Semi-degree threshold for anti-directed Hamiltonian cycles

Louis DeBiasio *

Department of Mathematics
Miami University
Oxford, Ohio U.S.A.

debiasld@miamioh.edu

Theodore Molla †

Department of Mathematics
University of Illinois at Urbana-Champaign
Urbana, Illinois, U.S.A

molla@illinois.edu

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Abstract

In 1960 Ghouila-Houri extended Dirac's theorem to directed graphs by proving that if D is a directed graph on n vertices with minimum out-degree and in-degree at least $n/2$, then D contains a directed Hamiltonian cycle. For directed graphs one may ask for other orientations of a Hamiltonian cycle and in 1980 Grant initiated the problem of determining minimum degree conditions for a directed graph D to contain an anti-directed Hamiltonian cycle (an orientation in which consecutive edges alternate direction). We prove that for sufficiently large even n, if D is a directed graph on n vertices with minimum out-degree and in-degree at least $\frac{n}{2}+1$, then D contains an anti-directed Hamiltonian cycle. In fact, we prove the stronger result that $\frac{n}{2}$ is sufficient unless D is one of two counterexamples. This result is sharp.

1 Introduction

A directed graph D is a pair $(V(D), E(D))$ where $E(D) \subseteq V(D) \times V(D)$. In this paper we will only consider loopless directed graphs, i.e. directed graphs with no edges of the type (v, v). An *anti-directed cycle (path)* is a directed graph in which the underlying graph forms a cycle (path) and no pair of consecutive edges forms a directed path. Note that an anti-directed cycle must have an even number of vertices. Let ADP, ADC stand for anti-directed path and anti-directed cycle respectively and let ADHP, ADHC stand for

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anti-directed Hamiltonian path and anti-directed Hamiltonian cycle respectively. Call an ADP $P = v_1 \dots v_d$ *proper* if d is even and $(v_1, v_2) \in E(P)$ and hence, $(v_{d-1}, v_d) \in E(P)$. Given an (undirected) graph G, let $\delta(G)$ be the minimum degree of G. If D is a directed graph, then $\delta(D)$ will denote the minimum degree of the underlying multigraph, i.e. the minimum total degree of D. For a directed graph D, let $\delta^+(D)$ and $\delta^-(D)$ be the minimum out-degree and minimum in-degree respectively. Finally, let $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$ and call this quantity the minimum semi-degree of G.

In 1952, Dirac [6] proved that if G is a graph on $n \geqslant 3$ vertices with $\delta(G) \geqslant n/2$, then G contains a Hamiltonian cycle. In 1960, Ghouila-Houri extended Dirac's theorem to directed graphs.

Theorem 1.1 (Ghouila-Houri [8]). *Let D be a directed graph on n vertices. If $\delta^0(D) \geqslant n/2$, then D contains a directed Hamiltonian cycle.*

(His original statement actually says that $\delta(D) \geqslant n$ is sufficient if D is strongly connected.)

In 1973, Thomassen proved that every tournament on $2n \geqslant 50$ vertices contains an ADHC [16]. Since the total degree of every vertex in a tournament on $2n$ vertices is $2n-1$, Grant wondered if all digraphs on $2n$ vertices with total degree $2n-1$ have an ADHC. So in 1980, Grant made the weaker conjecture (replacing total degree by semi-degree) that if D is a directed graph on $2n$ vertices with $\delta^0(D) \geqslant n$, then D contains an ADHC [9]. However, in 1983, Cai [2] gave a counterexample to Grant's conjecture (see Figure 1b).

Example 1.2 (Cai 1983). *For all n, there exists a directed graph D on $2n$ vertices with $\delta^0(D) = n$ such that D does not contain an ADHC.*

We define for each even n, a family of digraphs with minimum semi-degree $n/2 - 1$ which have no anti-directed cycle on n vertices. From this family, we define two digraphs with minimum semi-degree $n/2$ which have no anti-directed cycle on n vertices (see Figure 1).

Definition 1.3. *Let $n \geqslant 2$ be even and let $0 \leqslant k \leqslant \frac{n}{2}$. Let $F_{n,k}$ be a digraph on n vertices where $\{X_1, X_2, Y_1, Y_2\}$ is a partition of the vertex set with $|X_1| = |X_2| = \frac{n}{2} - k$ and $|Y_1| = |Y_2| = k$ and where (u, v) is an edge if and only if $u \neq v$ and*

(i) *$u \in Y_i$ and $v \in Y_i \cup X_i$ for $i \in [2]$ or*

(ii) *$u \in X_i$ and $v \in Y_{3-i} \cup X_{3-i}$ for $i \in [2]$.*

Let F_n^1 be the digraph obtained from $F_{n,1}$ by adding the edges (y_1, y_2) and (y_2, y_1), where y_i is the unique vertex in Y_i.

Let F_n^2 be the digraph obtained from $F_{n,1}$ by adding the edges (y_1, y_2), (y_2, x), and (x, y_1), where y_i is the unique vertex in Y_i and x is an arbitrary vertex in X_1.

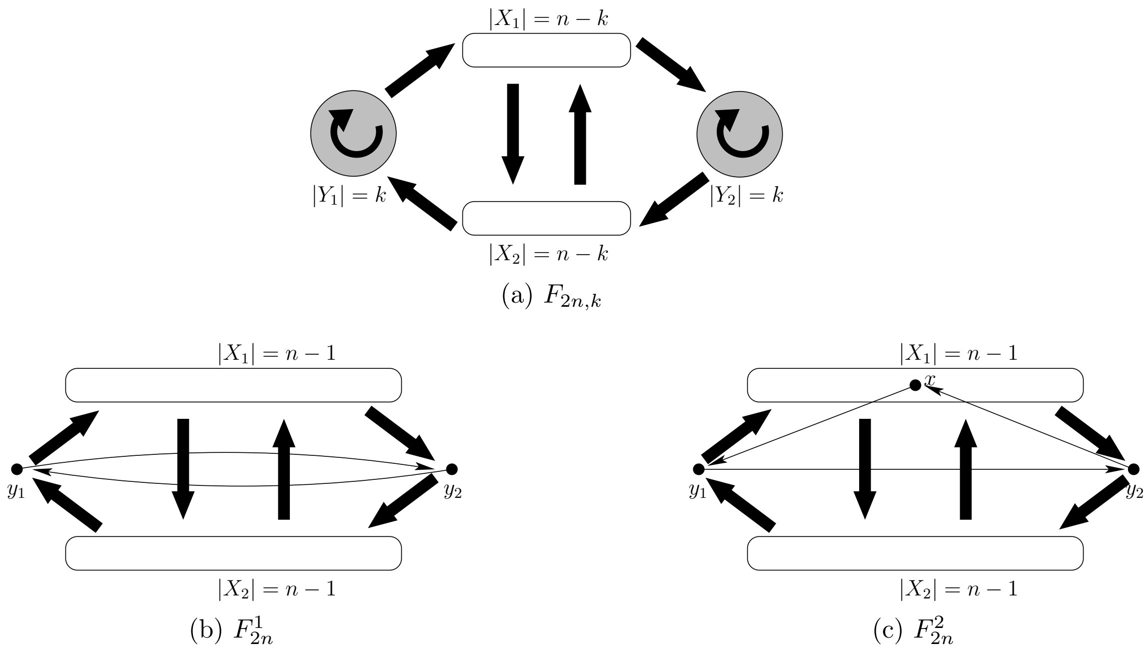

Figure 1: The solid arrows indicate all possible edges in the designated direction. The shaded sets with the curved arrows indicate all possible directed edges.

One can easily check that F^1_{2n} and F^2_{2n} have no ADHC and are edge maximal with respect to this property. Cai's example (F^1_{2n} above) and our modification of his example (F^2_{2n} above) shows that the semi-degree threshold for an ADHC in a directed graph on $2n$ vertices is at least $n + 1$. There have been a sequence of partial results which have improved the threshold from the upper end. In 1980, Grant proved that if D is a directed graph on $2n$ vertices and $\delta^0(D) \geqslant \frac{4}{3}n + 2\sqrt{n \log n}$, then D has an ADHC [9]. In 1995, Häggkvist and Thomason proved the very general result that if D is a directed graph on n vertices then the semi-degree threshold for all orientations of a cycle on n vertices is asymptotically $n/2$ (we conjecture an exact bound in Section 5).

Theorem 1.4 (Häggkvist, Thomason [10])**.** *For sufficiently large n, if D is a directed graph on n vertices and $\delta^0(D) \geqslant \frac{n}{2} + n^{5/6}$, then D contains every orientation of a cycle on n vertices.*

Then in 2008, Plantholt and Tipnis improved upon Grant's result by showing that if D is a directed graph on $2n$ vertices and $\delta^0(D) \geqslant \frac{4}{3}n$, then D has an ADHC [15] (note that this is for all n). Finally in 2011, Busch, Jacobson, Morris, Plantholt, Tipnis improved upon all the results for ADHC's by showing that if D is a directed graph on $2n$ vertices and $\delta^0(D) \geqslant n + \frac{14}{3}\sqrt{n}$, then D has an ADHC [1].

The main goal of this paper is to determine, for sufficiently large n, the exact semi-degree threshold for an ADHC. However, we actually prove something stronger which in effect shows that there are only two counterexamples to Grant's conjecture.

Theorem 1.5. *Let D be a directed graph on $2n$ vertices. If n is sufficiently large and $\delta^0(D) \geqslant n$, then D contains an anti-directed Hamiltonian cycle unless D is isomorphic to F^1_{2n} or F^2_{2n}.*

Since $\delta^0(F^1_{2n}) = \delta^0(F^2_{2n}) = n$, we obtain the following corollary.

Corollary 1.6. *Let D be a directed graph on $2n$ vertices. If n is sufficiently large and $\delta^0(D) \geqslant n + 1$, then D contains an anti-directed Hamiltonian cycle.*

Since we have determined the semi-degree threshold for ADHC's, we go back and modify the original conjecture that Grant hinted at.

Conjecture 1.7. *Let D be a directed graph on $2n$ vertices. If $\delta(D) \geqslant 2n + 1$, then D contains an anti-directed Hamiltonian cycle.*

An *anti-directed* 2*-factor* on n vertices is a directed graph in which the underlying graph forms a 2-factor and no pair of consecutive edges forms a directed path (once again note that n must be even for an anti-directed 2-factor to exist). Diwan, Frye, Plantholt, and Tipnis conjectured that if D is a directed graph on $2n \geqslant 8$ vertices and $\delta^0(D) \geqslant n$, then D contains an anti-directed 2-factor [7]. Since it can be easily shown that F^1_n and F^2_n each contain an anti-directed 2-factor with two cycles we also obtain the following corollary of Theorem 1.5, which implies their conjecture for sufficiently large n.

Corollary 1.8. *Let D be a directed graph on $2n$ vertices. If n is sufficiently large and $\delta^0(D) \geqslant n$, then D contains an anti-directed* 2*-factor with at most two cycles.*

Let L_n be the graph on vertex set $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ such that $\{u_i, v_j\} \in E(L_n)$ if and only if $|i - j| \leqslant 1$. We call L_n a *ladder* and note that L_n contains every bipartite 2-factor on $2n$ vertices. Let $\vec{L}_n$ be the directed graph obtained from L_n by orienting every edge $\{u_i, v_j\}$ from u_i to v_j. We call $\vec{L}_n$ an *anti-directed ladder* and note that $\vec{L}_n$ contains every anti-directed 2-factor on $2n$ vertices.

Czygrinow and Kierstead determined the minimum degree threshold for a balanced bipartite graph to contain a spanning ladder.

Theorem 1.9 (Czygrinow, Kierstead [4])**.** *There exists n_0 such that if G is a balanced bipartite graph on $2n \geqslant 2n_0$ vertices with $\delta(G) \geqslant \frac{n}{2} + 1$, then $L_n \subseteq G$.*

We make the following conjecture which would strengthen Corollary 1.6.

Conjecture 1.10. *Let D be a directed graph on $2n$ vertices. If n is sufficiently large and $\delta^0(D) \geqslant n + 1$, then $\vec{L}_n \subseteq D$. In particular D contains every possible anti-directed* 2*-factor.*

We note that Conjecture 1.10 holds asymptotically.

Observation 1.11. *For all $\varepsilon > 0$, there exists n_0 such that if D is a directed graph on $2n \geqslant 2n_0$ vertices with $\delta^0(D) \geqslant (1+\varepsilon)n$, then $\vec{L}_n \subseteq D$.*

Proof. Let X_1, X_2 be a random balanced bipartition of $V(D)$. We expect

$$\delta^+(x, X_2), \delta^-(x, X_1) \geqslant \frac{1}{2}(1+\varepsilon)n \text{ for all } x \in X_1 \cup X_2,$$

so by Chernoff's inequality there exists such a partition X_1, X_2 which satisfies

$$\delta^+(X_1, X_2) \geqslant \frac{n}{2} + 1 \text{ and } \delta^-(X_2, X_1) \geqslant \frac{n}{2} + 1.$$

Let G be an X_1, X_2-bipartite graph such that $\{u, v\} \in E(G)$ if and only if $u \in X_1$, $v \in X_2$ and $(u, v) \in \vec{E}_D(X_1, X_2)$. Note that G is a balanced bipartite graph on $2n$ vertices with $\delta(G) \geqslant \frac{n}{2} + 1$ and thus by Theorem 1.9, G contains a spanning ladder L_n which corresponds to a spanning anti-directed ladder $\vec{L}_n$ in D. □

2 Overview

Note that Observation 1.11 also implies that Theorem 1.5 holds asymptotically. To get the exact result, we use the now common stability technique where we split the proof into two cases depending on whether D is "close" to an extremal configuration or not (see Figure 1a). If D is close to an extremal configuration, then we use some ad-hoc techniques which rely on the exact minimum semi-degree condition and if D is not close to an extremal configuration then we use the recent absorbing method of Rödl, Ruciński, and Szemerédi (as opposed to the regularity/blow-up method).

To formally say what we mean by "close" to an extremal configuration we need the following definition.

Definition 2.1. *Let D be a directed graph on $2n$ vertices. We say D is α-extremal if there exists $A, B \subseteq V(D)$ such that $(1-\alpha)n \leqslant |A|, |B| \leqslant (1+\alpha)n$ and $\Delta^+(A, B) \leqslant \alpha n$ and $\Delta^-(B, A) \leqslant \alpha n$.*

This definition is more restrictive than simply bounding the number of edges, thus it will help make the extremal case less messy. However, a non-extremal set still has many edges from A to B.

Observation 2.2. *Let $0 < \alpha \ll 1$. Suppose D is not α-extremal, then for $A, B \subseteq V(D)$ with $(1-\alpha/2)n \leqslant |A|, |B| \leqslant (1+\alpha/2)n$, we have $\vec{e}(A, B) \geqslant \frac{\alpha^2}{2}n^2$.*

Proof. Let $A, B \subseteq V(D)$ with $(1-\alpha/2)n \leqslant |A|, |B| \leqslant (1+\alpha/2)n$. Since D is not α-extremal, there is some vertex $v \in A$ with $\deg^+(v, B) \geqslant \alpha n$ or $v \in B$ with $\deg^-(v, A) \geqslant \alpha n$. Either way, we get at least αn edges. Now delete v, and apply the argument again to get another αn edges. We may repeat this until $|A|$ or $|B|$ drops below $(1-\alpha)n$, i.e. for at least $\frac{\alpha}{2}n$ steps. This gives us at least $\frac{\alpha^2}{2}n^2$ edges in total. □

Finally, we make two more observations which will be useful when working with non-extremal graphs.

Observation 2.3. *Let $0 < \lambda \leqslant \alpha \ll 1$ and let D be a directed graph on n vertices. If D is not α-extremal and $X \subseteq V(D)$ with $|X| \leqslant \lambda n$, then $D' = D - X$ is not $(\alpha - \lambda)$-extremal.*

Proof. Let $A', B' \subseteq V(D') \subseteq V(D)$ with $(1-\alpha+\lambda)|D'| \leqslant |A'|, |B'| \leqslant (1+\alpha-\lambda)|D'|$. Note that

$$(1-\alpha)n \leqslant (1-\alpha+\lambda)(1-\lambda)n \leqslant (1-\alpha+\lambda)|D'| \leqslant |A'|, |B'| \leqslant (1+\alpha-\lambda)|D'| \leqslant (1+\alpha)n$$

thus there exists $v \in A'$ such that $\deg^+(v, B') \geqslant \alpha n \geqslant (\alpha-\lambda)|D'|$ or $v \in B'$ such that $\deg^-(v, A') \geqslant \alpha n \geqslant (\alpha - \lambda)|D'|$. □

Lemma 2.4. *Let $X, Y \subseteq V(D)$. If $\vec{e}(X,Y) \geqslant c|X||Y|$, then there exists*

(i) *$X' \subseteq X$, $Y' \subseteq Y$ such that $X' \cap Y' = \emptyset$ and $\delta^+(X', Y') \geqslant \frac{c}{8}|Y|, \delta^-(Y', X') \geqslant \frac{c}{8}|X|$ and*

(ii) *a proper anti-directed path in $D[X \cup Y]$ on at least $\frac{c}{4} \cdot \min\{|X|, |Y|\}$ vertices.*

Proof. *(i)* Let $X^* = X \setminus Y$ and $Y^* = Y \setminus X$. Delete all edges not in $\vec{E}(X,Y)$. Choose a partition $\{X'', Y''\}$ of $X \cap Y$ which maximizes $\vec{e}(X^* \cup X'', Y^* \cup Y'')$ and set $X_0 = X^* \cup X''$ and $Y_0 = Y^* \cup Y''$. Note that $\vec{e}(X_0) + \vec{e}(Y_0) + \vec{e}(X_0, Y_0) + \vec{e}(Y_0, X_0) = \vec{e}(X,Y)$. We have that

$$\vec{e}(X_0) = \sum_{v \in X_0} \deg^+(v, X_0) = \sum_{v \in X''} \deg^-(v, X_0) \leqslant \sum_{v \in X''} \deg^+(v, Y_0) \leqslant \vec{e}(X_0, Y_0)$$

where the inequality holds since if $\deg^-(v, X_0) > \deg^+(v, Y_0)$ for some $v \in X''$, then we could move v to Y'' and increase the number of edges across the partition. Similarly, $\vec{e}(X_0, Y_0) \geqslant \vec{e}(Y_0)$. Thus $\vec{e}(X_0, Y_0) \geqslant \frac{1}{4}\vec{e}(X,Y) \geqslant \frac{c}{4}|X||Y|$.

If there exists $v \in X_0$ such that $\deg^+(v, Y_0) < \frac{c}{8}|Y|$ or $v \in Y_0$ such that $\deg^-(v, X_0) < \frac{c}{8}|X|$, then delete v and set $X_1 = X_0 \setminus \{v\}$ and $Y_1 = Y_0 \setminus \{v\}$. Repeat this process until there no vertices left to delete. This process must end with a non-empty graph because fewer than $|X|\frac{c}{8}|Y| + |Y|\frac{c}{8}|X| = \frac{c}{4}|X||Y|$ edges are deleted in this process. Finally, let X' and Y' be the sets of vertices which remain after the process ends.

(ii) Apply Lemma 2.4.(i) to obtain sets $X' \subseteq X$, $Y' \subseteq Y$ such that $X' \cap Y' = \emptyset$ and $\delta^+(X', Y') \geqslant \frac{c}{8}|Y|$ and $\delta^-(Y', X') \geqslant \frac{c}{8}|Y|$. Let G be an auxiliary bipartite graph on X', Y' with $E(G) = \{\{x,y\} : (x,y) \in \vec{E}(X', Y')\}$. Note that $\delta(G) \geqslant \frac{c}{8}\min\{|X|, |Y|\}$ and thus G contains a path on at least $2\delta(G) \geqslant \frac{c}{4} \cdot \min\{|X|, |Y|\}$ vertices, which starts in X. This path contains a proper anti-directed path in D on at least $\frac{c}{4} \cdot \min\{|X|, |Y|\}$ vertices. □

3 Non-extremal Case

In this section we will prove that if D satisfies the conditions of Theorem 1.5 and D is not α-extremal, then D has an ADHC. We actually prove a stronger statement which in some sense shows that the extremal condition is "stable," i.e. graphs which do not satisfy the extremal condition do not require the tight minimum semi-degree condition.

Theorem 3.1. *For any $\alpha \in (0, 1/32)$ there exists $\varepsilon > 0$ and n_0 such that if $D = (V, E)$ is a directed graph on $2n \geqslant 2n_0$ vertices, D is not α-extremal and $\delta^0(D) \geqslant (1-\varepsilon)n$, then D contains an anti-directed Hamiltonian cycle.*

Lemma 3.2. *For all $0 < \epsilon \ll \beta \ll \lambda \ll \alpha \ll 1$ there exists n_0 such that if $n \geqslant n_0$, D is a directed graph on $2n$ vertices, $\delta^0(D) \geqslant (1-\varepsilon)n$, and D is not α-extremal, then there exists a proper anti-directed path P^* with $|P^*| \leqslant \lambda n$ such that for all $W \subseteq V(D) \setminus V(P^*)$ with $2w := |W| \leqslant \beta n$, $D[V(P^*) \cup W]$ contains a spanning proper anti-directed path with the same endpoints as P^*.*

Lemma 3.3. *For all $0 < \epsilon \ll \beta \ll \lambda \ll \alpha \ll 1$ there exists n_0 such that if $n \geqslant n_0$, D is a directed graph on $2n$ vertices, $\delta^0(D) \geqslant (1-\varepsilon)n$, D is not α-extremal, and P^* is a proper anti-directed path with $|P^*| \leqslant \lambda n$, then D contains an anti-directed cycle on at least $(2-\beta)n$ vertices which contains P^* as a segment.*

First we use Lemma 3.2 and Lemma 3.3 to prove Theorem 3.1.

Proof. Let $\alpha \in (0, 1/32)$ and choose $0 < \epsilon \ll \beta \ll \lambda \ll \sigma \ll \alpha$. Let n_0 be large enough for Lemma 3.2 and Lemma 3.3. Let D be a directed graph on $2n$ vertices with $\delta^0(D) \geqslant (1-\varepsilon)n$. Apply Lemma 3.2 to obtain an anti-directed path P^* having the stated property. Now apply Lemma 3.3 to obtain an anti-directed cycle C^* which contains P^* as a segment. Let $W = D - C^*$ and note that since C^* is an anti-directed cycle, $|C^*|$ is even which implies $|W|$ is even, since $|D|$ is even. Finally apply the property of P^* to the set W to obtain an ADHC in D. □

3.1 Absorbing

To prove Lemma 3.2 we will use the following more general statement.

Lemma 3.4. *Let $m, d \in \mathbb{N}$, $a > 0$, $b \in (0, \frac{a}{2d})$ and $c \in \left(0, 2b\left(\frac{a}{2d} - b\right)\right)$. There exists n_0 such that when V is a set of order $n \geqslant n_0$ the following holds. For every $S \in \binom{V}{m}$, let $f(S)$ be a subset of V^d. Call $T \in V^d$ a* good tuple *if $T \in f(S)$ for some $S \in \binom{V}{m}$. If $|f(S)| \geqslant an^d$ for every $S \in \binom{V}{m}$ then there exists a set $\mathcal{F}$ of at most bn/d good tuples such that $|f(S) \cap \mathcal{F}| \geqslant cn$ for every $S \in \binom{V}{m}$ and the images of distinct elements of $\mathcal{F}$ are disjoint.*

Proof. Pick $\varepsilon > 0$ so that

$$(1+a)\varepsilon < \frac{ab}{d} - 2b^2 - c.$$

Let $b' := \frac{b}{d}$, $p := b' - \varepsilon$ and $c' := c + (d^2+1)p^2$. Let $\mathcal{F}'$ be a random subset of V^d where each $T \in V^d$ is selected independently with probability pn^{1-d}. Let

$$\mathcal{O} := \left\{ \{T, T'\} \in \binom{V^d}{2} : \operatorname{im}(T) \cap \operatorname{im}(T') \neq \emptyset \right\}$$

and $\mathcal{O}_{\mathcal{F}'} := \mathcal{O} \cap \binom{\mathcal{F}'}{2}$.

We only need to show that, for sufficiently large n_0, with positive probability $|\mathcal{O}_{\mathcal{F}'}| < (d^2+1)p^2 n$, $|\mathcal{F}'| < b'n$ and $|f(S) \cap \mathcal{F}'| > c'n$ for every $S \in \binom{V}{m}$. We can then remove at most $(d^2+1)p^2 n$ tuples from such a set $\mathcal{F}'$ so that the images of the remaining tuples are disjoint. After also removing every $T \in \mathcal{F}'$ for which there is no $S \in \binom{V}{m}$ for which $f(S) = T$, the resulting set $\mathcal{F}$ will satisfy the conditions of the lemma.

Clearly,

$$|\mathcal{O}| \leqslant n \cdot d^2 \cdot n^{2d-2} = d^2 n^{2d-1},$$

and for any $\{T, T'\} \in \binom{V^d}{2}$, $\Pr(\{T, T'\} \subseteq \mathcal{F}') = p^2 n^{2-2d}$. Therefore, by the linearity of expectation, $\mathbb{E}[|\mathcal{O}_{\mathcal{F}'}|] < d^2 p^2 n$. So, by Markov inequality,

$$\Pr\left(|\mathcal{O}_{\mathcal{F}'}| \geqslant (d^2+1)p^2 n\right) \leqslant \frac{d^2}{d^2+1}.$$

Note that $\mathbb{E}[|\mathcal{F}'|] = pn$ and $pn \geqslant \mathbb{E}[|f(S) \cap \mathcal{F}'|] \geqslant apn$ for every $S \in \binom{V}{m}$. Therefore, by the Chernoff inequality, $\Pr(|\mathcal{F}'| \geqslant b'n) \leqslant e^{-\varepsilon^2 n/3}$ and, since

$$ap - c' = \frac{ab}{d} - a\varepsilon - (d^2+1)\left(\frac{b}{d} - \varepsilon\right)^2 - c \geqslant \frac{ab}{d} - 2b^2 - c - a\varepsilon > \varepsilon,$$

$\Pr(|\mathcal{F}' \cap f(S)| \leqslant c'n) < e^{-\varepsilon^2 n/3}$ for every $S \in \binom{V}{m}$. Therefore, for sufficiently large n_0,

$$\Pr\left(|\mathcal{O}_{\mathcal{F}'}| \geqslant (d^2+1)p^2\right) + \Pr(|\mathcal{F}'| \geqslant b'n) + \sum_{S \in \binom{V}{m}} \Pr(|\mathcal{F}' \cap f(S)| \leqslant c'n) < 1. \qquad \square$$

Let $\mathcal{P} := V^2 - \{(x,x) : x \in V\}$. For any $(x,y) \in \mathcal{P}$, call $(a,b,c,d) \in V^4$ an (x,y)-*absorber* if $abcd$ is a proper anti-directed path and $axcbyd$ is a proper anti-directed path (see Figure 2) and call $(a,b) \in V^2$ an (x,y)-*connector* if $xaby$ is an anti-directed path where (a,b) is an edge (note that specifying one edge dictates the directions of all the other edges).

Note that if $(x',x), (y,y') \in \vec{E}(D)$ and (a,b) is an (x,y)-connector disjoint from $\{x', y'\}$ then $x'xabyy'$ is an anti-directed path.

For all $(x,y) \in \mathcal{P}$, let $f_{\text{abs}}(x,y) = \{T \in V^4 : T \text{ is an } (x,y)\text{-absorber}\}$ and $f_{\text{con}}(x,y) = \{T \in V^2 : T \text{ is an } (x,y)\text{-connector}\}$.

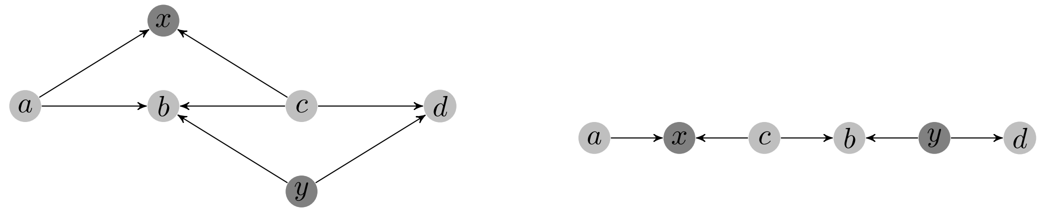

Figure 2: (a, b, c, d) is an (x, y)-absorber

Claim 3.5. *Let D satisfy the conditions of Lemma 3.2. For all $(x, y) \in \mathcal{P}$ we have*

(i) $|f_{abs}(x, y)| \geqslant \alpha^{12} n^4$ *and*

(ii) $|f_{con}(x, y)| \geqslant \alpha^3 n^2$.

Proof. Let $(x, y) \in \mathcal{P}$ and let $A = N^-(x)$ and $B = N^+(y)$.

(i) By Observation 2.2 and Lemma 2.4, there exists $A' \subseteq A$ and $B' \subseteq B$ such that $A' \cap B' = \emptyset$ and $\delta^+(A', B'), \delta^-(B', A') \geqslant \frac{\alpha^2}{16}(1 - \varepsilon)n \geqslant \alpha^3 n + 1$. For all $(c, b) \in \vec{E}(A', B')$, we have $|N^-(b) \cap A'| \geqslant \alpha^3 n + 1$ and $|N^+(c) \cap B'| \geqslant \alpha^3 n + 1$. So there are more than $(\alpha^3 n)^2$ choices for (b, c), $\alpha^3 n$ choices for a and $\alpha^3 n$ choices for d, i.e. $|f_{abs}(x, y)| \geqslant \alpha^{12} n^4$.

(ii) Similarly, by Observation 2.2, we have $\vec{e}(A, B) \geqslant \frac{\alpha^2}{2} n^2 \geqslant \alpha^3 n^2$, each of which is a connector.

□

Claim 3.6 (Connecting-Reservoir). *For all $0 < \gamma \ll \alpha$ and $D' \subseteq D$ such that $|D'| \geqslant (2 - \lambda)n$, there exists a set of pairwise disjoint ordered pairs $\mathcal{R}$ such that if $R = \cup_{(a,b) \in \mathcal{R}} \{a, b\}$, then $R \subseteq V(D')$, $|R| \leqslant \gamma n$ and for all distinct $x, y \in V(D)$, $|f_{con}(x, y) \cap \mathcal{R}| \geqslant \gamma^2 n$.*

Proof. For every $(x, y) \in \mathcal{P}$

$$|\{(a, b) \in f_{con}(x, y) : a, b \in V(D')\}| \geqslant |f_{con}(x, y)| - 2|D - D'|n \geqslant \alpha^3 n^2/2.$$

Therefore, we can apply Lemma 3.4 to obtain a set $\mathcal{R}$ of disjoint good ordered pairs such that $|\mathcal{R}| \leqslant \gamma n/2$ and $|f_{con}(x, y) \cap \mathcal{R}| \geqslant \gamma \alpha^3 n/4 - 2\gamma^2 n \geqslant \gamma^2 n$ and $\mathcal{R} \subseteq V(D')^2$. □

Now we prove Lemma 3.2.

Proof. Since $|f_{abs}(x, y) \cap \mathcal{P}(V')| \geqslant \alpha^{12} n^4$ we apply Lemma 3.4 to D obtain a set $\mathcal{A}$ of disjoint good 4-tuples $\{A_1, \ldots, A_\ell\}$ such that $|\mathcal{A}| \leqslant \lambda n/8$ and $|f_{abs}(x, y) \cap \mathcal{A}| \geqslant \lambda \alpha^{12} n/8 - 2(\lambda/2)^2 n \geqslant \lambda^2 n$. Let $A = \cup_{(a,b,c,d) \in \mathcal{A}} \{a, b, c, d\}$ and note that $|A| \leqslant \lambda n/2$.

Let $(a_i, b_i, c_i, d_i) := A_i$ for every $i \in [l]$, so $a_i b_i c_i d_i$ is a proper ADP. Note that there are less than $2|A|n$ ordered pairs that contain a vertex from A, so since $\lambda \ll \alpha$, we can greedily choose vertex disjoint $(x_i, y_i) \in f_{con}(d_i, a_{i+1})$ for each $i \in [l-1]$ such that $x_i, y_i \notin A$. Set $P^* := A_1 x_1 y_1 A_2 x_2 y_2 A_2 \dots A_{l-1} x_{l-1} y_{l-1} A_l$ and note that $|P^*| \leqslant \lambda n$ and $|P^*|$ is a proper ADP.

To see that P^* has the desired property, let $W \subseteq V \setminus V(P^*)$ such that $2w = |W| \leqslant \beta n$. Arbitrarily partition W into pairs and since $\beta \ll \lambda$, we can greedily match the disjoint pairs from W with 4-tuples in $\mathcal{A}$. By the way we have defined an (x, y)-absorber, $D[V(P') \cup W]$ contains a spanning proper anti-directed path starting with an out-edge from a_1 and ending with an in-edge to d_ℓ. □

3.2 Covering

The main challenge in the proof of Lemma 3.3 is to show that if a maximum length anti-directed path is not long enough, then we can build a constant number of vertex disjoint anti-directed paths whose total length is sufficiently larger.

Claim 3.7. *Under the conditions of Lemma 3.3, suppose P^* is a proper anti-directed path with $|P^*| \leqslant \lambda n$. For all $R \subseteq V(D - P^*)$ with $|R| \leqslant \beta^2 n$, if P is a proper anti-directed path in $D - R$ with beginning segment P^* such that $|P| < (2-\beta)n$, then there exist disjoint proper anti-directed paths $Q_1, \dots, Q_r \subseteq D - R$, such that $r \leqslant 6$, Q_1 contains P^* as an initial segment and*

$$|Q_1| + \cdots |Q_r| \geqslant |P| + \varepsilon \left\lceil \frac{1}{4} \log n \right\rceil .$$

First we show how this implies Lemma 3.3.

Proof. Let n be large enough so that we can apply Claim 3.6 and so that if $m := \lceil \frac{1}{4} \log n \rceil$, then

$$n \geqslant \frac{4m2^{2m}}{\varepsilon^2 \beta} \text{ and } m > 10\beta^{-4}\varepsilon^{-1}. \tag{1}$$

Let P^* be a proper anti-directed path with $|P^*| \leqslant \lambda n$. Let $D' = D - P^*$. Now apply Claim 3.6 to D' with $\gamma = \beta^2$ to get $\mathcal{R}$ and R such that $|f_{con}(x, y) \cap \mathcal{R}| \geqslant \beta^4 n$ for every $(x, y) \in \mathcal{P}$ and $|R| \leqslant \beta^2 n$.

Let P be a maximum length proper anti-directed path in $D - R$ that begins with P^*. If $|P| < (2-\beta)n$, then we apply Claim 3.7. Now connect $Q_1, \dots, Q_r$ into a longer path using at most 5 pairs from $\mathcal{R}$. Delete these vertices from R and reset $\mathcal{R}$. We may repeat this process as long as there are sufficiently many pairs remaining in $\mathcal{R}$. On each step, $|f_{con}(x, y) \cap \mathcal{R}|$ may be reduced by at most 5. However, in less than $\frac{2n}{\varepsilon m}$ steps, we will have a path of length greater than $(2-\beta)n$ in which case we would be done. By (1), $5 \cdot \frac{2n}{\varepsilon m} < \beta^4 n$, so we can repeat the process sufficiently many times. Once we have a path P with $|P| \geqslant (2-\beta)n$, we use one more pair from $\mathcal{R}$ to connect the endpoints of P to form an anti-directed cycle C, which is possible since $|P|$ is even. Note that C contains P^* as a segment by construction. □

Proof of Claim 3.7. Let n and m be as in (1). Let P be a maximum length proper ADP in $D - R$ containing P^* as an initial segment. Let $\hat{P}$ be the shortest segment of P immediately following P^* so that $P' := v_1 \dots v_p = P - (P^* \cup \hat{P})$ is a multiple of $2m$; thus $|\hat{P}| < 2m$. Set $T := V \setminus (V(P) \cup V(R))$, and $P_i := v_{2m(i-1)+1} \dots v_{2mi}$ for $i \in [s]$ where $s := \frac{p}{2m}$ (which is an integer by the choice of $\hat{P}$). Note that $|P_i| = 2m$ for every $i \in [s]$. Assume $|T| > \beta n - |R| > \beta n/2$.

Claim 3.8. *Let $c \in (\varepsilon^2 - 1, 1)$, $d \in [\varepsilon^2, 1 + c)$, and $b := \lceil (1 + c - d)m \rceil$. If $\vec{e}(T, P_i) \geqslant (1 + c)m|T|$, then there exists $X_i \subseteq V(P_i)$ and $Y_i \subseteq T$ such that $|X_i| = b$, $|Y_i| \geqslant 2m$ and $X_i \subseteq N^+(y)$ for every $y \in Y_i$. In particular, $D[V(P_i) \cup T]$ contains a proper anti-directed path on $2b$ vertices.*

Proof. Let $T' = \{v \in T : \deg^+(v, P_i) \geqslant b\}$ and since

$$(1 + c)m|T| \leqslant \vec{e}(T, P_i) \leqslant (|T| - |T'|)(b - 1) + |T'|2m \leqslant |T|(1 + c - d)m + |T'|2m$$

we have $|T'| \geqslant \frac{d}{2}|T|$. Together with (1) this gives us

$$|T'| \geqslant \frac{d}{2}|T| \geqslant \varepsilon^2 \beta n/2 \geqslant 2m2^{2m} > 2m\binom{2m}{b},$$

which by the pigeonhole principle implies that there exists $X_i \subseteq V(P_i)$ with $|X_i| = b$ and $Y_i \subseteq T'$ such that $|Y_i| \geqslant 2m$ and $X_i \subseteq N_H(y)$ for every $y \in Y_i$. □

By Claim 3.8, if $\vec{e}(T, P_i) \geqslant (1 + \varepsilon)|T|m$ there exists a proper anti-directed path Q_3 of length

$$2\left\lceil (1 + \varepsilon - \varepsilon^2)m \right\rceil > (2 + \varepsilon)m \text{ in } D[T \cup P_i].$$

Letting $Q_1 := P^* \hat{P} P_1 \cdots P_{i-1}$ and $Q_2 := P_{i+1} \cdots P_q$ then satisfies the condition of the lemma. Therefore, we can assume that,

$$\vec{e}(T, P_i) < (1 + \varepsilon)|T|m \text{ for every } i \in [s]. \tag{2}$$

We can also assume that

$$\vec{e}(T, T) < \varepsilon |T|^2. \tag{3}$$

Otherwise by Lemma 2.4.(ii) there exists a proper anti-directed path Q_2 of length $\frac{\varepsilon}{4}|T| \geqslant \varepsilon m$ in $D[T]$. Then $Q_1 := P$ and Q_2 satisfy the condition of the lemma.

So (3) implies that

$$\begin{aligned} \vec{e}(T, P') &\geqslant (1 - \varepsilon)n|T| - (|P^*| + |\hat{P}| + |R|)|T| - \vec{e}(T, T) \\ &\geqslant (1 - \varepsilon - \lambda - \beta^2)n|T| - 2m|T| - \varepsilon|T|^2 \geqslant (1 - 2\lambda)n|T| \end{aligned} \tag{4}$$

Let $\lambda \ll \sigma \ll \alpha$ and let

$$I := \{i \in [s] : \vec{e}(T, P_i) \geqslant (1 - \sigma)|T|m\}.$$

By (2) and (4),

$$(1-2\lambda)n|T| \leqslant \vec{e}(T,P') \leqslant (1-\sigma)m(s-|I|)|T|+(1+\varepsilon)m|I||T| \leqslant (1-\sigma)n|T|+(\sigma+\varepsilon)m|I||T|$$

which implies that $m|I| \geqslant \frac{\sigma-2\lambda}{\sigma+\varepsilon}n > (1-\frac{\alpha}{2})n$. Also note that $n \geqslant |P|/2 \geqslant m|I|$.

For every $i \in I$, let $X_i \subseteq P_i$ and $Y_i \subseteq T$ be the sets guaranteed by Claim 3.8 with $c := -\sigma$, $d := \sigma$ and $b := \lceil (1-2\sigma)m \rceil$. Let $Z_i := V(P_i) \setminus X_i$ for $i \in [I]$ and let $Z := \bigcup_{i \in I} Z_i$. Note that $|Z_i| = 2m - b$ for every $i \in I$ so $|Z| = (2m-b)|I|$ and

$$\left(1+\frac{\alpha}{2}\right)n > (1+2\sigma)n \geqslant (2m-b)|I| \geqslant m|I| > \left(1-\frac{\alpha}{2}\right)n.$$

Therefore by Observation 2.2, $\vec{e}(Z,Z) \geqslant \frac{\alpha^2}{2}|Z|^2$. Because

$$\frac{\alpha^2}{2} \leqslant \frac{\vec{e}(Z,Z)}{|Z|^2} = \frac{1}{|I|^2}\sum_{i\in I}\sum_{j \in I}\frac{\vec{e}(Z_i,Z_j)}{(2m-b)^2},$$

there exists $i,j \in I$ such that $\vec{e}(Z_i,Z_j) \geqslant \alpha^2(2m-b)^2/2$. Removing P_i and P_j divides P into three disjoint proper anti-directed paths (note that some of these paths may be empty). Label these paths Q_1, Q_2 and Q_3 so that $P^*\hat{P} \subseteq Q_1$. By Lemma 2.4.(ii) there exists a proper anti-directed path Q_4 of length at least $(\alpha^2/8)(2m-b) \geqslant (\alpha^2/8)m$ in $D[Z_i \cup Z_j]$. By Claim 3.8, there also exists a proper anti-directed path $Q_5 \subseteq D[X_i \cup Y_i]$ such that $|Q_5| \geqslant 2(1-2\sigma)m$.

If $i = j$ then $Q_4 \subseteq D[Z_i]$ and $|Q_1|+|Q_2|+|Q_3| = |P|-2m$. Therefore it is enough to observe that $|Q_4|+|Q_5| \geqslant 2(1-2\sigma)m + (\alpha^2/8)m \geqslant 2m + \varepsilon m$.

If $i \neq j$, then $Y'_j := Y_j \setminus V(Q_4)$ has order at least $2m - b \geqslant m$. So there exists a proper anti-directed path $Q_6 \subseteq D[X_j \cup Y'_j]$ such that $|Q_6| \geqslant 2(1-2\sigma)m$. Since $|Q_1|+|Q_2|+|Q_3| = |P|-4m$ and $|Q_4|+|Q_5|+|Q_6| \geqslant 4(1-2\sigma)m + (\alpha^2/8)m \geqslant 4m + \varepsilon m$, the proof is complete. □

4 Extremal Case

Let $0 < \alpha \ll \beta \ll \gamma \ll 1$. Let D be a directed graph on $2n$ vertices with $\delta^0(D) \geqslant n$ and suppose that D satisfies the extremal condition with parameter α and that D is not isomorphic to F_n^1 or F_n^2. We will first partition $V(D)$ in the preprocessing section, then we will handle the main proof. In this section we sometime use uv to denote the edge (u,v).

4.1 Preprocessing

The point of this section is to make the following statement precise: If D satisfies the extremal condition, then D is very similar to the digraph in Figure 1a.

Proposition 4.1. *If there exists an α-extreme pair of sets $A, B \subseteq V(G)$, then there exists a partition $\{X'_1, X'_2, Y'_1, Y'_2, Z\}$ of $V(G)$ such that*

(i) $|Z| \leqslant 3\alpha^{2/3}n$, $||X'_1| - |X'_2||, ||Y'_1| - |Y'_2|| \leqslant 3\alpha^{2/3}n$ *and*

(ii) $\delta^0(X'_{3-i}, X'_i), \delta^-(Y'_{3-i}, X'_i), \delta^+(Y'_i, X'_i) \geqslant |X'_i| - 2\alpha^{1/3}n$ *and* $\delta^0(Y'_i, Y'_i), \delta^-(X'_i, Y'_i), \delta^+(X'_{3-i}, Y'_i) \geqslant |Y'_i| - 2\alpha^{1/3}n$ *for* $i = 1, 2$.

Proof. Let $A, B \subseteq V(D)$ such that $(1-\alpha)n \leqslant |A|, |B| \leqslant (1+\alpha)n$, $\Delta^+(A, B) \leqslant \alpha n$, and $\Delta^-(B, A) \leqslant \alpha n$. We have that

$$\delta^+(A, \overline{B}) \geqslant (1-\alpha)n, \text{ and} \tag{5}$$
$$\delta^-(B, \overline{A}) \geqslant (1-\alpha)n. \tag{6}$$

Set $\widetilde{X_1} = V \setminus (A \cup B)$, $\widetilde{X_2} = A \cap B$, $\widetilde{Y_1} = A \setminus B$, $\widetilde{Y_2} = B \setminus A$. Note that $\widetilde{Y_1} \cup \widetilde{X_2} = A$ and $\widetilde{Y_2} \cup \widetilde{X_2} = B$. Therefore, $||\widetilde{Y_1}| - |\widetilde{Y_2}|| \leqslant 2\alpha n$, and $||\widetilde{X_1}| - |\widetilde{X_2}|| \leqslant 2\alpha n$, because $|\widetilde{X_1}| - |\widetilde{X_2}| = |V| - |A| - |B|$.

Let

$$\begin{aligned}
\hat{Y_1} &= \{v \in \widetilde{Y_1} : \deg^-(v, \widetilde{X_2}) < |\widetilde{X_2}| - \alpha^{1/3}n \text{ or } \deg^-(v, \widetilde{Y_1}) < |\widetilde{Y_1}| - \alpha^{1/3}n\},\\
\hat{Y_2} &= \{v \in \widetilde{Y_2} : \deg^+(v, \widetilde{X_2}) < |\widetilde{X_2}| - \alpha^{1/3}n \text{ or } \deg^+(v, \widetilde{Y_2}) < |\widetilde{Y_2}| - \alpha^{1/3}n\},\\
\hat{X_1} &= \{v \in \widetilde{X_1} : \deg^-(v, \widetilde{Y_1}) < |\widetilde{Y_1}| - \alpha^{1/3}n \text{ or } \deg^+(v, \widetilde{Y_2}) < |\widetilde{Y_2}| - \alpha^{1/3}n \text{ or}\\
&\qquad \deg^0(v, \widetilde{X_2}) < |\widetilde{X_2}| - \alpha^{1/3}n\},
\end{aligned}$$

$\hat{B} = \hat{Y_1} \cup \hat{X_1}$ and $\hat{A} = \hat{Y_2} \cup \hat{X_1}$. Note that $\hat{B} \subseteq \overline{B}$ and $\hat{A} \subseteq \overline{A}$. Now we show that each of these sets are small.

Claim 4.2. $|\hat{Y_1}|, |\hat{Y_2}|, |\hat{X_1}| \leqslant 2\alpha^{2/3}n$ *and* $|\hat{Y_1}| + |\hat{Y_2}| + |\hat{X_1}| \leqslant 3\alpha^{2/3}n$

Proof. By (5) and the definition of $\hat{X_1}, \hat{Y_1}$, we have

$$|\widetilde{Y_1} \cup \widetilde{X_2}|(1-\alpha)n = |A|(1-\alpha)n \leqslant \vec{e}(A, \overline{B}) \leqslant (|\overline{B}| - |\hat{B}|)|A| + |\hat{B}|(|A| - 2\alpha^{1/3}n)$$

This implies

$$|\hat{Y_1} \cup \hat{X_1}| = |\hat{B}| \leqslant \frac{|A|(|\overline{B}| - (1-\alpha)n)}{2\alpha^{1/3}n} \leqslant \frac{(1+\alpha)n((1+\alpha)n - (1-\alpha)n)}{2\alpha^{1/3}n} = (1+\alpha)\alpha^{2/3}n$$

Now using (6), the same calculation (with the symbol A exchanged with the symbol B) gives that $|\hat{Y_2} \cup \hat{X_1}| = |\hat{A}| \leqslant (1+\alpha)\alpha^{2/3}n$.

Thus $|\hat{Y_1}| + |\hat{Y_2}| + |\hat{X_1}| \leqslant 2(1+\alpha)\alpha^{2/3}n \leqslant 3\alpha^{2/3}n$. □

Let $X'_1 = \widetilde{X_1} \setminus \hat{X_1}$, $X'_2 = \widetilde{X_2}$, $Y'_i = \widetilde{Y_i} \setminus \hat{Y_i}$ for $i = 1, 2$, and $Z = \hat{X_1} \cup \hat{Y_1} \cup \hat{Y_2}$. Note that $|Z| \leqslant 3\alpha^{2/3}n$ and $||X'_1| - |X'_2||, ||Y'_1| - |Y'_2|| \leqslant 2\alpha n + 2\alpha^{2/3}n < 3\alpha^{2/3}n$. The required degree conditions all follow from (5) and (6); the definitions of $\hat{X_1}$, $\hat{Y_1}$ and $\hat{Y_2}$; and Claim 4.2. □

4.2 Preliminary results

The following facts immediately follow from the Chernoff bound for the hypergeometric distribution [13].

Lemma 4.3. *For any $\varepsilon > 0$, there exists n_0 such that if D is a digraph on $n \geqslant n_0$ vertices, $S \subseteq V(D)$, $m \leqslant |S|$ and $c := m/|S|$ then there exists $T \subseteq S$ of order m such that for every $v \in V$*

$$||N^{\pm}(v) \cap T| - c|N^{\pm}(v) \cap S|| \leqslant \varepsilon n \quad \textit{and}$$
$$||N^{\pm}(v) \cap (S \setminus T)| - (1-c)|N^{\pm}(v) \cap S|| \leqslant \varepsilon n.$$

We will need the following theorem and corollary and an additional lemma.

Theorem 4.4 (Moon, Moser [14])**.** *If G is a balanced bipartite graph on n vertices such that for every $1 \leqslant k \leqslant n/4$ there are less than k vertices v such that $\deg(v) \leqslant k$ then G has a Hamiltonian cycle.*

Corollary 4.5. *Let G be a U, V-bipartite graph on n vertices such that n is sufficiently large and $0 \leqslant |U| - |V| \leqslant 1$ and let $C \geqslant 3$ be a positive integer. If n is even, let $a \in U$ and $b \in V$ and if n is odd, let $a, b \in U$. If $\delta(G) > 2C$ and $\deg(v) > 2n/5$ for all but at most C vertices v then G has a Hamiltonian path with ends a and b.*

Proof. If n is even then iteratively pick $v_0 \in N(b) - a$, $v_1 \in N(v_0) - b$ and $v_2 \in N(a) - b - v_1$ and set $R = \{a, b, v_0, v_1, v_2\}$. If n is odd then iteratively pick $v_1 \in N(a)$ and $v_2 \in N(b) - v_1$. and set $R = \{a, b, v_1, v_2\}$. In both cases, we can select v_1, v_2 to have degree greater than $2n/5$. Applying Theorem 4.4 to the graph formed by removing R from the graph and adding a new vertex to V which is adjacent to $N(v_1) \cap N(v_2) \setminus R$ completes the proof. □

Definition 4.6. *Let S be a star with k leaves. If every edge of S is oriented away from the center, we say S is a k-out star, if every edge is oriented towards the center, we say S is a k-in star.*

Lemma 4.7. *Let G be a directed graph on n vertices and let $1 \leqslant d \leqslant D \ll n$. If $\delta^+(G) \geqslant d$ and $\Delta^-(G) \leqslant D$, then G has at least $\frac{(d-1)n-4(d-1+D)}{3(d+D-1)}$ disjoint 2-in-stars together with two independent edges.*

Proof. We start by noting that since $\delta^+(G) \geqslant d \geqslant 1$ and $\Delta^-(G) \leqslant D$ there is a matching of size at least 2. Let M be a maximum collection of two independent edges together with $m \geqslant 0$ vertex disjoint 2-in stars and let $L = V(G) \setminus V(M)$. Note that $\sum_{v \in L} \deg^+(v, L) \leqslant |L| = n - 3m - 4$ otherwise $\sum_{v \in L} \deg^-(v, L) = \sum_{v \in L} \deg^+(v, L) > |L|$ would give a 2-in star disjoint from M. Thus

$$(d-1)(n-3m-4) \leqslant d(n-3m-4) - \sum_{v \in L} \deg^+(v, L) \leqslant \vec{e}(L, M) \leqslant (3m+4)D$$

which gives $m \geqslant \frac{(d-1)n-4(d-1+D)}{3(d+D-1)}$. □

4.3 Finding the ADHC

Looking ahead (in what will be the main case), we are going to distribute vertices from Z to the sets X_1', X_2', Y_1', Y_2' to make sets X_1, X_2, Y_1, Y_2. Then we are going to partition each of the sets $X_1 = X_1^1 \cup X_1^2$, $X_2 = X_2^1 \cup X_2^2$, $Y_1 = Y_1^1 \cup Y_1^2$, and $Y_2 = Y_2^1 \cup Y_2^2$ (so that each set is approximately split in half). Then we are going to look at the bipartite graphs induced by edges from $X_2^1 \cup Y_1^1$ to $X_1^1 \cup Y_1^2$ and from $X_1^2 \cup Y_2^2$ to $X_2^2 \cup Y_2^1$ respectively (see Figure 3). By the degree conditions for X_1', X_2', Y_1', Y_2', these bipartite graphs will be nearly complete, however we must be sure that the vertices from Z each have degree at least γn in the bipartite graph. This next claim shows that the vertices of Z can be distributed so that this condition is satisfied.

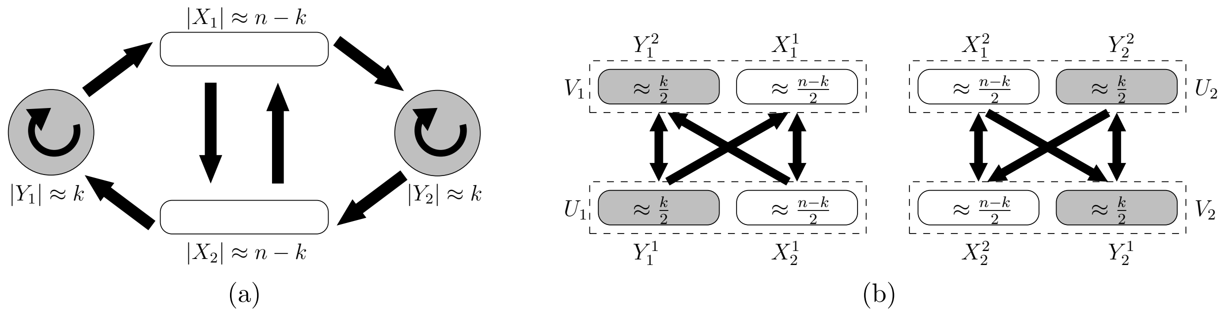

Figure 3: The objective partition, before and after.

Definition 4.8. *For $z \in Z$ and $A, B \in \{X_1', X_2', Y_1', Y_2'\}$, we say $z \in Z(A, B)$ if*

$$\deg^+(z, B) \geqslant 5\gamma n \text{ and } \deg^-(z, A) \geqslant 5\gamma n.$$

Claim 4.9. *Every vertex in Z belongs to at least one of the following sets:*

(i) $Z(X_i', X_i')$,

(ii) $Z(Y_i', Y_i')$,

(iii) $Z(X_i', X_{3-i}')$,

(iv) $Z(Y_i', Y_{3-i}')$,

(v) $Z_1 := \bigcap_{1 \leqslant i,j \leqslant 2} Z(Y_i', X_j')$ *or*

(vi) $Z_2 := \bigcap_{1 \leqslant i,j \leqslant 2} Z(X_i', Y_j')$.

Proof. Let $v \in Z$ and suppose that v is in none of the sets $(i) - (iv)$. Note that v must have at least $(n - |Z|)/4$ out-neighbors in some set $A \in \{X_1', X_2', Y_1', Y_2'\}$.

Assume $A = X_i'$ for some $i = 1, 2$. Because of the degree condition and the fact that v is in none of the sets $(i) - (iv)$, we have

$$\deg^-(v, Y_1 \cup Y_2) \geqslant n - 10\gamma n - |Z| \geqslant (1 - 11\gamma)n, \text{ which implies}$$
$$\deg^+(v, X_1 \cup X_2) \geqslant n - 10\gamma n - |Z| \geqslant (1 - 11\gamma)n.$$

This implies, $||X_1 \cup X_2| - n|, ||Y_1 \cup Y_2| - n| \leqslant 11\gamma n$. With Proposition 4.1, we have that $(1/2 - 6\gamma)n \leqslant |X_1|, |X_2|, |Y_1|, |Y_2| \leqslant (1/2 + 6\gamma)n$ so $v \in Z_1$.

If $A = Y_i'$ for some $i = 1, 2$, the previous argument (with the symbol X exchanged with the symbol Y) gives us that $v \in Z_2$. □

Since a vertex may be in multiple sets $(i) - (vi)$, we assign each vertex to the first set it is a member of in the ordering $(i) - (vi)$. Now we distribute vertices from Z.

Procedure 4.10. *(Distributing the vertices from Z) For $1 \leqslant i \leqslant 2$, set*

- $X_i := X_i' \cup Z(X_{3-i}', X_{3-i}') \cup Z(Y_i', Y_{3-i}')$ *and*
- $Y_i := Y_i' \cup Z(Y_i', Y_i') \cup Z(X_{3-i}', X_i') \cup Z_i$.

By Claim 4.9, $\{X_1, X_2, Y_1, Y_2\}$ is a partition of V. (We allow empty sets in our partitions). Note that the vertices from $Z_1 \cup Z_2$ have no obvious place to be distributed, thus our choice is arbitrary.

Call a partition of a set into two parts *nearly balanced* if the sizes of the two part differ by at most $2\beta n$. Call a partition $\bigcup_{1 \leqslant i,j \leqslant 2} \{X_i^j, Y_i^j\}$ of V a *splitting* of D if $\{X_i^1, X_i^2\}$ is a nearly balanced partition of X_i and $\{Y_i^1, Y_i^2\}$ is a nearly balanced partition of Y_i. Define $U_i := X_{3-i}^i \cup Y_i^i$ and $V_i := X_i^i \cup Y_i^{3-i}$ (see Figure 3). Note that, with Proposition 4.1, $||A| - n/2| \leqslant 3\beta n$ for any $A \in \{U_1, U_2, V_1, V_2\}$. Furthermore, if $u \in U_i \setminus Z$, by Proposition 4.1, $\deg^+(u, X_i' \cup Y_i') \geqslant |X_i' \cup Y_i'| - 4\alpha^{1/3}n$, so

$$\deg^+(u, V_i) \geqslant |V_i| - 4\alpha^{1/3}n - |Z| \geqslant |V_i| - 2\beta n. \tag{7}$$

Similarly, if $v \in V_i \setminus Z$, then

$$\deg^-(v, U_i) \geqslant |U_i| - 2\beta n. \tag{8}$$

Let G be the bipartite graph on vertex sets $U_1 \cup U_2, V_1 \cup V_2$ such that $\{u, v\} \in E(G)$ if and only if $u \in U_1 \cup U_2, v \in V_1 \cup V_2$, and $(u, v) \in E(D)$. Let $G_i := G[U_i, V_i]$ and $Q_i = \{v \in V(G_i) : \deg_G(v) < (1 - \gamma)n/2\}$. Call a splitting *good* if $\delta(G_i) \geqslant \gamma n$ and $|Q_i| \leqslant \beta n$ for $i \in 1, 2$.

The following claim says that we can obtain a good splitting of the graph obtained after Procedure 4.10 even if there are a small number of vertices which must be assigned to certain sets in the splitting.

Claim 4.11. *For all $1 \leqslant i, j \leqslant 2$, let $x_i^j, p(x_i^j), y_i^j, p(y_i^j)$ be non-negative integers and let $\mathcal{P}(X_i^j) \subseteq X_i$ and $\mathcal{P}(Y_i^j) \subseteq Y_i$ such that $|\mathcal{P}(X_i^j)| = p(x_i^j)$ and $|\mathcal{P}(Y_i^j)| = p(y_i^j)$. If*

(i) $\sum_{1 \leqslant i,j \leqslant 2} p(x_i^j) + p(y_i^j) \leqslant \beta n$

(ii) $x_i^j \geqslant p(x_i^j)$ *and* $y_i^j \geqslant p(y_i^j)$*;*

(iii) $x_i^1 + x_i^2 = |X_i|$ *and* $y_i^1 + y_i^2 = |Y_i|$*; and*

(iv) $||X_i|/2 - x_i^j|, ||Y_i|/2 - y_i^j| \leqslant \beta n$,

then there exists a good splitting $\{X_1^1, X_1^2, X_2^1, X_2^2, Y_1^1, Y_1^2, Y_2^1, Y_2^2\}$ of D such that for all $1 \leqslant i, j \leqslant 2$, $|X_i^j| = x_i^j$ and $\mathcal{P}(X_i^j) \subseteq X_i^j$, and $|Y_i^j| = y_i^j$ and $\mathcal{P}(Y_i^j) \subseteq Y_i^j$.

Proof. Set $\mathcal{P} = \cup_{1 \leqslant i,j \leqslant 2} \mathcal{P}(X_i^j) \cup \mathcal{P}(Y_i^j)$. Note that conditions (i)-(iv) allow for a splitting of D which satisfies the conclusions, so we are left to show that there exists such a splitting which is good.

When $|X_i| \geqslant 5\gamma n$, by Lemma 4.3, we can also ensure that for every $v \in V$,

$$\begin{aligned} |N^{\pm}(v) \cap X_i^j| &\geqslant |N^{\pm}(v) \cap (X_i \setminus \mathcal{P})| \frac{x_i^j - p(x_i^j)}{|X_i \setminus \mathcal{P}|} - \alpha n \\ &\geqslant \big(|N^{\pm}(v) \cap X_i| - \beta n\big) \left(1/2 - 2\beta n / |X_i|\right) - \alpha n \\ &\geqslant |N^{\pm}(v) \cap X_i|/2 - \gamma n, \end{aligned}$$

since $2\beta/5\gamma \ll \gamma$. By a similar calculation, if $|Y_i| \geqslant 5\gamma n$ we can partition Y_i so that $|N^{\pm}(v) \cap Y_i^j| \geqslant |N^{\pm}(v) \cap Y_i|/2 - \gamma n$ for every $v \in V$.

Let $v \in V(G_i)$ for some $i \in \{1, 2\}$. If $v \in Z$, by the previous calculation, Claim 4.9 and Procedure 4.10, $d_{G_i}(v) \geqslant \gamma n$. If $v \notin Z$, by (7) and (8), $d_{G_i}(v) \geqslant (1-\gamma)n/2$. Therefore, $\delta(G_i) \geqslant \gamma n$ and $|Q_i| \leqslant \beta n$. □

Proposition 4.12. *If there exists a good splitting of D and two independent edges uv and $u'v'$ such that either*

(i) $u \in U_1$, $v \in V_2$, $u' \in U_2$, $v' \in V_1$ *and* $|U_i| = |V_i|$ *for* $i = 1, 2$*; or*

(ii) *there exists* $i = 1, 2$ *such that* $u, u' \in U_i$, $v, v' \in V_{3-i}$, $|U_i| = |V_i| + 1$ *and* $|V_{3-i}| = |U_{3-i}| + 1$

then D contains an ADHC.

Proof. Apply Corollary 4.5 to get a Hamiltonian path P_i in G_i so that the ends of P_1 and P_2 are the vertices $\{u, u', v, v'\}$. These paths and the edges uv and $u'v'$ correspond to an ADHC in D. □

Note that the edges uv and $u'v'$ played a special role in the previous proposition. Now we discuss what properties these edges must have and how we can find them (this will be the bottleneck of the proof in each case and is the only place where the exact degree condition will be needed).

Definition 4.13. *Let uv be an edge in D. We call uv a* connecting edge *if for some $i = 1, 2$, $u \in X_i$ and either $v \in X_i$ or $v \in Y_i$; or $u \in Y_i$ and either $v \in Y_{3-i}$ or $v \in X_{3-i}$.*

Basically, connecting edges are edges which do not behave like edges in the graph shown in Figure 3a.

The following simple inequalities are used to help find connecting edges and follow directly from the degree condition. For any $A \subseteq V$ and $v \in A$

$$\deg^0(v, A) \geqslant n - |\overline{A}| \tag{9}$$

$$\deg^0(v, \overline{A}) \geqslant n - (|A| - 1) = n + 1 - |A|. \tag{10}$$

At this point, we split the proof into two main cases depending on the order of the sets Y_1 and Y_2.

Case 1: $\min\{|Y_1|, |Y_2|\} > \beta n$

Without loss of generality, suppose $|X_1 \cup Y_1| \geqslant |X_2 \cup Y_2|$.

If $|X_1 \cup Y_1| > n$ and $|X_1| \leqslant 2$, then let $X_1'' \subseteq \{v \in X_1 : \deg^-(v, Y_2 \cup X_1'') \geqslant 5\gamma n\}$ be as large as possible subject to $|X_1''| \leqslant |X_1 \cup Y_1| - n$. Reset $X_1 := X_1 \setminus X_1''$ and $Y_2 := Y_2 \cup X_1''$. If $|X_1 \cup Y_1| = n$ and $|X_1| = 1$, say $X_1 = \{v_1\}$, then if $\deg^-(v_1, Y_2) \geqslant 5\gamma n$ and there exists $v_2 \in X_2$ such that $\deg^-(v_2, Y_1) \geqslant 5\gamma n$, then we reset $X_i := X_i \setminus \{v_i\}$ and $Y_i := Y_i \cup \{v_{3-i}\}$ for $i = 1, 2$.

It is easy to check that the conclusions of Claim 4.11 still hold with the possibly redefined sets $\{X_1, X_2, Y_1, Y_2\}$. Furthermore, after these modifications, we still have that $|X_1 \cup Y_1| \geqslant |X_2 \cup Y_2|$ and the following two conditions are satisfied:

If $|X_1| = 1$, then there exists $i \in [2]$ such that for all $v \in X_i$, $\deg^-(v, Y_{3-i}) < 5\gamma n$. (11)

If $|X_1 \cup Y_1| > n$ and $|X_1| \leqslant 2$, then for every $v \in X_1$, $\deg^-(v, Y_2) < 5\gamma n$. (12)

Claim 4.14. *For each $i = 1, 2$, there exists a partition of X_i as $\{X_i^1, X_i^2\}$ with $||X_i^1| - |X_i^2|| \leqslant \alpha n$ and $W_i := Y_i \cup X_1^i \cup X_2^i$ such that either*

(i) *$|W_1|$, $|W_2|$ are odd and there are two independent connecting edges directed from W_j to W_{3-j} for some $j = 1, 2$ such that both edges have at least one endpoint in $Y_1 \cup Y_2$; or*

(ii) *$|W_1|, |W_2|$ are even and there are two independent connecting edges, one directed from W_1 to W_2 and the other directed from W_2 to W_1 such that both edges have at least one endpoint in $Y_1 \cup Y_2$.*

Proof. **Case 1** ($|X_1 \cup Y_1| = |X_2 \cup Y_2|$). For all $u \in Y_1$ and $u' \in Y_2$, we have $\deg^0(u, X_2 \cup Y_2), \deg^0(u', X_1 \cup Y_1) \geqslant 1$ by (10). From this we get independent edges uv and $u'v'$ with $u \in Y_1$, $u' \in Y_2$, $v \in X_2 \cup Y_2$ and $v' \in X_1 \cup Y_1$. We would be done unless n is odd and $X_1 \subseteq \{v'\}$ and $X_2 \subseteq \{v\}$, as otherwise we could obtain the partition $W_i := Y_i \cup X_1^i \cup X_2^i$ for $i = 1, 2$ with $u, v' \in W_1$ and $v, u' \in W_2$ and $|W_1|, |W_2|$ even. If there exists $u'' \in Y_1$ having an outneighbor in $X_2 \cup Y_2$ different from v, then we would be done, likewise if there exists $u'' \in Y_2$ having an out-neighbor in $X_1 \cup Y_1$ different than v'. Therefore, we are done if $X_1 = \emptyset$, because then every $v'' \in Y_2$ with $v'' \neq v$ must have an in-neighbor in Y_1. So we must have that $X_1 = \{v'\}$ and $X_2 = \{v\}$ with $\deg^-(v, Y_1) \geqslant |Y_1|$ and $\deg^-(v', Y_2) \geqslant |Y_2|$, but this contradicts (11).
Case 2 ($|X_1 \cup Y_1| > |X_2 \cup Y_2|$). By the case, we can choose distinct $u, u' \in Y_2$ such that $\deg^0(u, X_1 \cup Y_1), \deg^0(u', X_1 \cup Y_1) \geqslant 2$ by (10). Thus we can choose distinct $v \in N^+(u) \cap (X_1 \cup Y_1)$ and $v' \in N^+(u') \cap (X_1 \cup Y_1)$, with a preference for choosing vertices in Y_1. For $i = 1, 2$, let $\{X_i^1, X_i^2\}$ be a partition of X_i such that $||X_i^1| - |X_i^2|| \leqslant \alpha n$ and $W_i := Y_i \cup X_1^i \cup X_2^i$ with $u, u' \in W_1$ and $v, v' \in W_2$. If this can be done so that $|W_1|$ and $|W_2|$ are odd then we are done, so suppose not. Then it must be the case that $X_2 = \emptyset$ and $X_1 \subseteq \{v, v'\}$. If $X_1 \neq \emptyset$, then every vertex in Y_2 has an out-neighbor in X_1 which implies that $\deg^-(v, Y_2) \geqslant |Y_2|/2$ for some $v \in X_1$, contradicting (12). So suppose $X_1 = \emptyset$. Now we can finish by choosing $v'' \in Y_2$ distinct from u and letting $u'' \in (N^-(v'') \cap Y_1) \setminus \{v\}$. □

Apply Claim 4.14 to get connecting edges $uv, u'v'$ and for $1 \leqslant i, j \leqslant 2$, set $x_i^j := |X_i^j|$. By Claim 4.14 and Proposition 4.1 for $i = 1, 2$ we have $|x_1^i - x_2^i| \leqslant \alpha n + 3\alpha^{2/3} n$ (note that we are comparing the size of a subset of X_1 and a subset of X_2, which is different from Claim 4.14 where we are comparing the size of two subsets of X_i). So since $|Y_i| \geqslant \beta n$ for $i = 1, 2$, we can choose integers y_i^1, y_i^2 so that $|(y_i^1 + x_2^i) - (y_i^2 + x_1^i)| \leqslant 1$ and that x_i^j, y_i^j satisfy the conditions of Claim 4.11 for all $1 \leqslant i, j \leqslant 2$ (see Figure 3b).

We will now show how to apply Claim 4.11 depending on the connecting edges and the integers x_i^j, y_i^j computed above (note that the partition obtained from Claim 4.14 is only being used to compute the integers x_i^j, y_i^j and that we will be applying Claim 4.11 to obtain a possibly different partition). Let uv and $u'v'$ be the connecting edges from Claim 4.14. Suppose Claim 4.14.(i) holds and fix $i \in \{1, 2\}$ so that $u, u' \in W_i$ and $v, v' \in W_{3-i}$. Assign u, u', v and v' to the sets $\mathcal{P}(X_i^j), \mathcal{P}(Y_i^j)$ so that after splitting D with Claim 4.11, $u, u' \in U_i$ and $v, v' \in V_{3-i}$. Since $|W_1|$ and $|W_2|$ are odd, we can ensure that $|U_i| = |V_i| + 1$ and $|V_{3-i}| = |U_{3-i}| + 1$. We can then apply Proposition 4.12.(ii) to find an ADHC. Now suppose Claim 4.14.(ii) holds and let $u, v' \in W_1$, $v, u' \in W_2$ so that uv and $u'v'$ are the connecting edges. Assign u, u', v and v' to the sets $\mathcal{P}(X_i^j), \mathcal{P}(Y_i^j)$ so that after splitting D with Claim 4.11, $u \in U_1$,$v \in V_2$, $u' \in U_2$ and $v' \in V_1$. Since $|W_1|$ and $|W_2|$ are even, we can apply Proposition 4.12.(i) to find an ADHC.
Case 2: $\min\{|Y_1|, |Y_2|\} \leqslant \beta n$

Without loss of generality, suppose $|X_1| \geqslant |X_2|$. If $|X_1 \cup Y_1| > n$, then let

$$X_1'' \subseteq \{v \in X_1 : \deg^-(v, X_1) \geqslant 5\gamma n\} \cup \{v \in Y_1 : \deg^-(v, X_1) \geqslant 5\gamma n\}$$

be as large as possible subject to $|X_1''| \leqslant |X_1 \cup Y_1| - n$. Reset $X_1 := X_1 \setminus X_1''$ and $Y_1 := Y_1 \setminus X_1''$ and $X_2 := X_2 \cup X_1''$. If we still have $|X_1 \cup Y_1| > n$, then because of how we distributed the vertices in Claim 4.9 and Procedure 4.10 together with how we reassigned the vertices of X_1'', we have

$$\Delta^-(X_1 \cup Y_1, X_1) < 5\gamma n. \tag{13}$$

By Proposition 4.1, $|X_1'| \leqslant n + 2\alpha^{2/3}$ and $|Z| \leqslant 3\alpha^{2/3}$, thus $|X_1''| \leqslant 5\alpha^{3/2} \ll \beta n$. Therefore, the conclusions of Claim 4.11 still hold with the redefined sets $\{X_1, X_2, Y_1, Y_2\}$.

Case 2.1: $|X_1| \leqslant n$. If $Y_1 = \emptyset$ or $Y_2 = \emptyset$, say $Y_1 = \emptyset$, then we can split $Y_2 = Y_2^1 \cup Y_2^2$ so that $|X_1 \cup Y_2^1| = n = |X_2 \cup Y_2^2|$. In this case we can directly find the ADHC by only considering edges from $X_1 \cup Y_2^1$ to $X_2 \cup Y_2^2$. So suppose $Y_1 \neq \emptyset$ and $Y_2 \neq \emptyset$.

Suppose $|X_1 \cup Y_1| = |X_2 \cup Y_2| = n$. We first note that two independent connecting edges $uv, u'v'$ will allow us to either assign u, u' so that $u \in U_1$ and $u' \in U_2$ and v, v' so that $v \in V_1$ and $v' \in V_2$, or assign u, u' so that $u, u' \in U_i$ and v, v' so that $v, v' \in V_{3-i}$ (this is possible since $Y_1, Y_2 \neq \emptyset$). In the first case we can apply Claim 4.11, so that $|U_1| + |V_2| = |U_2| + |V_1| = n$, $|U_1| = |V_1|$ and $|U_2| = |V_2|$; in the second case we can apply Claim 4.11 so that $|U_i| + |V_{3-i}| = n + 1$, $|U_{3-i}| = |V_{3-i}| + 1$ and $|V_i| = |U_i| + 1$. Applying Proposition 4.12.(i) or (ii) then gives the desired ADHC.

So in this case we show that D must contain two independent connecting edges (here is the only place where we make use of the fact that D is not isomorphic to F_{2n}^1 or F_{2n}^2). Note that:

$$\delta^+(Y_i, X_{3-i} \cup Y_{3-i}) \geqslant n - (|X_i \cup Y_i| - 1) = 1 \quad \text{for } i = 1, 2 \tag{14}$$

If there is an edge in $D[X_1]$ or an edge in $D[X_2]$; or $|Y_1| \geqslant 2$ and $|Y_2| \geqslant 2$, then we easily obtain two independent connecting edges using (14). If say $|Y_1| = 1$ and $|Y_2| \geqslant 2$, then $|Y_1 \cup X_2| \leqslant n - 1$ so $\delta^-(Y_1, X_1 \cup Y_2) \geqslant 2$ and $\delta^-(X_1, Y_2) \geqslant n - |Y_1 \cup X_2| \geqslant 1$, which together give two independent edges. Finally, if $|Y_1| = 1 = |Y_2|$, let $\{y_i\} = Y_i$ for $i = 1, 2$. If there exists $x_1 \in X_1$ and $x_2 \in X_2$ such that $x_i x_{3-i}$ is not an edge for some $i \in [2]$, then because of the semi-degree condition and the fact that X_1 and X_2 are independent sets, it must be that $x_i y_i$ and $x_{3-i} y_{3-i}$ are edges, giving us two independent connecting edges. If there exists $x_i \in X_i$ such that $y_i x_i$ is not an edge, then, by the semi-degree condition and the fact that X_i is an independent set, $y_{3-i} x_i$ is an edge. Also by the semi-degree condition, y_i must have an out-neighbor in X_{3-i} and, with the edge $y_{3-i} x_i$, this gives us two independent connecting edges. If there exists $x_i \in X_i$ such that $x_i y_{3-i}$ is not an edge, then an analogous argument gives two independent connecting edges. So we have proved that D contains a subgraph isomorphic to $F_{2n,1}$. Since $|X_1 \cup Y_2| = |X_2 \cup Y_1| = n$, the semi-degree condition implies that every vertex in $y \in Y_1 \cup Y_2$ is incident to at least two connecting edges: one oriented away from y and the other oriented towards y. If $\{y_1, y_2\}$ is an independent set, then we clearly have two connecting edges, so assume that $y_i y_{3-i}$ is an edge. If $y_{3-i}y_i$ is an edge, then since D is not isomorphic to F^1_{2n}, there must exist at least one more edge in D. Since F^1_{2n} is an edge-maximal graph without an ADHC, D must contain an ADHC. So we can assume $y_{3-i}y_i$ is not an edge, and thus y_i must have an in-neighbor x_i in X_i and y_{3-i} must have an out-neighbor x'_i in X_i. If $x_i \neq x'_i$, then we have two independent connecting edges. If $x_i = x'_i$, then D contains a subgraph isomorphic to F^2_{2n}, and as before since D is not isomorphic to F^2_{2n}, D must contain an ADHC.

Now suppose $|X_i \cup Y_i| > |X_{3-i} \cup Y_{3-i}|$ for some $i = 1, 2$. By (9), $\deg^0(u, X_i \cup Y_i) \geqslant 1$ for all $u \in X_i \cup Y_i$ and $\deg^+(u, X_i \cup Y_i) \geqslant n - (|X_{3-i} \cup Y_{3-i}| - 1) \geqslant 2$ for all $u \in Y_{3-i}$. Let $u \in Y_{3-i}$, let $v_1, v_2 \in N^+(u) \cap (X_i \cup Y_i)$, and let $u' \in X_i \setminus \{v_1, v_2\}$. Choose distinct $v \in N^+(u) \cap (X_i \cup Y_i)$ and $v' \in N^+(u') \cap (X_i \cup Y_i)$ with a preference for choosing v and v' in X_i (this can be done since we chose u' distinct from v_1, v_2). If $|X_i| \leqslant n - 2$, then $|X_i \cup \{u, u', v, v'\}| \leqslant n + 1$, and when $i = 2$, $|X_2 \cup Y_2| > |X_1 \cup Y_1|$, and $|Y_1| \geqslant 1$ imply that $n - 2 \geqslant |X_1| \geqslant |X_i|$. So suppose $i = 1$ and $n - 1 \leqslant |X_1| \leqslant n$. Note that in this case, for every $u \in X_1$, $\deg^+(u, X_1 \cup Y_1) \geqslant \max\{1, |Y_1| - 1\} \geqslant |Y_1|/2$, so the bound in implies that there are two disjoint edges in $G[X_1]$.

So we can assume, in all cases, that $|X_i \cup \{u, u', v, v'\}| \leqslant n + 1$. Therefore, after assigning u, u' to X^{3-i}_i and v, v' to X^i_i or Y^{3-i}_i as appropriate, we can apply Claim 4.11 to get $|U_{3-i}|+|V_i| = n+1$, $|U_{3-i}| = |V_{3-i}|+1$ and $|V_i| = |U_i|+1$. Applying Proposition 4.12.(ii) then completes this case.

Case 2.2: $|X_1| \geqslant n + 1$.

Set $d = |X_1| - n$ and recall that $d \ll \beta n$. By (9), $\delta^+(D[X_1]) \geqslant d$. By the case, $X''_1 \cap X_1 = \emptyset$, so $\Delta^-(D[X_1]) < 5\gamma n$ and $\frac{(d-1)n-4(d-1+5\gamma n)}{3(d-1+5\gamma n)} \geqslant d - 1$. Applying Lemma 4.7 gives two independent edges uv, $u'v'$ and a collection of $d - 1$ vertex disjoint 2-in stars $\{S_1, \ldots, S_{d-1}\}$ in $D[X_1]$. Assign u, u' and the vertices in $S_1, \ldots, S_{d-1}$ to X^2_1. Also, assign v and v' to X^1_1. Recall that $X^1_1 \cup X^2_1 \subseteq U_2 \cup V_1$, so we can use Claim 4.11, to get a good splitting of D such that $|U_2| = \lceil n/2 \rceil + d$, $|V_1| = \lfloor n/2 \rfloor$, $|V_2| = \lceil n/2 \rceil - d + 1$ and $|U_1| = \lfloor n/2 \rfloor - 1$. We then use Corollary 4.5, to find a Hamiltonian path P_1 in G_1 with ends v and v'.

We now move the roots of the stars $S_1, \ldots, S_{d-1}$ from U_2 to V_2 and then use Corollary 4.5 to complete the proof. More explicitly, we greedily find a matching M between the leaves of the stars $S_1, \ldots, S_{d-1}$ and the vertices in V_2 of degree at least $(1 - \gamma)n/2$ in G_2. For each $1 \leqslant i \leqslant d - 1$, let a_i and b_i be the vertices matched to the leaves of S_i and replace $V(S_i) \cup \{a_i, b_i\}$ in G_2 with a new vertex adjacent to $N_{G_2}(a_i) \cap N_{G_2}(b_i)$ minus the vertices of the stars. Apply Corollary 4.5 to get a Hamiltonian path P_2 in the resulting graph with ends u and u'. The stars $S_1, \ldots, S_{d-1}$; the edges in M; the paths P_1 and P_2; and the edges uv and $u'v'$ correspond to an ADHC in D.

5 Conclusion

We end with the following conjecture which along with Theorem 1.5 would provide a full generalization of Dirac's theorem to directed graphs with respect to minimum semi-degree.

Conjecture 5.1. *Let D be a directed graph on n vertices and let $\vec{C}$ be an orientation of a cycle on n vertices which is not anti-directed. If $\delta^0(D) \geqslant \frac{n}{2}$, then $\vec{C} \subseteq D$.*

We believe that the methods developed in this paper along with the ideas in [10] and [11] provide an approach to this problem. We intend to carry out this program in a subsequent paper.

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