p \rtimes \mathrm{Aut}(Z_p)$ is not 2-closed.

We also need the following theorem.

Theorem 2.3. [5, Theorem 5.1] *Let $G_1 \leqslant \mathrm{Sym}(\Omega_1)$ and $G_2 \leqslant \mathrm{Sym}(\Omega_2)$ be transitive permutation groups. Consider the natural product action of $G_1 \times G_2$ on $\Omega_1 \times \Omega_2$. Then $(G_1 \times G_2)^{(2)} = G_1^{(2)} \times G_2^{(2)}$.*

Finally, we fix the following notation. Let $A \leqslant \mathrm{Sym}(\Omega)$. Suppose that A_B is the setwise stabilizer of $B \subseteq \Omega$ and $g \in A_B$, we denote A_B^B to be the induced permutation group on B by A_B and denote g^B to be the induced permutation on B by g.

3 2-closed groups containing a normal regular cyclic group

In this section we classify 2-closed groups G that contain a normal regular cyclic group Z_n. With notation in Section 2, we may suppose that $G = Z_n \rtimes G_0 \leqslant Z_n \rtimes \mathrm{Aut}(Z_n)$ acting naturally on Z_n. We first handle the special case that n is a prime power in Subsection 3.1 and Subsection 3.2. The notation needed for the statement of Theorem 1.2 is given in Subsection 3.3.1 and the proof is given in Subsection 3.3.2.

3.1 The case $n = p^d$ with p an odd prime

Let $n = p^d$ where p is an odd prime and $d \geqslant 2$ is an integer. Then $\mathrm{Aut}(Z_n) = Z_{(p-1)} \times Z_{p^{d-1}}$ is a cyclic group. We take $\alpha \in \mathrm{Aut}(Z_n)$ such that $o(\alpha) = p$, then there exists $\gamma \in \mathrm{Aut}(Z_n)$ with order p^{d-1} such that $\alpha = \gamma^{p^{d-2}}$. We first look at the action of α on Z_n.

Let $H = Z_{p^{d-1}}$ be the unique subgroup of Z_n of order p^{d-1}. Let $N = Z_n \rtimes \mathrm{Aut}(Z_n)$. Then the cosets of H form a block system $\mathcal{B}$ of N on Z_n. Denote $\mathcal{B} = \{B_1 = H, B_2, \ldots, B_p\}$. Since the elements in $B_2, \ldots, B_p$ are of order p^d, γ fixes each block setwise and γ^{B_i} is a $p^{d-1}-$cycle for each $i \geqslant 2$. However, γ fixes the point $1 \in H = B_1$, so the order of γ^{B_1} is strictly less than p^{d-1}. It then follows that α fixes B_1 pointwise and is fixed point free on each B_i for $i \geqslant 2$.

On the other hand, let $N_{B_i}^{B_i}$ be the induced permutation group of the setwise stabilizer N_{B_i} on B_i. Then $N_{B_i}^{B_i} = \hat{Z}_{p^{d-1}} \rtimes K_i$ and $K_i \cong \mathrm{Aut}(Z_{p^{d-1}})$, ($\hat{Z}_{p^{d-1}}$ is defined in equation (1)). For each $i \geqslant 2$, since γ^{B_i} is fixed point free, we have that $\gamma^{B_i} = \hat{y}_i^{B_i}\tau$ where $1 \neq y_i \in H \leqslant Z_n$ and $\tau \in K_i$. Since τ normalizes $\hat{Z}_{p^{d-1}}$, $(\gamma^{B_i})^2 = \hat{y}_i^{B_i}(\tau\hat{y}_i^{B_i}\tau^{-1})\tau\tau = a_{i2}\tau^2$ where a_{i2} is some element in $\hat{Z}_{p^{d-1}}$. By induction, we have that for each $k \geqslant 1$, $(\gamma^{B_i})^k = a_{ik}\tau^k$ where a_{ik} is some element in $\hat{Z}_{p^{d-1}}$. Since γ^{B_i} is of order p^{d-1} and $\hat{Z}_{p^{d-1}} \cap K_i = \{1\}$, we have that $\tau^{p^{d-1}} = 1$. Since $\tau \in \mathrm{Aut}(Z_{p^{d-1}}) = Z_{p-1} \times Z_{p^{d-2}}$, $\tau^{p^{d-2}} = 1$. Recall that $\alpha = \gamma^{p^{d-2}}$, it then follows that α^{B_i} is $\hat{x}_i^{B_i}$ for some $x_i \in Z_n$ with order p. Note that x_i may not equal x_j for $2 \leqslant i < j \leqslant p$, but they are all of order p. We have proved the following lemma.

Lemma 3.1. *Let $\alpha \in \mathrm{Aut}(Z_{p^d})$ with order p. Let $\mathcal{B} = \{B_1 = H, B_2, \ldots, B_p\}$ be the cosets of the subgroup H where $H < Z_{p^d}$ is of order p^{d-1}. Then α fixes $B_1 = H$ pointwise and for each $i \geqslant 2$, α^{B_i} is $\hat{x}_i^{B_i}$ for some $x_i \in Z_n$ with order p.*

Corollary 3.2. *Let $n = p^d$ and $Z_n = \langle z \rangle$. Let $Z_p \leqslant Z_n$ be the subgroup of order p. Suppose that $G = Z_n \rtimes G_0$ where $G_0 \leqslant \mathrm{Aut}(Z_n)$. Then the coset $zZ_p \subseteq z^{G_0}$ if and only if $p||G_0|$.*

Remark: Let $S = z^{G_0}$ and $\Gamma = \mathrm{Cay}(Z_n, S)$. If case (b) of Theorem 2.1 occurs for Γ, then $zZ_p \subseteq z^{G_0}$. That is why we consider this corollary.

Proof. Let $\mathrm{Aut}(Z_{p^d}) = \langle\mu\rangle \times \langle\gamma\rangle = Z_{p-1} \times Z_{p^{d-1}}$ and $\alpha = \gamma^{p^{d-2}}$. Then $p||G_0|$ if and only if $\alpha \in G_0$.

Let $\mathcal{B} = \{B_1 = H, B_2, \ldots, B_p\}$ be the cosets of the subgroup H where $H < Z_{p^d}$ is of order p^{d-1}. Then it is easy to show that μ fixes B_1 setwise, and permutes $B_2, \ldots, B_p$ as a $(p-1)$-cycle.

By Lemma 3.1, if $\alpha \in G_0$ then $zZ_p \subseteq z^{G_0}$. Conversely, suppose that $zZ_p \subseteq z^{G_0}$. Note that the generator $z \in B_k$ for some $k \geqslant 2$ and $zZ_p \subseteq B_k$. By the action of μ and γ, we conclude that $\alpha \in G_0$. $\square$

Proposition 3.3. *Let $n = p^d$ where p is an odd prime and $d \geqslant 2$. Let $G = Z_n \rtimes G_0 \leqslant Z_n \rtimes \mathrm{Aut}(Z_n)$ acting naturally on Z_n. Then G is 2-closed if and only if $G_0 \leqslant Z_{p-1}$. Moreover, if G is 2-closed then G can be represented by an arc-transitive circulant.*

Proof. As defined at the beginning of Subsection 3.1, let $\alpha \in \mathrm{Aut}(Z_{p^d})$ be an element of order p. Let $\mathcal{B} = \{B_1 = H, B_2, \ldots, B_p\}$ be the cosets of the subgroup H where $H < Z_{p^d}$ is of order p^{d-1}.

Suppose first that $G_0 \nleqslant Z_{p-1}$, that is $p||G_0|$, then $\alpha \in G_0$. By Lemma 3.1, α fixes $B_1 = H$ pointwise and for each $i \geqslant 2$, α^{B_i} is $\hat{x}_i^{B_i}$ for some $x_i \in Z_n$ with order p.

Let $1 \neq \beta \in \mathrm{Sym}(Z_n)$ such that β fixes every element of $B_1, \ldots, B_{p-1}$ and $\beta^{B_p} = \alpha^{B_p}$. That means $\beta^{B_p} = \hat{x}_p^{B_p}$, (recall that $\hat{x} : z \mapsto zx$ for any $z \in Z_n$). We claim that $\beta \in (Z_{p^d} \rtimes \langle\alpha\rangle)^{(2)}$ and so $\beta \in G^{(2)}$. Take any pair $(y_1, y_2) \in Z_n \times Z_n$. If both y_1 and y_2 belong to B_p, then $(y_1, y_2)^\beta = (y_1x_p, y_2x_p)$ is in the orbital $(y_1, y_2)^G$. Suppose next that exactly one of $\{y_1, y_2\}$ lies in B_p, say $y_2 \in B_p$. Since the stabilizer G_{y_1} is the conjugate of G_0 in G by an element in $\hat{Z}_n$, a conjugate of α, say ρ, is in G_{y_1}. Therefore β^{B_p} equals $(\rho^j)^{B_p}$ for some $j \in \{1, \ldots, p-1\}$, and so $(y_1, y_2)^\beta \in (y_1, y_2)^G$. It then follows that $\beta \in (Z_{p^d} \rtimes \langle\alpha\rangle)^{(2)} \leqslant G^{(2)}$. However, since β fixes B_1 and B_2 pointwise, $\beta \notin Z_{p^d} \rtimes \mathrm{Aut}(Z_{p^d})$, and so $\beta \notin G$ and G is not 2-closed.

Suppose next that $G_0 \leqslant Z_{p-1}$. Let $S = z^{G_0}$ where $z \in Z_{p^d}$ is an element of order p^d and let $\Gamma = \mathrm{Cay}(Z_n, S)$. Since $(p, |G_0|) = 1$, $p \nmid |S|$ and so S is not a union of cosets of any subgroup of Z_n. By Theorem 2.1, $\mathrm{Aut}(\Gamma) = G$ and so G is 2-closed. This completes the proof. $\square$

Remark: In above proof, note that β is in $(Z_{p^d} \rtimes \langle\alpha\rangle)^{(2)}$. Hence we actually proved that $(Z_{p^d} \rtimes \langle\alpha\rangle)^{(2)} \nleqslant Z_{p^d} \rtimes \mathrm{Aut}(Z_{p^d})$ where $\alpha \in \mathrm{Aut}(Z_{p^d})$ is of order p.

3.2 The case $n = 2^d$ for $d \geqslant 2$

Notation: For convenience, in this subsection we write Z_n additively as the group $\mathbb{Z}_n$ of integers modulo n, so in this case

$$\hat{Z}_n = \hat{\mathbb{Z}}_n = \{\hat{x} : g \to g + x \mid x \in Z_n\}.$$

Moreover $\mathrm{Aut}(Z_n)$ is the multiplicative group $\mathbb{Z}_n^*$ so that $i^* \in \mathrm{Aut}(Z_n)$ denotes the map $j \mapsto ij$.

3.2.1 $d = 2$:

In this case, $\mathrm{Aut}(Z_4) = \langle(-1)^*\rangle \cong Z_2$. We have the following result.

Lemma 3.4. *Suppose that $\hat{Z}_4 \leqslant G \leqslant \hat{Z}_4 \rtimes \langle(-1)^*\rangle \cong D_8$. Then G is 2-closed and is the full automorphism group of an arc-transitive circulant.*

Proof. Either $G \cong Z_4$ is regular or $G \cong D_8$. Note that $\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_4, \{1\})) = Z_4$ and $\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_4, \{1, -1\})) = D_8 = Z_4 \rtimes Z_2$, this proves the lemma. □

Remark: By [14, Lemma 2.3], a connected arc-transitive circulant Γ is both normal and of lexicographic product form if and only if $\Gamma = \mathrm{Cay}(\mathbb{Z}_4, \{1, -1\})$ and $\mathrm{Aut}(\Gamma) = Z_4 \rtimes \mathrm{Aut}(Z_4)$. In this case the orbit $1^{\mathrm{Aut}(\mathbb{Z}_4)} = \{1, 3\} = 1 + Z_2$ is a coset of Z_2.

3.2.2 $d \geqslant 3$:

In this case, $\mathrm{Aut}(Z_n) = \langle(-1)^*\rangle \times \langle 5^*\rangle \cong Z_2 \times Z_{2^{d-2}}$. Denote $N = \hat{\mathbb{Z}}_n \rtimes \mathbb{Z}_n^*$. Let H be the unique subgroup of $\mathbb{Z}_n$ with order 2^{d-2}. Let $B_0 = H, B_1 = 1+H, B_2 = 2+H, B_3 = 3+H$ be the cosets of H, then $\mathcal{B} = \{B_0, B_1, B_2, B_3\}$ forms a complete block system of N on $\mathbb{Z}_n$.

We first study the action of 5^*. By computation 5^* preserves each block B_i, we determine the induced permutation $(5^*)^{B_i}$ next. Since $B_1 \cup B_3$ consists of all elements of order 2^d, $(5^*)^{B_1}$ and $(5^*)^{B_3}$ are 2^{d-2}-cycles. As $B_0 = \langle 4\rangle = Z_{2^{d-2}}$ and $B_0 \cup B_2 = \langle 2\rangle = Z_{2^{d-1}}$, it is easy to deduce that $(5^*)^{B_2}$ is a product of two 2^{d-3}-cycles (if $d = 3$, then $(5^*)^{B_2}$ is trivial). Therefore the orders of $(5^*)^{B_1}$ and $(5^*)^{B_3}$ are 2^{d-2}, the order of $(5^*)^{B_2}$ is 2^{d-3}, and the order of $(5^*)^{B_0}$ is 2^{d-4} (if $d = 3$, then the order is 1).

Case 1: $d = 3$

In this case, $n = 8$ and $\mathrm{Aut}(Z_8) = \langle(-1)^*\rangle \times \langle 5^*\rangle \cong Z_2 \times Z_2$. By computation, 5^* fixes B_0 and B_2 pointwise, and the induced action $(5^*)^{B_1} = \hat{4}^{B_1}$ and $(5^*)^{B_3} = \hat{4}^{B_3}$. The element $(-1)^*$ fixes B_0 pointwise and $((-1)^*)^{B_2} = \hat{4}^{B_2}$.

Lemma 3.5. *Let $\mathbb{Z}_8 = \langle z\rangle$. Suppose that $G = \mathbb{Z}_8 \rtimes G_0$ where $G_0 \leqslant \mathrm{Aut}(\mathbb{Z}_8) = \langle(-1)^*\rangle \times \langle 5^*\rangle$. Then the coset $z + Z_2 \subseteq z^{G_0}$ if and only if $5^* \in G_0$ where $Z_2 = \langle 4\rangle$ is the subgroup of order 2.*

Proof. Note that both z and $z + Z_2$ are contained in B_1 or B_3 and $(-1)^*$ interchanges two blocks B_1 and B_3. The result follows from the analysis of the actions of $(-1)^*$ and 5^* easily. □

Proposition 3.6. *With above notation, let $G = Z_8 \rtimes G_0$ where $G_0 \leqslant \mathrm{Aut}(Z_8) = \langle(-1)^*\rangle \times \langle 5^*\rangle$. Then*

1. *if $G_0 = \mathrm{Aut}(Z_8)$ then G is not 2-closed.*
2. *if $G_0 \lneqq \mathrm{Aut}(Z_8)$ and $G_0 \neq \langle 5^*\rangle$, then G is 2-closed and can be represented by an arc-transitive circulant.*
3. *if $G_0 = \langle 5^*\rangle$, then G is 2-closed and can be represented by a circulant.*

Proof. (1) Suppose first that $G_0 = \mathrm{Aut}(Z_8)$. Let $\beta \in S_8$ such that β fixes B_0, B_1 and B_3 pointwise and $\beta^{B_2} = \hat{4}^{B_2}$. Take any pair $(y_1, y_2) \in Z_8 \times Z_8$. If both y_1 and y_2 belong to B_2, then $(y_1, y_2)^\beta = (y_1, y_2)^{\hat{4}}$ is in the orbital $(y_1, y_2)^G$. Suppose next that exactly one of $\{y_1, y_2\}$ belongs to B_2, say $y_2 \in B_2$. It is straightforward to check that $(y_1, y_2)^\beta = (y_1, y_2)^{(-1)^*}$ if $y_1 \in B_0$. Let G_1 be the point stabilizer of point 1, then G_1 is the conjugate of G_0 by $\hat{1} \in \hat{\mathbb{Z}}_n$. Let α_1 be the corresponding conjugate of 5^* in G_1. It follows that $(y_1, y_2)^\beta = (y_1, y_2)^{\alpha_1}$ if $y_1 \in B_1 \cup B_3$. Hence $\beta \in G^{(2)}$. However since β fixes 0 and 1, $\beta \notin G$ and so G is not 2-closed.

(2) In this case, $5^* \notin G_0$. Let $S = 1^{G_0}$ and let $\Gamma = \mathrm{Cay}(\mathbb{Z}_8, S)$. It follows from Lemma 3.5 and Theorem 2.1 that $G = \mathrm{Aut}(\Gamma)$ and is 2-closed.

(3) Finally we show that $\mathbb{Z}_8 \rtimes \langle 5^* \rangle$ is 2-closed. Let $S_1 = 1^{\langle 5^* \rangle} = \{1, 5\}$ and $S_2 = 2^{\langle 5^* \rangle} = \{2\}$. Let $\Gamma = \mathrm{Cay}(\mathbb{Z}_8, S_1 \cup S_2)$. By [12, Theorem 1.3], it is easy to deduce that Γ is not arc-transitive. Suppose $g \in \mathrm{Aut}(\Gamma)$ such that g fixes 0 and 1, it is straightforward to check that $g = 1$. We conclude that $\mathrm{Aut}(\Gamma) = \mathbb{Z}_8 \rtimes \langle 5^* \rangle$ as required. ☐

Case 2: $d \geqslant 4$

Let $\alpha = (5^*)^{2^{d-4}}$ be an element of order 4 in $\langle 5^* \rangle$. By the analysis of action of 5^*, we deduce that α fixes B_0 pointwise and $o(\alpha^{B_2}) = 2$, $o(\alpha^{B_1}) = o(\alpha^{B_3}) = 4$.

Suppose first that $d = 4$, then $\alpha = 5^*$. By direct computation, $\alpha^{B_2} = \hat{8}^{B_2}$, $\alpha^{B_1} = \hat{4}^{B_1}$ and $\alpha^{B_3} = \widehat{-4}^{B_3}$.

Next suppose $d \geqslant 5$. Denote $N = \hat{\mathbb{Z}}_n \rtimes \mathbb{Z}_n^*$. Note that $N_{B_i}^{B_i} \cong \hat{Z}_{2^{d-2}} \rtimes K_i$ where $K_i \cong \mathrm{Aut}(Z_{2^{d-2}})$ for each $i \in \{1, 2, 3\}$. Since $(5^*)^{B_i}$ is fixed point free on B_i for $i = 1, 2, 3$, $(5^*)^{B_i} = \hat{y}_i^{B_i} \tau_i$ where $0 \neq y_i \in \mathbb{Z}_n$ and $\tau_i \in K_i$. Since τ_i normalizes $\hat{Z}_{2^{d-2}}$, $((5^*)^{B_i})^2 = \hat{y}_i^{B_i} (\tau_i \hat{y}_i^{B_i} \tau_i^{-1}) \tau_i \tau_i = a_{i2} \tau_i^2$ where a_{i2} is some element in $\hat{Z}_{2^{d-2}}$. By induction, we have that for each $k \geqslant 1$, $((5^*)^{B_i})^k = a_{ik} \tau_i^k$ where a_{ik} is some element in $\hat{Z}_{2^{d-2}}$. Since $\tau_i \in \mathrm{Aut}(Z_{2^{d-2}})$ and $d \geqslant 5$, $\tau_i^{2^{d-4}} = 1$. By the order of α^{B_i}, we have that $\alpha^{B_i} = \hat{x}_i^{B_i}$, where $x_1, x_3 \in Z_n$ are of order 4 and $x_2 = 2^{d-1}$ is the unique involution in Z_n. In addition, $2x_1 = 2x_3 = 2^{d-1}$. Therefore we have proved the following lemma.

Lemma 3.7. *Suppose $d \geqslant 4$. With above notation, let $\alpha = (5^*)^{2^{d-4}}$ be an element of order 4 in $\langle 5^* \rangle$. Then α fixes B_0 pointwise, $\alpha^{B_2} = (\widehat{2^{d-1}})^{B_2}$, $\alpha^{B_1} = \hat{x}_1^{B_1}$ for some $x_1 \in Z_n$ with order 4 and $\alpha^{B_3} = \hat{x}_3^{B_3}$ for some $x_3 \in Z_n$ with order 4.*

Corollary 3.8. *Let $n = 2^d$ for $d \geqslant 4$ and let $Z_n = \langle z \rangle$. Suppose that $G = Z_n \rtimes G_0$ where $G_0 \leqslant \mathrm{Aut}(Z_n) = \langle (-1)^* \rangle \times \langle 5^* \rangle$. Let $\alpha \in \langle 5^* \rangle$ be of order 4. Then*

1. *the coset $z + Z_4 \subseteq z^{G_0}$ if and only if $\alpha \in G_0$ where $Z_4 \leqslant Z_n$ is the subgroup of order 4.*
2. *the coset $z + Z_2 \subseteq z^{G_0}$ if and only if $\alpha^2 \in G_0$ where $Z_2 \leqslant Z_n$ is the subgroup of order 2.*

Proof. By Lemma 3.7, we have that $z + Z_4 \subseteq z^{G_0}$ if $\alpha \in G_0$ and $z + Z_2 \subseteq z^{G_0}$ if $\alpha^2 \in G_0$.

With the notation in Lemma 3.7, suppose that $z + Z_4 \subseteq z^{G_0}$. Note that $z \in B_1$ or B_3 and $z + Z_4 \subseteq B_1$ or B_3 respectively. Since $(-1)^*$ interchanges B_1 and B_3, it is easy to deduce that $\alpha \in G_0$. Similarly, if $z + Z_2 \subseteq z^{G_0}$ then $\alpha^2 \in G_0$. ☐

Proposition 3.9. *With above notation, let* $G = Z_n \rtimes G_0 \leqslant Z_n \rtimes \mathrm{Aut}(Z_n)$ *where* $n = 2^d$ *for* $d \geqslant 4$. *If* $\alpha = (5^*)^{2^{d-4}} \in G_0$, *then* $(Z_n \rtimes \langle\alpha\rangle)^{(2)} \nleqslant Z_n \rtimes \mathrm{Aut}(Z_n)$. *In particular, G is not 2-closed on* Z_n.

Proof. Let $1 \neq \beta \in Sym(Z_{2^d})$ such that β fixes B_0, B_2, B_3 pointwise and $\beta^{B_1} = \widehat{(2^{d-1})}^{B_1}$ is of order 2. Therefore $\beta^{B_1} = (\alpha^2)^{B_1}$. We will show next that $\beta \in (Z_{2^d} \rtimes \langle\alpha\rangle)^{(2)} \leqslant G^{(2)}$.

Take any pair $(y_1, y_2) \in Z_n \times Z_n$. If both y_1 and y_2 belong to B_1, then $(y_1, y_2)^\beta = (y_1, y_2)^{\widehat{2^{d-1}}}$ is in the orbital $(y_1, y_2)^G$. Suppose next that exactly one of $\{y_1, y_2\}$ belongs to B_1, say $y_2 \in B_1$. By Lemma 3.7, $(y_1, y_2)^\beta = (y_1, y_2)^{\alpha^2}$ if $y_1 \in B_0$ or B_2. Let G_3 be the point stabilizer of point 3, then G_3 is the conjugate of G_0 by $\hat{3} \in \hat{Z}_n$. Let α_3 be the corresponding conjugate of α in G_3, it follows from Lemma 3.7 that $(y_1, y_2)^\beta = (y_1, y_2)^{\alpha_3}$ if $y_1 \in B_3$. Thus $\beta \in (Z_{2^d} \rtimes \langle\alpha\rangle)^{(2)} \leqslant G^{(2)}$. However since β fixes B_0 and B_3 pointwise, $\beta \notin Z_{2^d} \rtimes \mathrm{Aut}(Z_{2^d})$ and so $(Z_{2^d} \rtimes \langle\alpha\rangle)^{(2)} \nleqslant Z_{2^d} \rtimes \mathrm{Aut}(Z_{2^d})$. In particular G is not 2-closed. □

Next we will show that if $\alpha \notin G_0$ then G is 2-closed. Note that $\alpha \notin G_0$ is equivalent to the condition that either $|G_0| \leqslant 2$ or $|G_0| = 4$ and $G_0 \nleqslant \langle 5^* \rangle$.

We first discuss the case that $\alpha^2 \notin G_0$.

Lemma 3.10. *With above notation, let* $n = 2^d$ *for* $d \geqslant 4$. *Let* $G = Z_n \rtimes G_0$. *Suppose* $\alpha^2 \notin G_0$. *Then G is the full automorphism group of an arc-transitive circulant and so G is 2-closed.*

Proof. Let $S = 1^{G_0}$ be the orbit of 1 under G_0, and let $\Gamma = \mathrm{Cay}(Z_n, S)$. Since $\alpha^2 \notin G_0$, it follows from corollary 3.8 that S is not a union of cosets of any subgroup of Z_n. By Theorem 2.1, $\mathrm{Aut}(\Gamma) = G$ as required. □

It remains to show that if $G = Z_n \rtimes G_0$ where $\alpha^2 \in G_0$ but $\alpha \notin G_0$ then G is the full automorphism group of some circulant. We will prove this in Proposition 3.15 when we handle the more general case.

3.3 The general case.

3.3.1 The notation for the main theorem.

We explain Conditions 1.1 in more detail first.

Let

$$n = 2^{d_1} p_2^{d_2} \cdots p_t^{d_t}, \ \ d_1 \geqslant 0, \ d_2, \ldots, d_t \geqslant 1, \ t \geqslant 1$$

where $p_2, \ldots, p_t$ are distinct odd primes. For convenience, we also write $p_1 = 2$. In addition, the notion $p_i^{d_i} || n$ means $p_i^{d_i} | n$ but $p_i^{d_i+1} \nmid n$.

Let $G = \hat{Z}_n \rtimes G_0$ acting on Z_n naturally where $G_0 \leqslant \mathrm{Aut}(Z_n)$. In order to reduce the proof in the general case to the prime power case, we choose the product action form to describe G. Let Z_m be the unique subgroup of Z_n of order m for $m|n$. Then we may write

$$Z_n = Z_{2^{d_1}} \times Z_{p_2^{d_2}} \times \cdots \times Z_{p_t^{d_t}} = \{(z_1, \ldots, z_t) = z_1 z_2 \cdots z_t | z_i \in Z_{p_i^{d_i}}, \text{ where } p_1 = 2\}.$$

For any $g = (g_1, \ldots, g_t) \in Z_n$, we have $\hat{g} : (z_1, \ldots, z_t) \mapsto (z_1 g_1, \ldots, z_t g_t)$. Moreover,

$$\mathrm{Aut}(Z_n) = \mathrm{Aut}(Z_{2^{d_1}}) \times \cdots \times \mathrm{Aut}(Z_{p_t^{d_t}}) = D_1 D_2 \cdots D_t,$$

where D_i is the direct factor subgroup of $\mathrm{Aut}(Z_n)$ that fixes each component of the elements of Z_n except for the i-th component. So $D_i \cong \mathrm{Aut}(Z_{p_i^{d_i}})$.

In fact D_i induces a faithful action on the subgroup $Z_{p_i^{d_i}}$. With notation in §3.2, if $d_1 \geqslant 3$ then the induced action D_1 on the subgroup $Z_{2^{d_1}}$ is permutation isomorphic to $\langle (-1)^* \rangle \times \langle 5^* \rangle (d_1 \geqslant 3)$, the multiplicative group of units of the ring $\mathbb{Z}_{2^{d_1}}$ acting on the additive group $\mathbb{Z}_{2^{d_1}}$. Let $\phi : \langle (-1)^* \rangle \times \langle (5)^* \rangle \to D_1$ be the corresponding group isomorphism.

The normalizer of $\hat{Z}_n$ in $\mathrm{Sym}(Z_n)$ is

$$N = \hat{Z}_n \rtimes \mathrm{Aut}(Z_n) = (\hat{Z}_{2^{d_1}} \rtimes \mathrm{Aut}(Z_{2^{d_1}})) \times \cdots \times (\hat{Z}_{p_t^{d_t}} \rtimes \mathrm{Aut}(Z_{p_t^{d_t}}))$$

acting on Z_n by the natural product action. Therefore $G = \hat{Z}_n \rtimes G_0 \leqslant N$ has the natural product action.

We need the following two easy observations in the proof below.

(1) Note that when $i \geqslant 2$, $\mathrm{Aut}(Z_{p_i^{d_i}}) = Z_{p-1} \times Z_{p_i^{d_i-1}}$. Conditions 1.1 [ii] is equivalent to $\alpha_i \notin G_0$ where $\alpha_i \in D_i \cong Z_{p_i-1} \times Z_{p_i^{d_i-1}}$ is of order p_i.

(2) When $i = 1$ and $d_1 \geqslant 4$, denote $\alpha_1 = \phi((5^*)^{2^{d_1-4}}) \in D_1$, then the order of α_1 is 4. Conditions 1.1 [iv] is equivalent to $\alpha_1 \notin G_0$.

3.3.2 The proof of Theorem 1.2.

Lemma 3.11. *With notation in Subsection 3.3.1, suppose $G = \hat{Z}_n \rtimes G_0$ where $G_0 \leqslant \mathrm{Aut}(Z_n)$. If G_0 fails to satisfy one of conditions 1.1, then G is not 2-closed.*

Proof. If condition (i) does not hold, then there exists an odd prime $p_i \geqslant 5$ where $i \geqslant 2$ such that $p_i || n$ and $D_i \leqslant G_0$. In this case we take $K = \hat{Z}_{p_i} \rtimes D_i$. By hypothesis, K is the subgroup of G which fixes each component of elements of Z_n except for the i-th component. Hence the action of K on Z_n is the product action of $\bar{K} \times \{1\}$ on $Z_n = Z_{p_i} \times Z_{\frac{n}{p_i}}$ where $\bar{K} \cong K$ acts on Z_{p_i} naturally. It follows from Theorem 2.3 that $K^{(2)} = (\overline{K})^{(2)} \times \{1\}$. By the remark after Corollary 2.2, $(\overline{K})^{(2)} \nsubseteq Z_{p_i} \rtimes \mathrm{Aut}(Z_{p_i})$. Since $G^{(2)} \geqslant K^{(2)}$, we have that G is not 2-closed in this case.

If condition (ii) does not hold, then there exists an odd prime p_i where $i \geqslant 2$ such that $p_i^{d_i} || n$ and $d_i \geqslant 2$. Since $\alpha_i \in G_0$ in this case, we take $K = \hat{Z}_{p_i^{d_i}} \rtimes \langle \alpha_i \rangle \leqslant G$. Hence the action of K on Z_n is the product action of $\bar{K} \times \{1\}$ on $Z_n = Z_{p_i^{d_i}} \times Z_{\frac{n}{p_i^{d_i}}}$ where $\bar{K} \cong K$ acts on $Z_{p_i^{d_i}}$ naturally. By the remark after Proposition 3.3, $(\overline{K})^{(2)} \nsubseteq Z_{p_i^{d_i}} \rtimes \mathrm{Aut}(Z_{p_i^{d_i}})$. The same argument as above proves that G is not 2-closed in this case either.

Suppose $2^{d_1} || n$ and $d_1 \geqslant 3$, suppose also that either condition (iii) or (iv) fails. Take $K = \widehat{Z_8} \rtimes D_1$ if $d_1 = 3$ and take $K = \widehat{Z_{2^{d_1}}} \rtimes \langle \alpha_1 \rangle$ if $d_1 \geqslant 4$. By the same argument as above, it follows from Proposition 3.6(1) and Proposition 3.9 that G is not 2-closed. □

Lemma 3.12. *With notation in Subsection 3.3.1, suppose $G = \hat{Z}_n \rtimes G_0$ where $G_0 \leqslant \mathrm{Aut}(Z_n)$ and G_0 satisfies Conditions 1.1. Let $S = z^{G_0}$ where $Z_n = \langle z \rangle$, and let $\Gamma = \mathrm{Cay}(Z_n, S)$. Then exactly one of the following holds.*

1. *G is the full automorphism group of Γ and so G is 2-closed and can be represented by an arc-transitive circulant.*
2. *$2^{d_1}||n$, $d_1 \geqslant 4$, and $\alpha_1^2 \in G_0 \cap D_1$.*
3. *$2^3||n$, and $D_1 \cap G_0 = \langle \phi(5^*) \rangle \cong Z_2$.*
4. *$n = 4m$ where $m > 1$ is odd. $D_1 \cap G_0 = D_1 \cong Z_2$, that is $G_0 = \mathrm{Aut}(Z_4) \times K$ where $K \leqslant \mathrm{Aut}(Z_m)$.*

Moreover, in the latter three cases, $\Gamma = \Sigma[\overline{K}_2]$ is a lexicographic product and the pointwise stabilizer of $\{1, z\}$ in $\mathrm{Aut}(\Gamma)$ preserves each coset of Z_2.

Proof. Suppose that G is not the full automorphism group of Γ. By the condition (i), for any odd prime $p_i \geqslant 5$ such that $p_i||n$, we have $G_0 \neq \mathrm{Aut}(Z_{p_i}) \times H$ for some $H \leqslant \mathrm{Aut}(Z_{n/p_i})$. It then follows from Theorem 2.1 that case (b) of Theorem 2.1 occurs for Γ. That is $n = bk > 4$ where $b \geqslant 2$ and $\Gamma = \Sigma[\overline{K}_b]$. Moreover, the group Z_n has a subgroup Y of order b and S is a union of Y-cosets each consisting of generators for Z_n.

Recall that $n = 2^{d_1} p_2^{d_2} \cdots p_t^{d_t}$. Suppose that $p_j | b$ for some $j \in \{1, \ldots, t\}$. Then $zZ_{p_j} \subseteq S$ where Z_{p_j} is the subgroup of order p_j and $d_j \geqslant 2$ by Theorem 2.1 (b). Let $z = (z_1, \ldots, z_t)$ where z_i is a generator of $Z_{p_i^{d_i}}$ for each i. Thus $zZ_{p_j} \subseteq S = z^{G_0}$ implies that $z_j Z_{p_j} \subseteq z_j^{D_j \cap G_0}$ in the j-th component. By Corollary 3.2, the condition (ii) implies that $b = 2^l$ is a power of 2. Similarly, by Corollary 3.8, Lemma 3.5 and the action of $\mathrm{Aut}(Z_4)$, the condition (iii) and (iv) imply that b must be 2 and one of cases 2-4 happens.

Suppose next that one of cases 2-4 occurs. Thus $\Gamma = \Sigma[\overline{K}_2]$ where $\Sigma = \mathrm{Cay}(Z_n/Z_2, \overline{S})$ and $\overline{S} = \{sZ_2 | s \in S\}$. Moreover, by [14, Lemma 2.3], the set $\{xZ_2 | x \in Z_n\}$ forms a block system of $\mathrm{Aut}(\Gamma)$, and so $\mathrm{Aut}(\Gamma) = Z_2 \wr \mathrm{Aut}(\Sigma)$.

Let $\overline{G}_0 = G_0/\langle \alpha_1^2 \rangle$ in case 2, and let $\overline{G}_0 = G_0/(D_1 \cap G_0)$ in case 3 or 4. Then $\overline{G}_0 \leqslant \mathrm{Aut}(Z_n/Z_2)$ and $\overline{S} = (zZ_2)^{\overline{G}_0}$. Note that G_0 satisfies Conditions 1.1, it follows that $\overline{S}$ is not the union of cosets of any subgroup of Z_n/Z_2. By Theorem 2.1, Σ is normal and $\mathrm{Aut}(\Sigma) = (Z_n/Z_2) \rtimes \overline{G}_0$. Therefore the pointwise stabilizer of $\{1, z\}$ in $\mathrm{Aut}(\Gamma)$ preserves each coset of Z_2. □

Remark: Suppose G satisfies Conditions 1.1. By the above lemma, G can be represented by an arc-transitive circulant if and only if G does not arise in any of the cases 2-4 of Lemma 3.12.

Next we will show that if one of cases 2-4 occurs then there exists a circulant Γ which is not arc-transitive such that $\mathrm{Aut}(\Gamma) = G$. We discuss case 4 first.

Lemma 3.13. *Suppose $n = 4m$ where $m > 1$ is odd and $G = Z_{4m} \rtimes G_0$ where $G_0 = \mathrm{Aut}(Z_4) \times K$ and $K \leqslant \mathrm{Aut}(Z_m)$. Suppose further that G_0 satisfies Conditions 1.1. Then G is 2-closed and can be represented by a circulant.*

Proof. Let $z = z_1 z_2 \in Z_{4m}$ where z_1 is a generator of Z_4 and z_2 is a generator of Z_m. Let $S_1 = z^{G_0}$ and $\Gamma_1 = \mathrm{Cay}(Z_{4m}, S_1)$. By Lemma 3.12, S_1 is the union of some cosets of $Z_2 = \langle z_1^2 \rangle$. Let $S_2 = z_2^{G_0} \subseteq Z_m$ and $\Gamma_2 = \mathrm{Cay}(Z_{4m}, S_2)$. Thus $B_0 = Z_m, B_1 = zZ_m, B_2 = z^2 Z_m$ and $B_3 = z^3 Z_m$ are the connected components of Γ_2.

Let $S = S_1 \cup S_2$ and $\Gamma = \mathrm{Cay}(Z_{4m}, S)$. Suppose first that Γ is arc-transitive. Note that S_1 consists of elements of order $4m$ and S_2 contains elements of order m. We observe that S is not the union of cosets of any subgroup. By [12, Theorem 1.3], $\Gamma = \Sigma[\overline{K}_b] - b.\Sigma$ where $n = br$, $4 \leqslant b < n$ and $\gcd(b, r) = 1$. Thus writing $Z_n = Y \times M$ with $Y \cong Z_b$ and $M \cong Z_r$, we have that $S = Y \setminus \{1\} \times T$ and $T \subseteq M \setminus \{1\}$. Analyzing the orders of elements of S, we have that $b = p_i$ is prime, $p_i \geqslant 5$ and $p_i || m$ as $(b, r) = 1$. As $z^{G_0} \subset Y \setminus \{1\} \times T$, $D_i \cong \mathrm{Aut}(Z_{p_i}) \subseteq G_0$, contradicting the condition (i). Thus Γ is not arc-transitive.

Let P be the point stabilizer of $\mathrm{Aut}(\Gamma)$ on vertex 1. Since $P \geqslant G_0$, P has two orbits S_1 and S_2 and so $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1) \cap \mathrm{Aut}(\Gamma_2)$

Assume that $g \in \mathrm{Aut}(\Gamma)$ fixing $1 \in B_0$ and $z \in B_1$. Consider $z^2 \in B_2 \cap zS_1$ which is adjacent to z. It follows from Lemma 3.12 that g fixes each coset of $Z_2 = \langle z_1^2 \rangle$. Hence $(z^2)^g \in \{z^2, z^2 z_1^2\} = z^2 Z_2$ and g fixes both $z \in B_1$ and $zz_1^2 \in B_3$. Moreover, as $g \in \mathrm{Aut}(\Gamma_2)$, we conclude that g must fix B_0, B_1, B_2 and B_3 setwise. Therefore, g fixes z^2. Continuing in this fashion, we conclude that g fixes $z^3, z^4, \ldots$ and so on. Thus $g = 1$ and $P = G_0$. It follows that $\mathrm{Aut}(\Gamma) = G$ as required. □

It remains to handle case 2 and case 3 in Lemma 3.12. By Lemma 3.12, we may suppose that $8|n$ and $G = \hat{Z}_n \rtimes G_0$ where $G_0 \leqslant \mathrm{Aut}(Z_n)$. Let $S_1 = z^{G_0}$ where $Z_n = \langle z \rangle$ and let $S_2 = (z^2)^{G_0} \subseteq Z_{n/2} = \langle z^2 \rangle$. We construct $\Gamma = \mathrm{Cay}(Z_n, S_1 \cup S_2)$. We will show that Γ can represent G in both case 2 and case 3. In order for proving this, let $\Gamma_1 = \mathrm{Cay}(Z_n, S_1)$ and $\Gamma_2 = \mathrm{Cay}(Z_n, S_2)$ we need to study Γ_1 and Γ_2. Note that Γ_1 has been studied in Lemma 3.12. We study Γ_2 in the following lemma.

Lemma 3.14. *Suppose that case 2 or 3 of Lemma 3.12 occurs. With above notation, we have that* $\Gamma_2 = 2.\mathrm{Cay}(\langle z^2 \rangle, S_2)$. *Let* $A_3 = \mathrm{Aut}(\mathrm{Cay}(\langle z^2 \rangle, S_2))$ *and* $A_2 = \mathrm{Aut}(\Gamma_2)$. *Then* $A_2 = A_3 \wr Z_2$. *Moreover,* $\mathrm{Cay}(\langle z^2 \rangle, S_2)$ *is a normal arc-transitive circulant and* $A_3 = \langle z^2 \rangle \rtimes G_0^{\langle z^2 \rangle}$.

Proof. Let $\Delta_1 = \langle z^2 \rangle$ and $\Delta_2 = z\langle z^2 \rangle$. Then $\Gamma_2 = 2.\mathrm{Cay}(\langle z^2 \rangle, S_2)$ such that Δ_1 and Δ_2 are two connected components of Γ_2. Thus $A_2 = A_3 \wr Z_2$.

Let $\overline{G}_0 = G_0 / \langle \alpha_1^2 \rangle$ in case 2, and let $\overline{G}_0 = G_0 / (D_1 \cap G_0)$ in case 3. Note that G_0 preserves Δ_1, it is easy to check that the induced permutation group $G_0^{\Delta_1} \cong \overline{G}_0$ and $G_0^{\Delta_1} \leqslant \mathrm{Aut}(\langle z^2 \rangle)$. Also $S_2 = (z^2)^{G_0^{\Delta_1}}$ is an orbit of $G_0^{\Delta_1}$. Since G_0 satisfies conditions in Theorem 1.2, S_2 is not the union of cosets of any subgroup of $\langle z^2 \rangle$. By Theorem 2.1 and Conditions 1.1, we conclude that $\mathrm{Cay}(\langle z^2 \rangle, S_2)$ is normal and $\mathrm{Aut}(\mathrm{Cay}(\langle z^2 \rangle, S_2)) = \langle z^2 \rangle \rtimes G_0^{\Delta_1}$. □

Proposition 3.15. *With notation in Subsection 3.3.1, suppose* $G = \hat{Z}_n \rtimes G_0$ *where* $G_0 \leqslant \mathrm{Aut}(Z_n)$ *and* G_0 *satisfies Conditions 1.1. Suppose further that case 2 or 3 of Lemma 3.12 occurs. Let* $S_1 = z^{G_0}$ *where* $Z_n = \langle z \rangle$ *and let* $S_2 = (z^2)^{G_0}$. *Let* $\Gamma = \mathrm{Cay}(Z_n, S_1 \cup S_2)$ *and let* P *be the point stabilizer of vertex 1 in* $\mathrm{Aut}(\Gamma)$. *Then*

1. Γ *is not arc-transitive, and* S_1, S_2 *are two orbits of* P.

2. *For any* $g \in \mathrm{Aut}(\Gamma)$ *such that* g *fixes 1 and* z, *we have that* $g = 1$.

3. $\mathrm{Aut}(\Gamma) = G = Z_n \rtimes G_0$. *So* Γ *is normal and* G *is 2-closed.*

Proof. (1) Suppose, to the contrary, that Γ is arc-transitive. Note that S_1 consists of elements of order n and S_2 contains elements of order $n/2 \neq n$. Also observe that S is not the union of cosets of any subgroup. By [12, Theorem 1.3], $\Gamma = \Sigma[\overline{K}_b] - b.\Sigma$, where $n = br$, $4 \leqslant b < n$ and $\gcd(b, r) = 1$. Thus writing $Z_n = Y \times M$ with $Y \cong Z_b$ and $M \cong Z_r$, we have that $S = Y \setminus \{1\} \times T$ and $T \subseteq M \setminus \{1\}$. Analyzing the orders of elements of S, by conditions (i)(ii) we have that $b = 4$. As $(b, r) = 1$, $4||n$, contradicting the fact that $8|n$. Thus Γ is not arc-transitive. As $P \geqslant G_0$, S_1, S_2 are two orbits of P.

(2) Let $\Gamma_1 = \mathrm{Cay}(Z_n, S_1)$, $\Gamma_2 = \mathrm{Cay}(Z_n, S_2)$ and $A_1 = \mathrm{Aut}(\Gamma_1)$, $A_2 = \mathrm{Aut}(\Gamma_2)$. It follows from (1) that $\mathrm{Aut}(\Gamma) = A_1 \cap A_2$.

Let $g \in \mathrm{Aut}(\Gamma)$ such that g fixes 1 and z. By Lemma 3.12, g preserves each coset of Z_2 and so $(z^2)^g \in \{z^2, z^2 z^{n/2}\}$. Moreover, since $z^2 \in S_2$ and g preserves S_2, we have $(z^2)^g \in S_2$. By the proof of Lemma 3.14, we have that $z^2 Z_2 \nsubseteq S_2$ and so $z^2 z^{n/2} \notin S_2$. Thus g fixes z^2. Let $\Delta_1 = \langle z^2 \rangle$ and $\Delta_2 = z\langle z^2 \rangle$ be two connected components of Γ_2. By Lemma 3.14, $g^{\Delta_1} \in \mathrm{Aut}(\langle z^2 \rangle)$ fixes Δ_1 pointwise. Now g fixes z and z^2 and consider $(z^3)^g$. Using the same argument we deduce that g fixes Δ_2 pointwise and so $g = 1$.

(3) It follows from (2) that $P = G_0$ and so $A = G = \hat{Z}_n \rtimes G_0$. Therefore Γ is normal and G is 2-closed on Z_n. □

Theorem 1.2 now follows from Lemma 3.11, Lemma 3.12, Lemma 3.13 and Proposition 3.15.

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