Lifespan in a strongly primitive Boolean linear dynamical system*

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Abstract

Let $\mathcal{F}$ be a set of k by k nonnegative matrices such that every "long" product of elements of $\mathcal{F}$ is positive. Cohen and Sellers (1982) proved that, then, every such product of length 2^k-2 over $\mathcal{F}$ must be positive. They suggested to investigate the minimum size of such $\mathcal{F}$ for which there exists a non-positive product of length 2^k-3 over $\mathcal{F}$ and they constructed one example of size 2^k-2. We construct one of size k and further discuss relevant basic problems in the framework of Boolean linear dynamical systems. We also formulate several primitivity properties for general discrete dynamical systems.

Keywords: discrete dynamical system, hitting time, lexicographic order, non-homogeneous matrix product, phase space, strong primitivity, primitivity, weak primitivity, Wielandt matrix.

1 Strongly primitive matrix sets

Let A be a set. A **word** w over A is a sequence of elements of A, say $w = a_1 \cdots a_\ell$ where $a_1, \ldots a_\ell \in A$. For any $a \in A$, we write $|w|_a$ for the number of occurrences of a in the sequence w. We call ℓ the length of the word w, which surely equals $\sum_{a\in A} |w|_a$ and will be denoted by $|w|$. We call w a nonempty word provided $|w| > 0$. The free semigroup generated by A, in symbols A^+, has all the nonempty words over A as its elements and has string concatenation as the semigroup multiplication.

For any nonnegative integer k, we write $[k]$ for the set $\{1, \ldots, k\}$. Note that $[0] = \emptyset$. Pick a positive integer k. We use the notation Mat_k to denote the multiplicative semigroup

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of all $k \times k$ nonnegative matrices. Let $\mathcal{M}$ be a set of $k \times k$ nonnegative matrices. For any word $w = w_1 \cdots w_\ell \in \mathcal{M}^\ell \subseteq \mathcal{M}^+$, we set $\mathcal{M}_w$ to be the product $w_1 \cdots w_\ell$ in Mat_k and call it a **product over $\mathcal{M}$ of length ℓ.** The map from $\mathcal{M}^+$ to Mat_k that sends w to $\mathcal{M}_w$ is a semigroup morphism and the image of it is called the **semigroup of $\mathcal{M}$**. Take a map $\mathtt{t}$ from $\mathcal{M}$ to nonnegative integers. Define the length of $\mathtt{t}$ be be $|\mathtt{t}| = \sum_{M\in\mathcal{M}} \mathtt{t}(M)$. When $|\mathtt{t}| > 0$, the **Hurwitz product of $\mathcal{M}$ of type $\mathtt{t}$** [14] is the matrix

$$\mathcal{M}^{\mathtt{t}} := \sum_{w} \mathcal{M}_w,$$

where the sum runs over all those sequences $w \in \mathcal{M}^+$ such that $|w|_M = \mathtt{t}(M)$ holds for every $M \in \mathcal{M}$. The **strong primitivity exponent of $\mathcal{M}$**, denoted by $\mathrm{g}(\mathcal{M})$, is the minimum positive integer ℓ such that all products over $\mathcal{M}$ of length at least ℓ is positive; the **primitivity exponent of $\mathcal{M}$**, denoted by $\mathrm{g'}(\mathcal{M})$, is the minimum positive integer ℓ such that there exists a positive product over $\mathcal{M}$ of length ℓ; and the **weak primitivity exponent of $\mathcal{M}$**, denoted by $\mathrm{g''}(\mathcal{M})$, is the minimum positive integer ℓ such that we can find a positive Hurwitz product of length ℓ over $\mathcal{M}$. If the relevant positive integer does not exist, our convention is to set the exponent to be ∞. It is clear that

$$\mathrm{g}(\mathcal{M}) \geqslant \mathrm{g'}(\mathcal{M}) \geqslant \mathrm{g''}(\mathcal{M}).$$

We say that $\mathcal{M}$ is **strongly primitive**, **primitive**, or **weakly primitive**, respectively, if $\mathrm{g}(\mathcal{M}) < \infty$, $\mathrm{g'}(\mathcal{M}) < \infty$, or $\mathrm{g''}(\mathcal{M}) < \infty$.

The weak primitivity property was introduced by Fornasini and Valcher [15] and its rich properties and applications were further developed in [3, 4, 5, 16, 30, 33]. Olesky, Shader and Van den Driessche [30, Theorem 7] showed for every positive integer d that

$$\max\{\mathrm{g''}(\mathcal{M}) :\ \mathcal{M} \in \binom{\mathrm{Mat}_k}{d} \text{ is weakly primitive}\} = \Theta(k^{d+1}).$$

A matrix set $\mathcal{M} \in \binom{\mathrm{Mat}_k}{d}$ is **irreducible** provided

$$\cup_{M\in\mathcal{M}}\{(i,j) \in ([k] \setminus A) \times A :\ M(i,j) \neq 0\} \neq \emptyset$$

for every $A \in 2^{[k]} \setminus \{\emptyset, [k]\}$. If $\mathcal{M} \in \binom{\mathrm{Mat}_k}{d}$ is irreducible and no matrix of $\mathcal{M}$ contains a zero row, Protasov [33, Theorem 2] found a polynomial time algorithm to decide if $\mathcal{M}$ is weakly primitive.

The concept of primitivity property for matrix sets was pioneered by Protasov and Voynov [34] and has attracted quite some attention [1, 6, 34, 39]. Blondel, Jungers and Olshevsky [6, Theorem 6] proved that recognizing primitivity is decidable but NP-hard as soon as there are three matrices in the set. They [6, Theorem 10] also showed that the primitivity exponent of a primitive $k \times k$ matrix set $\mathcal{M}$ could be superpolynomial in the size of the instance $k^2|\mathcal{M}|$. A nonnegative matrix is **essential** if it has no zero lines. Protasov and Voynov [34] designed a polynomial time algorithm to test if a given irreducible set of essential matrices is primitive. Voynov discovered that the primitivity exponent of a primitive $k \times k$ essential matrix set is $O(k^3)$ [6, 39].

As pointed out by Cohen and Sellers [12, p. 187], a strongly primitive matrix set is the combinatorial part of Hajnal's [20] concept of ergodic set, and is relevant to inhomogeneous Markov chains, automata, mathematical demography and linear switching systems [17, 19, 21, 22, 30, 31, 43]. It also worths mentioning that strong primitivity property plays a crucial role in approximation theory and functional analysis such as solvability of refinement functional equations with column stochastic matrices [10, 29] and the convergence of subdivision schemes with nonnegative masks [27, 41, 47].

In 1982, Cohen and Sellers [12, Theorem 1, Theorem 2] established the following result.

Theorem 1 (Cohen-Sellers)**.** *Let $\mathcal{M}$ be a strongly primitive set of $k \times k$ nonnegative matrices. Then*

$$\mathrm{g}(\mathcal{M}) \leqslant 2^k - 2. \tag{1}$$

Moreover, one can choose a set $\mathcal{M}$ of size $2^k - 2$ to guarantee the equality in Eq. (1)*.*

Theorem 1 tells us that the strong primitivity exponent $\mathrm{g}(\mathcal{M})$ can grow exponentially in dimension k. To understand the strong primitivity property, an important question arising in this context is whether the strong primitivity exponent $\mathrm{g}(\mathcal{M})$ can grow subexponentially with respect to the size of the instance $k^2|\mathcal{M}|$. For any $k \geqslant 2$, the first main result of this paper, Theorem 2, asserts that there exists a size-k strongly primitive set $\mathcal{M}$ of $k \times k$ (essential) matrices satisfying $\mathrm{g}(\mathcal{M}) = 2^k - 2$, thereby giving a positive answer to the previous question. This may be a sign that the strong primitivity property can hardly be decided in polynomial time. Cohen and Sellers concluded the paper [12] with the question on determining the smallest size of $\mathcal{M}$ for which equality is valid in Eq. (1). Thus, our Theorem 2 is also an effort towards answering their question.

For each nonnegative matrix M, we can replace all its positive entries by 1 and thus form a $(0, 1)$ matrix $\overline{M}$. For M^t to be a positive matrix, it is equivalent to have $\overline{M}^t$ to be the all ones matrix according to the Boolean algebra operation. With this in mind, one can check that all the properties addressed above on nonnegative matrices are indeed properties on Boolean matrices. In mathematics, especially in various kinds of representation theory, to understand the structure of an object, we often let it act on other objects and then turn to the study of this action. In this paper and its subsequent work [40, 44, 45], we suggest a genetic approach to understand the structure of a Boolean matrix set or even Boolean tensor set, in which we recast everything in the language of phase spaces. This is to let the Boolean matrices act on Boolean row vectors and then study the corresponding Boolean linear dynamical systems. We believe that this approach leads us to a series of fundamental mathematics structures and problems. In particular, this approach suggests us to consider many natural problems which, at first sight, may look to be far away from the above primitivity problems for nonnegative matrix sets.

In §2, we address the above-mentioned question of Cohen and Sellers in the framework of Boolean linear dynamical systems. We hope that the phase space approach adopted in §2 will make some readers interested enough to keep reading into §3 where we set forth our formal description of primitive discrete dynamical systems, including a definition of phase space. In §4 we discuss several aspects of strongly primitive essential Boolean linear dynamical systems and try to show that it is a good playground for mathematicians. The last section, §5, is devoted to the proofs of those results claimed in §3 and §4.

2 The question of Cohen and Sellers

A **digraph** D consists of its vertex set $\mathrm{V}(D)$ and its arc set $\mathrm{A}(D)$ where $\mathrm{A}(D) \subseteq \mathrm{V}(D) \times \mathrm{V}(D)$. A **walk** in a digraph D of length k is a sequence of vertices $v_1, \ldots, v_{k+1}$ such that $(v_i, v_{i+1}) \in \mathrm{A}(D)$ for all $i \in [k]$. We often write this walk as $v_1 \to v_2 \to \cdots \to v_k \to v_{k+1}$ and call this walk a **closed walk** when $v_1 = v_{k+1}$. A **path** in a digraph is a walk without repeated vertices. Every arc is a walk of length 1 but not necessarily a path. A digraph D is **transitive** provided $u \to w$ is an arc in D whenever $u \to v \to w$ is a walk in D. A digraph D is **acyclic** if it has no closed walk of positive length. Recall that a **partially ordered set** (poset) is just an acyclic and transitive digraph, while its arc set is typically called a **partial order** on the vertex set. A **totally ordered set**, also known as a **linearly ordered set**, is a partially ordered set where exactly one of $u \to w$ and $w \to u$ is an arc for every two distinct vertices u and w. The arc set of a totally ordered set is called a **total order** or a **linear o rder**.

Fix a positive integer k. Let Set_k denote $2^{[k]} \setminus \{\emptyset\}$. We could naturally identify Set_k with the set of nonzero Boolean row vectors (binary string) of length k, viewing each $A \in \mathrm{Set}_k$ as the $(0,1)$ string of length k whose ith position is occupied by a 1 if and only if $i \in A$. Let $\mathcal{M}$ be a set of $k \times k$ Boolean matrices. We write $\mathbb{P}_{\mathcal{M}}$ for the digraph with vertex set Set_k of which $A \to B$ is an arc if and only if $\{A, B\} \in \binom{\mathrm{Set}_k}{2}$ and there exists a matrix M from the semigroup of $\mathcal{M}$ such that $AM = B$. Let $\mathrm{g}(\mathcal{M})$ be the length of a longest path in $\mathbb{P}_{\mathcal{M}}$. We call $\mathcal{M}$ **strongly primitive** provided $\mathbb{P}_{\mathcal{M}}$ is acyclic and $[k]$ is the unique sink vertex in $\mathbb{P}_{\mathcal{M}}$. If $\mathcal{M}$ is strongly primitive, $\mathrm{g}(\mathcal{M})$ is known as the **strong primitivity exponent** of $\mathcal{M}$.

Take $k \geqslant 2$ and $i \in [k]$. We define $M_{k,i}$ to be the $k \times k$ Boolean matrix whose (p,q)-entry is 1 if and only if $p = i$ or $q = i$ or $p = q \in [i-1]$. We display some examples below to illustrate the definition.

$$M_{6,1} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad M_{6,4} = \begin{pmatrix} \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{pmatrix} \quad M_{6,6} = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{pmatrix}$$

Let ϕ_k be the bijection from Set_k to $[2^k - 1]$ that sends $A \in \mathrm{Set}_k$ to the integer

$$\phi_k(A) := \sum_{i \in A} 2^{k-i}. \tag{2}$$

The **lexicographic order** on Set_k is the total order π_k in which the following holds:

$$\phi_k^{-1}(1) \to \phi_k^{-1}(2) \to \cdots \to \phi_k^{-1}(2^k - 1) = [k]. \tag{3}$$

The next result not only improves the bound of the matrix set size in Theorem 1 from $2^k - 2$ to k, but also indicates the special role of the lexicographic order π_k.

Theorem 2. *Take $k \geqslant 2$ and let $\mathcal{M}_k = \{M_{k,1}, \ldots, M_{k,k}\}$ be a size-k set of $k \times k$ Boolean matrices. Then $\mathrm{g}(\mathcal{M}_k) = 2^k - 2$ and $\mathbb{P}_{\mathcal{M}_k}$ is the lexicographic order on Set_k as given in Eq.* (3).

Proof. Pick $i \in [k]$ and $A \in \mathrm{Set}_k$. If $i \in [k] \setminus A$, we can check that $AM_{k,i} = \{i\} \cup ([i-1] \cap A)$ and thus $\phi_k(A) < \phi_k(AM_{k,i})$; if $i \in A$, we can check that $AM_{k,i} = [k]$ and thus $\phi_k(A) \leqslant \phi_k(AM_{k,i})$ where equality holds if and only if $A = [k]$. Especially, if $s \in [2^k - 2]$ and $i = \max\{j \in [k] : \ j \notin \phi_k^{-1}(s)\}$, then $\phi_k^{-1}(s)M_{k,i} = \phi_k^{-1}(s+1)$. We illustrate this with two examples below.

$$\begin{pmatrix} * & * & * & 0 & \mathbf{1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 0 \end{pmatrix} = \begin{pmatrix} * & * & * & \mathbf{1} & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} * & * & * & * & * & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} * & * & * & * & * & \mathbf{1} \end{pmatrix}$$

The above discussion implies that

$$\phi_k^{-1}(1) \to \phi_k^{-1}(2) \to \cdots \to \phi_k^{-1}(2^k - 1) = [k]$$

is a path of length $2^k - 2$ in $\mathbb{P}_{\mathcal{M}_k}$ and that the arcs in $\mathbb{P}_{\mathcal{M}_k}$ are all of the form $\phi_k^{-1}(s) \to \phi_k^{-1}(t)$ where $1 \leqslant s < t \leqslant 2^k - 1$. This proves that $\mathrm{g}(\mathcal{M}_k) = 2^k - 2$ and that $\mathbb{P}_{\mathcal{M}_k}$ is the claimed total order, as was to be shown. $\square$

3 Discrete dynamical systems

Let S be a nonempty set and let $\mathcal{F}$ be a family of maps from S to S. Viewing the maps in $\mathcal{F}$ as a set of time-evolution laws and S the set of possible states, the pair $\mathcal{D} = (S, \mathcal{F})$ forms a **discrete dynamical system** (**DDS**) on the ground set S, where the dynamics are given by iterating the maps in $\mathcal{F}$ and hence time changes in discrete steps. The fundamental role of the mathematical operation of function compositions makes DDS a very basic mathematical object which models many practical systems [13, 26].

We have used prefix notation for function values, that is, we put the function name followed by the input name inside parentheses. When talking about the functions in a

DDS, we will use postfix notation instead. That is, when considering the DDS $(S, \mathcal{F})$, for any $f \in \mathcal{F}$ and $s \in S$, we adopt the notation s^f for the image of s under f. When s is a row vector and f is a matrix that acts on x by multiplication on the right, we often directly write s^f as sf.

The **phase space** of the DDS $(S, \mathcal{F})$, which encodes all the phase transitions of the system, is the digraph with vertex set S and arc set $\{s \to s^f : s \in S, f \in \mathcal{F}\}$, and will be denoted by $\mathcal{PS}_{S,\mathcal{F}}$ or simply $\mathcal{PS}_{\mathcal{F}}$. We call $b \in S$ a **black hole** of $(S, \mathcal{F})$ provided b is fixed by all $f \in \mathcal{F}$ and that for every $s \in S$, every long enough walk starting from s in $\mathcal{PS}_{\mathcal{F}}$ will end at b. A DDS $\mathcal{D}$ is **strongly primitive** if it has a black hole. Clearly, every strongly primitive DDS $\mathcal{D}$ has exactly one black hole, which we often denote by $\mathbb{1} = \mathbb{1}_{\mathcal{D}}$, and one may imagine that the so-called 'time's arrow' just points towards this black hole $\mathbb{1}$.

Example 3. Let S be the set of positive integers. Let $f \in S^S$ be given by

$$s^f := \begin{cases} 1, & \text{if } s = 1; \\ 3s+1, & \text{if } s \text{ is odd and } s > 1; \\ \frac{s}{2}, & \text{if } s \text{ is even.} \end{cases}$$

The renowned Collatz conjecture [24] just asserts that (S, f) is strongly primitive with 1 being its black hole!

For a strongly primitive DDS $\mathcal{D} = (S, \mathcal{F})$ and every $s \in S$, the **longest duration** for s is the length of a longest path from s to the black hole of $\mathcal{D}$ and this length is denoted by $\mathrm{ld}(\mathcal{D}, s)$ or simply $\mathrm{ld}(s)$. Similarly, we define the **shortest duration** for s to be the length of a shortest path from s to the black hole of $\mathcal{D}$ and write it as $\mathrm{sd}(\mathcal{D}, s)$ or simply $\mathrm{sd}(s)$. The **transient** (**strong primitivity exponent**) of a strongly primitive system $(S, \mathcal{F})$ is the minimum nonnegative integer T such that every length T walk in $\mathcal{PS}_{\mathcal{F}}$ ends at the black hole of $(S, \mathcal{F})$, which we denote by $\mathrm{g}(S, \mathcal{F})$ or simply $\mathrm{g}(\mathcal{F})$. The **hitting time** of a strongly primitive DDS $(S, \mathcal{F})$ is the minimum nonnegative integer T such that every $s \in S$ can reach the black hole in T steps in $\mathcal{PS}_{\mathcal{F}}$, which we denote by $\underline{\mathrm{g}}(\mathcal{D})$, or $\underline{\mathrm{g}}(S, \mathcal{F})$, or simply $\underline{\mathrm{g}}(\mathcal{F})$. In other words,

$$\mathrm{g}(\mathcal{F}) = \overline{\lim}_{s \in S} \mathrm{ld}(s), \quad \underline{\mathrm{g}}(\mathcal{F}) = \overline{\lim}_{s \in S} \mathrm{sd}(s).$$

If you view each vertex as a birth place and each directed path as the journey of life of someone with the black hole death, then $\mathrm{g}(\mathcal{F})$ is the longest life of any person while $\underline{\mathrm{g}}(\mathcal{F})$ is the minimum time elapsed in evolution until at least one person from each birth place will have died. Clearly,

$$\underline{\mathrm{g}}(\mathcal{F}) \leqslant \mathrm{g}(\mathcal{F}) \leqslant |S| - 1. \tag{4}$$

We will thus call $|S| - 1$ the **absolute upper bound** of the life expectancy in $(S, \mathcal{F})$.

Example 4. Let S be the set of all nonnegative integers and let f be the map which sends i to $\max(0, i-1)$. Then (S, f) is a strongly primitive DDS with infinite transient.

A **pointed discrete dynamical system** $\mathcal{D}' = (\mathcal{D}, s)$ is a DDS $\mathcal{D} = (S, \mathcal{F})$ together with a **base point** $s \in S$. Note that under function compositions the set S^S forms a semigroup. For any word $w = w_1 \cdots w_\ell \in \mathcal{F}^+$, $\mathcal{F}_w$ is the element of S^S obtained by composing functions $w_1, \ldots, w_\ell$ from left to right in this order. The map from $\mathcal{F}^+$ to S^S that sends $w \in \mathcal{F}^+$ to $\mathcal{F}_w \in S^S$ is a semigroup morphism and the image of it is called the semigroup of $\mathcal{F}$. We say that $w \in \mathcal{F}^+$ is a **synchronizing word** for $\mathcal{D}' = (\mathcal{D}, s)$ if $\mathcal{F}_w$ maps all elements of S to s. The pointed DDS $\mathcal{D}'$ is **primitive** provided it has a synchronizing word and the shortest length of such synchronizing words is the **primitivity exponent** of $\mathcal{D}'$, denoted by g'($\mathcal{D}'$). By taking $S = \mathrm{Set}_k$, $s = [k]$ and $\mathcal{F} \subseteq \mathrm{Mat}_k$, since matrix multiplication is nothing but the composition of linear maps, it is easy to see that our definition of primitivity property here is a generalization of the primitivity property of Boolean matrix set. This viewpoint on primitivity property may also support the observation [2] that primitivity property and synchronizing property have close connections.

Example 5. Let S be a nonempty set with $s \in S$, let $k = |S|$ and let $\mathcal{F}$ be a nonempty subset of S^S. Assume that the pointed DDS $\mathcal{D}' = ((S, \mathcal{F}), s)$ is primitive. The famous Černý conjecture [11, 38] says that there exists $s' \in S$ such that the primitivity exponent of $((S, \mathcal{F}), s')$ is at most $(k-1)^2$. If Černý conjecture can be proved true, then we can deduce that the primitive exponent of $\mathcal{D}'$ is at most $(k-1)k$.

Take a positive integer k. A map f from $2^{[k]}$ to $2^{[k]}$ is a **Boolean linear map on $\mathbf{2^{[k]}}$** [18, 44] provided

- $A^f \cup B^f = (A \cup B)^f$ for all $A, B \in [k]$, and
- $\emptyset^f = \emptyset$.

A **Boolean linear dynamical system** is a DDS of the form $(2^{[k]}, \mathcal{F})$ where $\mathcal{F}$ is a set of Boolean linear maps on $2^{[k]}$. De Morgan defined the important concept of "relation" in mathematics. Closely related to this basic object is the hemimorphisms between Boolean algebras [23, p. 45]. The Boolean linear dynamical system is just a place for the compositions of hemimorphisms and is a special case of the general tropical linear dynamical system [9, 28].

For any positive integer k and a set of Boolean linear maps $\mathcal{F}$ on $2^{[k]}$, let us consider the pointed DDS $\mathcal{D} = ((2^{[k]}, \mathcal{F}), \emptyset)$, which arises in considering the zero controllability of positive switched systems [35]. It is NP-complete to decide if $\mathcal{D}$ is primitive and it is NP-hard to calculate g'($\mathcal{D}$) [7]. The set of synchronizing words for $\mathcal{D}$ naturally corresponds to paths from $[k]$ to $\emptyset$ in the digraph $\mathcal{PS}_{2^{[k]}, \mathcal{F}}$. This observation leads to the fact that g'($\mathcal{D}$) $\leqslant 2^k - 1$ whenever $\mathcal{D}$ is primitive. Like the problem of Cohen and Sellers, we define γ'_k to be the smallest size of a set $\mathcal{F}$ of Boolean linear maps on $2^{[k]}$ which ensures g'($\mathcal{D}$) $= 2^k - 1$ and want to determine γ'_k. The next result says that g'($\mathcal{D}$) can grow sub-exponentially with respect to the size of the instance $k^2|\mathcal{F}|$.

Theorem 6. *It holds $\gamma'_k \leqslant k$ for every $k \geqslant 2$.*

There may exist many reasonable ways of extending the weak primitivity property of nonnegative matrix sets. We illustrate one of them here. Suppose the set S is endowed with a poset structure $(S, \prec)$ and we say that x is bigger than y whenever $y \prec x$ in $(S, \prec)$. Take a map $\mathtt{t}$ from $\mathcal{F}$ to nonnegative integers and assume $|\mathtt{t}| > 0$. We say that a pointed DDS $\mathcal{D}' = ((S, \mathcal{F}), s)$ is **weakly primitive of type** $\mathtt{t}$ (with respect to $\prec$) if, for every $x \in S$, the set

$$\{x^{\mathcal{F}_w} : \ w \in \mathcal{F}^+, |w|_u = \mathtt{t}(u), \forall u \in S\}$$

has s as the unique least upper bound in the poset $(S, \prec)$. The **weak primitivity exponent** of a weakly primitive pointed DDS $\mathcal{D}'$ with respect to the partial order $\prec$ is the minimum positive integer ℓ such that $\mathcal{D}'$ is weakly primitive of type $\mathtt{t}$ for some function $\mathtt{t}$ with $|\mathtt{t}| = \ell$.

The next section will be mainly concerned with the strong primitivity property. Let $\mathcal{D} = (S, \mathcal{F})$ be a strongly primitive DDS. It turns out that the construction of $\mathbb{P}_{\mathcal{M}}$ in §2 gives rise to an important poset structure now for studying strong primitivity property. That is, we can construct a partial order $\prec_{\mathcal{D}}$ on S by setting $(x, y) \in \prec_{\mathcal{D}}$, also denoted by $x \prec_{\mathcal{D}} y$, if and only if $x \neq y$ and there is a path leading from x to y in $\mathcal{PS}_{\mathcal{D}}$. We write $\mathbb{P}_{\mathcal{D}}$ to stand for this poset $(S, \prec_{\mathcal{D}})$. It is not hard to see that every finite poset with a unique maximal element will arise in this way. Recall that a linear extension of a finite poset $(S, \prec)$ is a bijective map π from $[|S|]$ to S such that $i < j$ holds whenever $\pi(i) \prec \pi(j)$. If it happens that $\mathrm{g}(\mathcal{D}) = |S| - 1 < \infty$, we will have a longest path in $\mathcal{PS}_{\mathcal{D}}$ of length $|S| - 1$, say

$$x_1 \to x_2 \to \cdots \to x_{|S|-1} \to x_{|S|} = \mathbb{1},$$

which means that the poset $\mathbb{P}_{\mathcal{D}}$ is the total order

$$x_1 \prec_{\mathcal{D}} x_2 \prec_{\mathcal{D}} \cdots \prec_{\mathcal{D}} x_{|S|-1} \prec_{\mathcal{D}} x_{|S|} = \mathbb{1}.$$

Actually, when S is a finite set, $\mathrm{g}(\mathcal{D}) = |S| - 1$ if and only if $\mathbb{P}_{\mathcal{D}}$ is a total order if and only if $\mathbb{P}_{\mathcal{D}}$ allows a unique linear extension.

4 Strongly primitive Boolean linear dynamical systems

Take a positive integer k. A map f from Set_k to Set_k is **essential** provided

- $A^f \cup B^f = (A \cup B)^f$ for all $A, B \in \mathrm{Set}_k$, and
- $[k]^f = [k]$.

An essential map from Set_k to Set_k is the combinatorial counterpart of a k by k matrix without zero lines. Indeed, we could think of such a map f as the k by k Boolean matrix M_f whose (i, j)-entry is 1 if and only if $j \in i^f$. We thus do not distinguish between an essential map f and the corresponding matrix M_f and so the image A^f of $A \in \mathrm{Set}_k$ under the map f could be conveniently written as $A\,\mathrm{M}_f$ or even just Af. For ease of illustration, we use $A \xrightarrow{f} B$ to mean that $B = Af$ and we do not distinguish an element $i \in [k]$ and the corresponding singleton set $\{i\}$.

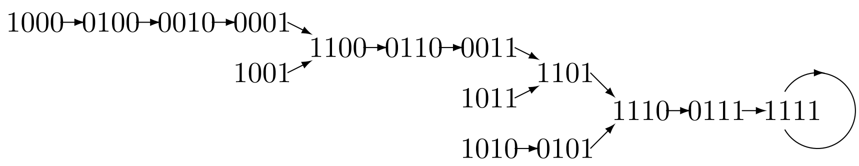

Figure 1: $\mathcal{PS}_{W_{4;1}}$.

Let $\mathcal{F}$ be a family of essential maps on Set_k. We call $(\mathrm{Set}_k, \mathcal{F})$ an **essential Boolean linear dynamical system**. We refer to $\mathcal{F}$ as a **strongly primitive set of essential maps** on Set_k of size t if $|\mathcal{F}| = t$ and $(\mathrm{Set}_k, \mathcal{F})$ is strongly primitive. If $(\mathrm{Set}_k, \mathcal{F})$ is strongly primitive, it is clear that its black hole must be $[k]$. By identifying essential maps on Set_k with $k \times k$ Boolean matrices without zero lines, the strong primitivity concept defined here coincides with the one given in §2. If $\mathcal{F}$ is a singleton set, being strongly primitive and being primitive are the same. Thus, we call a $k \times k$ Boolean matrix M **primitive** if $(\mathrm{Set}_k, \mathcal{F})$ is (strongly) primitive for $\mathcal{F} = \{M\}$.

Choose $k \geqslant 2$ and $i \in [k-1]$. We adopt the notation $W_{k;i}$ for the essential map f on Set_k satisfying $i^f := i+1$ for $i \in [k-1]$ and $k^f := \{1, 1+i\}$. A **Wielandt matrix** is a matrix which is permutation similar to $W_{k;1}$ [42]. In general, a **Wielandt-type** matrix, as introduced in [40], is a matrix which is permutation similar to a matrix $W_{k;i}$ where $\gcd(k,i) = 1$. One can check that $\mathrm{g}(W_{k;i}) = k + (k-2)(k-i)$ when $\gcd(k,i) = 1$. We display several Wielandt-type matrices below and present the phase space of $W_{4;1}$ in Fig. 1.

$$W_{4;1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}, W_{4;3} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, W_{5;4} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 7 (Wielandt [42])**.** *For every primitive matrix M of order $k \geqslant 2$, it holds $\mathrm{g}(M) \leqslant (k-1)^2 + 1$, with equality if and only if M is a Wielandt matrix.*

For any positive integers k and t, let Prim_k be the set of all strongly primitive sets of essential maps on Set_k and let $\mathrm{Prim}_{k,t}$ be the set of those elements from Prim_k of size t. Theorem 7 gives an upper bound for the transients of elements of $\mathrm{Prim}_{k,1}$. It is natural to wonder which kind of upper bound can be obtained when we turn to $\mathrm{Prim}_{k,t}$ for $t > 1$, namely we are interested in the dynamical behavior of those non-homogeneous products of matrices. Indeed, Protasov [32] explicitly suggested to estimate the parameter $\mathrm{g}_{k,t}$, where

$$\mathrm{g}_{k,t} := \max\{\mathrm{g}(\mathcal{F}) : \ \mathcal{F} \in \mathrm{Prim}_{k,t}\}.$$

Analogously, we let

$$\underline{\mathrm{g}}_{k,t} = \max\{\underline{\mathrm{g}}(\mathcal{F}) : \ \mathcal{F} \in \mathrm{Prim}_{k,t}\},$$

which is surely not greater than $\mathrm{g}_{k,t}$. We make the convention that the maximum over an empty set equals $-\infty$ and the minimum over an empty set is $+\infty$. In view of Theorem 7, we have the absolute upper bound

$$\underline{\mathrm{g}}_{k,t} \leqslant (k-1)^2 + 1; \tag{5}$$

On the other hand, Eq. (4) gives the absolute upper bound

$$\mathrm{g}_{k,t} \leqslant 2^k - 2. \tag{6}$$

Besides Eqs. (5) and (6), very little is known about $\mathrm{g}_{k,t}$ and $\underline{\mathrm{g}}_{k,t}$. Let us mention that the effort to understand the behavior of $\mathrm{g}_{k,t}$ and $\underline{\mathrm{g}}_{k,t}$ from $t = 1$ to $t > 1$ is part of a nonparametric version of the following proposal of Hajnal [19, p. 67]:

> Discussions of Markov chains ... are generally restricted to the homogeneous case ... In particular, the ergodic properties have only been established in this case. It seems natural to consider to what extent these properties hold for non-homogeneous chains.

As recorded in Theorem 1, Cohen and Sellers [12, Theorem 1] already found Eq. (6). Upon this discovery, they suggested to estimate the parameter γ_k, where

$$\gamma_k := \min\{t : \ \mathrm{g}_{k,t} = 2^k - 2\}.$$

Due to Eq. (5), we can accordingly define

$$\beta_k := \max\{t : \ \underline{\mathrm{g}}_{k,t} = (k-1)^2 + 1\}.$$

The parameters β and γ indicate the extremal size of the strongly primitive matrix sets attaining the absolute upper bounds. In particular, the parameter γ_k tells you how much freedom you will need for the possibility of living as long as the absolute upper bound for a strongly primitive essential Boolean linear dynamical system. Pick $k \geqslant 2$. Cohen and Sellers [12, Theorem 2] showed that you will need at most $2^k - 2$ operators to produce a strongly primitive essential Boolean linear system on Set_k where you can find someone with a life of maximum possible length, namely $\gamma_k \leqslant 2^k - 2$. By Theorem 2, we can formulate the following improvement, namely, to guarantee a lifespan achieving the absolute upper bound, much less freedom will already do the job!

Theorem 8. *For every integer $k \geqslant 2$, $\gamma_k \leqslant k$ holds.*

We can check that $\gamma_2 = 1, \gamma_3 = 2, \gamma_4 = 3, \gamma_5 \in \{2, 3, 4\}$ [40]. It is not clear if $\gamma_k \leqslant k-1$ holds for all $k \geqslant 2$. We still lack some technique to yield a nontrivial lower bound of γ_k and we even do not know if

$$\gamma_2 \leqslant \gamma_3 \leqslant \gamma_4 \leqslant \cdots$$

is valid.

Take $k \geqslant 2$ and let $\pi = (\mathrm{Set}_k, <_\pi)$ be a total order on Set_k. We say that a set $\mathcal{M}$ of essential maps on Set_k is **compatible** with π if for every $A \in \mathrm{Set}_k \setminus \{[k]\}$ and every $M \in \mathcal{M}$, it holds

$$A <_\pi AM,$$

namely, if π is a linear extension of the poset $\mathbb{P}_{\mathcal{M}}$. We define MC^π to be the set of primitive matrices of order k which are compatible with π. It is easy to see that all subsets of MC^π belong to Prim_k and a set $\mathcal{M}$ of essential maps on Set_k is compatible with π if and only if $\mathcal{M} \subseteq \mathrm{MC}^\pi$. We may think of the total order π as some kind of given ranking and then we want to understand the lifespan in a world where such a ranking is respected. This leads us to the parameter

$$\mathrm{g}(\pi) := \mathrm{g}(\mathrm{MC}^\pi),$$

which is $\max \mathrm{g}(\mathcal{M})$ where $\mathcal{M}$ runs through all sets of essential maps on Set_k which are compatible with π, and, for any positive integer t, the parameter

$$\mathrm{g}_t(\pi) := \max\{\mathrm{g}(\mathcal{M}) :\ \mathcal{M} \in \binom{\mathrm{MC}^\pi}{t}\}.$$

In addition, let

$$\gamma(\pi) := \min\{t \geqslant 1 :\ \mathrm{g}_t(\pi) = 2^k - 2\},$$

which is the minimum number of choices a linear world should possess to produce a life trajectory π. This parameter is interesting as we have

$$\gamma_k = \min_{\pi} \gamma(\pi), \tag{7}$$

where π runs through all total orders on Set_k.

The **broken Boolean lattice $\mathbf{B}_k = (\mathbf{Set}_k, \subsetneq)$** is the poset on Set_k where A is said to be less than B if and only if A is a proper subset of B, namely $A \subsetneq B$.

Theorem 9. *Take $k \geqslant 2$ and let π be a total order on Set_k. Then the following are equivalent.*

(i) $\gamma(\pi) < \infty$.

(ii) π is a linear extension of the broken Boolean lattice $\mathbf{B}_k$.

(iii) $\gamma(\pi) \leqslant 2^k - 3$.

When $k = 2$, Eq. (3) becomes

$$01 \to 10 \to 11,$$

and so π_2 coincides with $\mathbb{P}_{\{f\}}$ for

$$f := \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

This means that $\gamma(\pi_2) = 1$.

Theorem 10. *Let k be an integer greater than 2 and let π_k be the lexicographic order on* Set_k*. Then $\gamma(\pi_k) = k$.*

To get a better bound than that of Theorem 8, by virtue of Eq. (7), we should expect the existence of a total order τ on Set_k such that $\gamma(\tau) < k$ holds. Theorem 10 says that this τ, if any, could not be the lexicographic order when $k \geqslant 3$. Note that our proof of Theorem 2, and hence Theorem 8, makes use of a strongly primitive matrix set compatible with the lexicographic order on Set_k and this explains the main difficulty in extending our work here for a possible better bound. It may be interesting to determine $\gamma(\tau)$ for some other natural total orders on Set_k.

Given an element $\mathcal{M}$ from Prim_k, how to prolong the longest lifespan by adding some more suitable essential maps without making someone live forever, namely without losing the strong primitivity property? Let $\gamma(\mathcal{M})$ be the minimum size of $\mathcal{F}$ such that $\mathrm{g}(\mathcal{F} \cup \mathcal{M}) = 2^k - 2$. As illustrated by the next theorem, it turns out that we can always enlarge a strongly primitive matrix set into a bigger one in which the longest life duration achieves the absolute upper bound (Eq. (6)), which means that $\gamma(\mathcal{M})$ is well-defined.

Theorem 11. *Let k be an integer not less than 2 and $\mathcal{M}$ an element from* Prim_k*. Then it holds $\gamma(\mathcal{M}) \leqslant 2^k - 3$.*

Theorem 11 establishes an absolute upper bound for $\gamma(\mathcal{M})$, which implies the first half of Theorem 1 by taking $\mathcal{M} = \emptyset$. It will be interesting to develop some tools to get the exact value of $\gamma(\mathcal{M})$ for some explicit $\mathcal{M}$, say some singleton set $\mathcal{M}$. Recently, Wang, Wu and Xiang [40] found that

$$\gamma(W_{k;k-1}) \leqslant \binom{k-2}{\lfloor (k-2)/2 \rfloor}$$

for all integers $k \geqslant 2$, from which $\gamma_5 \leqslant 4$ follows. For a set $\mathcal{M}$ of essential maps on Set_k and any positive integer t, let $\mathrm{g}_t(\mathcal{M}) = \max\{\mathrm{g}(\mathcal{F}) :\ \mathcal{M} \subseteq \mathcal{F}, |\mathcal{F} \setminus \mathcal{M}| = t\}$. Besides some trivial examples, it seems that no result about $\mathrm{g}_t(\mathcal{M})$ is known.

For each positive integer k, let $\alpha_k = \max\{t :\ \mathrm{Prim}_{k,t} \neq \emptyset\}$. It is obvious that α_k will never decrease when k becomes bigger. Indeed, for any strongly primitive set $\mathcal{F}$ of essential maps on Set_k, we can create a strongly primitive set $\mathcal{F}'$ of essential maps on Set_{k+1} of equal size by putting

$$\mathcal{F}' := \{f' :\ f \in \mathcal{F}\},$$

where f' is the essential map on Set_{k+1} such that $i^{f'} = i^f \cup \{k+1\}$ if $i \in [k]$ and $(k+1)^{f'} = [k+1]$ for all $f \in \mathcal{F}$.

Theorem 12. *For all positive integers k, $\alpha_k \geqslant 1 + \binom{2^k-2}{2}$.*

It is clear that Prim_k forms an abstract simplicial complex of dimension $\alpha_k - 1$. We wonder whether or not Prim_k is a pure simplicial complex, i.e., whether or not all maximal elements of Prim_k under the inclusion relationship are of the same size α_k. Note that if Prim_k is indeed a pure simplicial complex, the ensuing result will follow easily from this fact.

Theorem 13. *It holds for every positive integer k that*

$$\mathrm{g}_{k,1} \leqslant \mathrm{g}_{k,2} \leqslant \cdots \leqslant \mathrm{g}_{k,\alpha_k}. \tag{8}$$

Compared with the inequalities in (8), it is trivial to prove the following analogue for the hitting time $\underline{\mathrm{g}}$:

$$\underline{\mathrm{g}}_{k,1} \leqslant \underline{\mathrm{g}}_{k,2} \leqslant \cdots \leqslant \underline{\mathrm{g}}_{k,\alpha_k}.$$

Recall that Theorem 8 means that the absolute upper bound described in Eq. (6) can be achieved at a small t. Accordingly, regarding the absolute upper bound listed in Eq. (5), the following theorem states that it is achievable only when the strongly primitive matrix set is a very specific singleton set.

Theorem 14. *Take $k \geqslant 2$ and $\mathcal{F} \in \mathrm{Prim}_k$. Then $\underline{\mathrm{g}}(\mathcal{F}) \leqslant (k-1)^2 + 1$, with equality if and only if $\mathcal{F}$ is a singleton set consisting of a Wielandt matrix. In particular, we have $\beta_k = 1$.*

So far, we have been discussing the extremal behaviors of the indexes $\underline{\mathrm{g}}$ and g for strongly primitive set of essential Boolean linear maps. For a deeper understanding of the two indexes, we may want to understand the four index sets defined as below:

$$\begin{cases} \mathcal{G}_{k,t} := \{\mathrm{g}(\mathcal{F}) : \ \mathcal{F} \in \mathrm{Prim}_{k,t}\}; \\ \underline{\mathcal{G}}_{k,t} := \{\underline{\mathrm{g}}(\mathcal{F}) : \ \mathcal{F} \in \mathrm{Prim}_{k,t}\}; \\ \mathcal{G}_{k} := \{\mathrm{g}(\mathcal{F}) : \ \mathcal{F} \in \mathrm{Prim}_{k}\}; \\ \underline{\mathcal{G}}_{k} := \{\underline{\mathrm{g}}(\mathcal{F}) : \ \mathcal{F} \in \mathrm{Prim}_{k}\}. \end{cases}$$

The community of combinatorics matrix theory has made clear the structure of the set $\mathcal{G}_{k,1} = \underline{\mathcal{G}}_{k,1}$ [25, 37, 46]; see [8, §3.5]. Moreover, Shao determined the set $\mathcal{G}_k$ in his PhD thesis [36, pp. 120–124]. We report his result below and present a proof of it in §5. Our proof of Theorem 15 follows the same idea with that of Shao but is written in the language of our phase space approach.

Theorem 15 (Shao)**.** *For every positive integer k, $\mathcal{G}_k = [2^k - 2]$ holds.*

We have discussed above some results about the primitivity exponent and the hitting time of a strongly primitive essential Boolean linear dynamical system. In general, we may ask what is the shape of $\mathbb{P}_{\mathcal{D}}$ for a strongly primitive essential binary linear system $\mathcal{D}$, what can be said on the Möbius function of this poset, and so forth. It may be useful to develop more concepts and apparatus for thinking further about combinatorial properties of (inhomogeneous) products of Boolean matrices. Let us end this section with one question about the dynamics generated by a single matrix.

Pick $f \in \mathrm{Prim}_k$. For every $A \in \mathrm{Set}_k$, we say that a positive integer i is a **stride length** of f at A provided $Af^i \supsetneq A$ and we write $\mathrm{StL}_f(A)$ for the least stride length of f at A. Similarly, for every $A \in \mathrm{Set}_k$, we say that a positive integer i is a **weak stride length** of f at A provided $|Af^i| > |A|$ and we write $\mathrm{WStL}_f(A)$ for the least weak stride length of f at A. How to find some nontrivial upper bound for $\max_{A \in \mathrm{Set}_k \setminus \{[k]\}} \mathrm{StL}_f(A)$ and $\max_{A \in \mathrm{Set}_k \setminus \{[k]\}} \mathrm{WStL}_f(A)$?

Example 16 (Xinmao Wang)**.** For

$$f = \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ \mathbf{1} & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & \mathbf{1} & 0 & 0 \end{pmatrix},$$

we have

$$\max_{A \in \mathrm{Set}_5} \mathrm{StL}_f(A) = 7, \max_{A \in \mathrm{Set}_5} \mathrm{WStL}_f(A) = 6$$

and

$$\{\mathbf{2}, \mathbf{5}\} \to \{3\} \to \{4\} \to \{1, 5\} \to \{2, 3\} \to \{3, 4\} \to \{\mathbf{1}, \mathbf{4}, \mathbf{5}\} \to \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}\}$$

is a path showing that $\mathrm{StL}_f(A) = 7$ and $\mathrm{WStL}_f(A) = 6$ for $A = \{2, 5\}$.

5 Proofs

Proof of Theorem 6. For any $i \in [k]$, let $F_{k,i}$ be the Boolean linear map on $2^{[k]}$ such that

$$s^{F_{k,i}} := \begin{cases} s, & \text{if } s > i; \\ [i-1], & \text{if } s = i; \\ [k], & \text{if } s < i; \end{cases}$$

for every $s \in [k]$. Let $\mathcal{F}_k = \{F_{k,i} : \ i \in [k]\}$.

Recall the definition of ϕ_k in Eq. (2). Take $t \in [2^k - 2]$ and let j be the smallest element of $\phi_k^{-1}(t+1)$. For any $i \in [k]$, we can check that

$$(\phi_k^{-1}(t+1))^{F_{k,i}} = \begin{cases} \phi_k^{-1}(t+1), & \text{if } j > i; \\ (\phi_k^{-1}(t+1) \backslash \{j\}) \cup [j-1] = \phi_k^{-1}(t), & \text{if } j = i; \\ [k] = \phi_k^{-1}(2^k - 1), & \text{if } j < i. \end{cases}$$

This implies that there exists a word $w = w_1 \cdots w_{2^k-2}$ over $\mathcal{F}_k$ such that

$$\phi_k^{-1}(2^k - 1) = [k] \xrightarrow{w_1} \phi_k^{-1}(2^k - 2) \xrightarrow{w_2} \cdots \xrightarrow{w_{2^k-2}} \phi_k^{-1}(1) = \{1\} \xrightarrow{w_{2^k-1}} \emptyset$$

is the unique shortest path in $\mathcal{PS}_{\mathcal{F}_k}$ from $[k]$ to $\emptyset$. Note that g'$((2^{[k]}, \mathcal{F}_k), \emptyset)$ is just the length of the shortest path from $[k]$ to $\emptyset$ and so the proof is finished. □

The next lemma says that for every $\mathcal{M} \in \mathrm{Prim}_k$, there is a common linear extension of $\mathbf{B}_k$ and $\mathbb{P}_{\mathcal{M}}$.

Lemma 17. *Take a positive integer k and a set $\mathcal{M} \in \mathrm{Prim}_k$. The poset $\mathbb{P}_{\mathcal{M}}$ has a linear extension $(\mathrm{Set}_k, \lhd)$ such that $A \lhd B$ holds for every two elements from Set_k satisfying either $A \subsetneq B$ or $A \prec_{\mathcal{M}} B$.*

Proof. Consider the digraph D with $\mathrm{V}(D) = \mathrm{Set}_k$ and $\mathrm{A}(D) = \mathrm{A}_1 \cup \mathrm{A}_2$, where $\mathrm{A}_1 = \{A \to B : A, B \in \mathrm{Set}_k, A \subsetneq B\}$ and $\mathrm{A}_2 = \{A \to B : A, B \in \mathrm{Set}_k, A \prec_{\mathcal{M}} B\}$. If D is acyclic, then D possesses a linear extension and every such linear extension must be what we wanted.

Suppose on the contrary that D is not acyclic and so, considering that both $(\mathrm{Set}_k, \mathrm{A}_1)$ and $(\mathrm{Set}_k, \mathrm{A}_2)$ are transitive and acyclic digraphs, we can find a closed walk of positive length $2p$ in D:

$$C_1 \prec_{\mathcal{M}} B_1 \subsetneq C_2 \prec_{\mathcal{M}} B_2 \subsetneq \cdots \subsetneq C_p \prec_{\mathcal{M}} B_p \subsetneq C_1.$$

This means that there are matrices $M_1, \ldots, M_p$ from the multiplicative semigroup generated by $\mathcal{M}$ such that

$$C_1 M_1 = B_1 \subsetneq C_2, \ldots, C_p M_p = B_p \subsetneq C_1,$$

and hence

$$C_1 M_1 M_2 \cdots M_p \subsetneq C_1.$$

Consequently, starting from C_1 and applying the map $M_1 M_2 \cdots M_p$ repeatedly we will be always outside of the black hole $[k]$, which contradicts the assumption that $\mathcal{M}$ is strongly primitive. $\square$

Pick a positive integer k and any two elements $A, B \in \mathrm{Set}_k$. We write $f^{(k)}_{B \mapsto A}$ for the essential map on Set_k such that

$$C f^{(k)}_{B \mapsto A} := \begin{cases} A, & \text{if } C \subseteq B; \\ [k], & \text{if } C \setminus B \neq \emptyset. \end{cases} \tag{9}$$

Lemma 18. *Take $k \geqslant 2$ and $p \in [2^k - 3]$. Let π be a linear extension of the broken Boolean lattice $\mathbf{B}_k$. Let $\mathcal{F}_p = \{M_i : i \in [p]\}$ where $M_i = f^{(k)}_{\pi(i) \mapsto \pi(i+1)}$ for $i \in [p]$. Then $\mathcal{F}_p$ is strongly primitive, $\mathcal{PS}_{\mathcal{F}_p}$ contains the path $\pi(1) \to \pi(2) \to \cdots \to \pi(p+1) \to [k]$, and $\mathrm{g}(\mathcal{F}_p) = p + 1$.*

Proof. Since π is a linear extension of $\mathbf{B}_k$, from Eq. (9) we can see that

- each arc in $\mathcal{PS}_{\mathcal{F}_p}$ is either of the form $\pi(i) \to \pi(j)$ where $1 \leqslant i < j \leqslant p+1$, or of the form $\pi(i) \to \pi(2^k - 1)$ for $i \in [2^k - 1]$; and that
- $\pi(1) \xrightarrow{M_1} \pi(2) \xrightarrow{M_2} \cdots \xrightarrow{M_p} \pi(p+1) \xrightarrow{M_p} [k]$ is a path of length $p+1$ in $\mathcal{PS}_{\mathcal{F}_p}$.

The claims about $\mathcal{F}_p$ now follow directly. $\square$

Proof of Theorem 9. The implication (iii) to (i) is a triviality.

We now prove that (i) implies (ii). From $\gamma(\pi) < \infty$ we obtain the existence of $\mathcal{M} \subseteq \mathrm{MC}^{\pi}$ such that $\mathrm{g}(\mathcal{M}) = 2^k - 2$. Note that $\mathbb{P}_{\mathcal{M}}$ is the total order

$$\pi(1) < \cdots < \pi(2^k - 2) < \pi(2^k - 1) = [k].$$

By Lemma 17, this total order π has to be a linear extension of the broken Boolean lattice.

Finally, we need to establish the direction of (ii) $\Rightarrow$ (iii). By Lemma 18, $\mathcal{F}_{2^k-3} \in \mathrm{MC}^{\pi}$ and $\mathrm{g}(\mathcal{F}_{2^k-3}) = 2^k - 2$, proving the desired fact that $\gamma(\pi) \leqslant 2^k - 3$. $\square$

Proof of Theorem 10. Theorem 2 means that $\gamma(\pi_k) \leqslant k$. It suffices to show $\gamma(\pi_k) \geqslant k$. Let

$$a_k := \phi_k^{-1}(4) = \{k-2\} \to \{k-2, k\} = \phi_k^{-1}(5)$$

and

$$a_i := \phi_k^{-1}(2^k - 2^{i-1} - 1) = [k] \setminus \{i\} \to [i] = \phi_k^{-1}(2^k - 2^{i-1})$$

for $i \in [k-1]$. They are k arcs in the Hasse diagram of the total order π_k. By way of contradiction, let us assume $\gamma(\pi_k) < k$ and thus there should be a primitive matrix M of order k such that $\mathcal{PS}_M$ contains two arcs a_p and a_q where $1 \leqslant p < q \leqslant k$. From $a_p \in \mathrm{A}(\mathcal{PS}_M)$ we deduce that

$$([k] \setminus \{p\})M \subseteq [p]. \tag{10}$$

Considering that M is primitive and so $[k]M = [k]$ holds, we reach the conclusion that

$$[k] \setminus [p] \subseteq \{p\}M. \tag{11}$$

Case 1. $q \in [k-1]$.

We get Eq. (10) just from the fact that $p \in [k-1]$. So, we can also get from $q \in [k-1]$ that $k \notin ([k] \setminus \{q\})M$ and hence $k \notin \{p\}M$, contradicting Eq. (11).

Case 2. $q = k$.

From $a_q \in \mathrm{A}(\mathcal{PS}_M)$ we see that

$$\{k-2, k\} = \{k-2\}M. \tag{12}$$

Since $k \in \{k-2\}M$, Eq. (10) gives $p = k-2$. Applying Eq. (11) now yields $k-1 \in \{k-2\}M$, violating Eq. (12). □

Proof of Theorem 11. By Lemma 17, we can take π to be a common linear extension of $\mathbf{B}_k$ and $\mathbb{P}_{\mathcal{M}}$. Now, Theorem 9 guarantees the existence of a set $\mathcal{F} \in \binom{\mathrm{MC}^\pi}{t}$ for some $t \leqslant 2^k - 3$ such that $\mathcal{F} \cap \mathcal{M} = \emptyset$ and $\mathrm{g}(\mathcal{F} \cup \mathcal{M}) = 2^k - 2$. □

Proof of Theorem 12. Take any linear order π on Set_k which is a linear extension of the broken Boolean lattice $\mathbf{B}_k$, say

$$\pi(1) < \ldots < \pi(2^k - 2) < \pi(2^k - 1) = [k].$$

Let

$$\mathcal{F} := \{M_{i,j} :\ 1 \leqslant i < j \leqslant 2^k - 2\} \cup \{f^{(k)}_{[k]\mapsto[k]}\}, \tag{13}$$

where $M_{i,j} = f^{(k)}_{\pi(i)\mapsto\pi(j)}$ for those i, j satisfying $1 \leqslant i < j \leqslant 2^k - 2$. It follows from Eq. (9) that $\mathcal{PS}_{\mathcal{F}}$ is the total order π and so $\mathrm{g}(\mathcal{F}) = 2^k - 2$. It is clear that $\mathcal{F}$ consists of $1 + \binom{2^k-2}{2}$ different maps and so the theorem follows. □

For every integer $k \geqslant 2$, Theorem 12 tells us that

$$\alpha_k \geqslant 1 + \binom{2^k-2}{2} \geqslant 2^k - 2 \geqslant k, \tag{14}$$

where the second inequality is strict for $k \geqslant 3$. The subsequent two lemmas determine the values of $\mathrm{g}_{k,k}, \ldots, \mathrm{g}_{k,2^k-2}, \mathrm{g}_{k,2^k-1}, \ldots, \mathrm{g}_{k,\alpha_k}$.

Lemma 19. *Let k be an integer greater than 1. Then $\mathrm{g}_{k,k} = \mathrm{g}_{k,k+1} = \cdots = \mathrm{g}_{k,2^k-2} = 2^k - 2$.*

Proof. For $\mathcal{M}_k = \{M_{k,1}, \ldots, M_{k,k}\}$, Theorem 2 claims that $\mathrm{g}(\mathcal{M}_k) = 2^k - 2$ and $\mathbb{P}_{\mathcal{M}_k}$ is the lexicographic order π_k on Set_k.

Taking $\pi = \pi_k$, we now consider the family of essential maps $\mathcal{F}$ as given in Eq. (13). By Eq. (14), we have

$$|\mathcal{F} \setminus \mathcal{M}_k| \geqslant 2^k - 2 - k.$$

The lexicographic order $\pi_k = \mathbb{P}_{\mathcal{M}_k}$ is surely a linear extension of the broken Boolean lattice $\mathbf{B}_k$. This along with Eq. (9) tells us that $\mathrm{g}(\mathcal{F}' \cup \mathcal{M}_k) = 2^k - 2$ for every $\mathcal{F}' \subseteq \mathcal{F} \setminus \mathcal{M}_k$. It follows that $\mathrm{g}_{k,k} = \mathrm{g}_{k,k+1} = \cdots = \mathrm{g}_{k,2^k-2+k} = 2^k - 2$, finishing the proof. □

Lemma 20. *Let k be an integer greater than 2. Then $\mathrm{g}_{k,2^k-1} = \cdots = \mathrm{g}_{k,\alpha_k} = 2^k - 2$.*

Proof. Take $\mathcal{A} \in \mathrm{Prim}_{k,\alpha_k}$ and, thanks to Lemma 17, pick a common linear extension of $\mathbf{B_k}$ and $\mathbb{P}_{\mathcal{A}}$, say

$$(\mathrm{Set}_k, \lhd): \ A_1 \lhd A_2 \lhd \cdots \lhd A_{2^k-1} = [k] = \mathbb{1}.$$

Let

$$\mathcal{F} := \{M_i : \ i \in [2^k - 2]\},$$

where $M_i = f^{(k)}_{A_i \mapsto A_{i+1}}$ for $i \in [2^k-2]$. It follows from Eq. (9) that $\mathcal{F} \cup \mathcal{A}$ is strongly primitive and so the definition of α_k leads to $\mathcal{F} \subseteq \mathcal{A}$. Take an integer t satisfying $2^k - 1 \leqslant t \leqslant \alpha_k$. We can find a set $\mathcal{F}_t$ such that $\mathcal{F} \subseteq \mathcal{F}_t \subseteq \mathcal{A}$ and $|\mathcal{F}_t| = t$. It is clear that $\mathcal{F}_t \in \mathrm{Prim}_{k,t}$ and $\mathrm{g}(\mathcal{F}_t) = 2^k - 2$. Since $2^k - 2$ is the absolute upper bound for the strong primitivity exponent here, we conclude that $\mathrm{g}_{k,t} = 2^k - 2$, as desired. □

Proof of Theorem 13. When $k = 2$, the primitive matrices of order k can be listed as

$$\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}, \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}.$$

It is thus easy to see that $\alpha_2 = 2 = 2^2 - 2$.

Combining Lemma 19 and Lemma 20, our task is now reduced to establishing

$$\mathrm{g}_{k,1} \leqslant \cdots \leqslant \mathrm{g}_{k,k}\,. \tag{15}$$

We will be able to obtain Eq. (15) provided, for each $t \in [k-1]$ and each $\mathcal{A} \in \mathrm{Prim}_{k,t}$, we can find $\mathcal{B} \in \mathrm{Prim}_{k,t+1}$ such that $\mathcal{A} \subseteq \mathcal{B}$. Lemma 17 allows us to get a common linear extension of $\mathbf{B_k}$ and $\mathbb{P}_{\mathcal{A}}$, say

$$(\mathrm{Set}_k, \lhd): \ A_1 \lhd A_2 \lhd \cdots \lhd A_{2^k-1} = \mathbb{1}.$$

Let $\mathcal{F} = \{M_i : i \in [2^k-2]\}$ where $M_i = f^{(k)}_{A_i \mapsto A_{i+1}}$ for $i \in [2^k-2]$. It follows from Eq. (9) that $\mathcal{F} \cup \mathcal{A}$ is strongly primitive. Since $t+1 \leqslant k \leqslant 2^k-2$, we can get $F \in \mathcal{F} \setminus \mathcal{A}$. Letting $\mathcal{B} = \mathcal{A} \cup \{F\}$, it is not hard to check that $\mathcal{B} \in \mathrm{Prim}_{k,t+1}$ and hence we are done! $\square$

Proof of Theorem 14. Pick arbitrarily an element M from $\mathcal{F}$. Theorem 7 gives $\underline{\mathrm{g}}(M) \leqslant (k-1)^2+1$ and the equality holds if and only if M is permutation similar to $W_{k;1}$. It is obvious that $\underline{\mathrm{g}}(\mathcal{F}) \leqslant \underline{\mathrm{g}}(M) \leqslant (k-1)^2+1$. Consequently, to complete the proof, it suffices to demonstrate that $\{W_{k;1}, M\}$ is not strongly primitive for every Wielandt matrix M other than $W_{k;1}$. Let us assume that $M = PW_{k;1}P^{-1}$, where P is a permutation matrix which is not equal to the identity matrix.

Case 1. $k \xrightarrow{P} i \in [k-1]$.

In this case, we have $j \in [k-1]$ such that $k \xrightarrow{P} i \xrightarrow{W_{k;1}} i+1 \xrightarrow{P^{-1}} j$, namely $k \xrightarrow{M} j$. In $\mathcal{PS}_{W_{k;1}}$, we have a path $j \to j+1 \to \cdots \to k$, which, combined with the arc $k \to j$ from $\mathcal{PS}_M$, gives a cycle in $\mathcal{PS}_{W_{k;1},M}$. This shows that $\{W_{k;1}, M\}$ is not strongly primitive.

Case 2. $k \xrightarrow{P} k$.

As P is not the identity matrix, we can set $i = \max\{j : jP \neq j\} \in [k-1]$. It follows that $i \xrightarrow{P} g \leqslant i-1 \in [k-2]$. Henceforth, $i \xrightarrow{P} g \xrightarrow{W_{k;1}} g+1 \in [i]$. This in turns gives $i \xrightarrow{P} g \xrightarrow{W_{k;1}} g+1 \xrightarrow{P^{-1}} h < i \leqslant k-1$. Now, we can find the following cycle in the phase space of $\{W_{k;1}, M\}$:

$$h \xrightarrow{W_{k;1}} h+1 \xrightarrow{W_{k;1}} \cdots \xrightarrow{W_{k;1}} i \xrightarrow{M} h,$$

proving that $\{W_{k;1}, M\}$ is not strongly primitive, as desired. $\square$

Proof of Theorem 15. For $p=1$, it is immediate that $\mathrm{g}(\mathcal{F}) = p$ for $\mathcal{F} = \{f^{(k)}_{[k]\mapsto[k]}\}$ (c.f. Eq. (9)). If $2 \leqslant p \leqslant 2^k-2$, it follows from Lemma 18 that $p \in \mathcal{G}_k$. For any $p > 2^k-2$, Eq. (6) tells us $p \notin \mathcal{G}_k$. $\square$

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References

[1] Yu. A. Al'pin and V. S. Al'pina. Combinatorial properties of irreducible semigroups of nonnegative matrices. *Journal of Mathematical Sciences*, 191:4–9, 2013.

[2] D. S. Ananichev, M. V. Volkov and V. V. Gusev. Primitive digraphs with large exponents and slowly synchronizing automata. *Journal of Mathematical Sciences*, 192:263–278, 2013.

[3] L. B. Beasley and A. E. Guterman. The characterization of operators preserving primitivity for matrix k-tuples. *Linear Algebra and its Applications*, 430:1762–1777, 2009.

[4] L. B. Beasley and S. Kirkland. A note on k-primitive directed graphs. *Linear Algebra and its Applications*, 373:67–74, 2003.

[5] L. B. Beasley and S. Mousley. k-Primitivity of digraphs. *Linear Algebra and its Applications*, 449:512–519, 2014.

[6] V. D. Blondel, R. M. Jungers and A. Olshevsky. On primitivity of sets of matrices. *Automatica. A Journal of IFAC*, 61:80–88, 2015.

[7] V. D. Blondel and J. N. Tsitsiklis. When is a pair of matrices mortal? *Information Processing Letters*, 63:283–286, 1997.

[8] R. Brualdi and H. J. Ryser. Combinatorial Matrix Theory. Cambridge University Press, Cambridge, 1991.

[9] P. Butkovič. Max-linear Systems: Theory and Algorithms. *Springer Monographs in Mathematics*, Springer, 2010.

[10] A. S. Cavaretta, W. Dahmen and C. A. Micchelli. Stationary Subdivision. *Memoirs of the American Mathematical Society*, Vol. 93, AMS, 1991.

[11] J. Černý. Poznámka k homogénnym experimentom s konečnými automatami. (in Slovak). *Matematicko-fyziklny Časopis Slovenskej Akadémie Vied*, 14:208–216, 1964.

[12] J. E. Cohen and P. H. Sellers. Sets of nonnegative matrices with positive inhomogeneous products. *Linear Algebra and its Applications*, 47:185–192, 1982.

[13] O. Colón-Reyes, A. S. Jarrah, R. Laubenbacher and B. Sturmfels. Monomial dynamical systems over finite fields. *Complex Systems*, 16:333–342, 2006.

[14] C. Fleischhack and S. Friedland. Asymptotic positivity of Hurwitz product traces: Two proofs. *Linear Algebra and its Applications*, 432:1363–1383, 2010.

[15] E. Fornasini and M. Valcher. Primitivity of positive matrix pairs: algebraic characterization, graph-theoretic description, and 2D systems interpretation. *SIAM Journal on Matrix Analysis and Applications*, 19:71–88, 1998.

[16] E. Fornasini and M. Valcher. A polynomial matrix approach to the structural properties of 2D positive systems. *Linear Algebra and its Applications*, 413:458–473, 2006.

[17] E. Fornasini and M. Valcher. Reachability of a class of discrete-time positive switched systems. *SIAM Journal on Control and Optimization*, 49:162–184, 2011.

[18] V. S. Grinberg. Wielandt-type bounds for primitive mappings of partially ordered sets. *Mathematical Notes*, 45:450–454, 1989.

[19] J. Hajnal. The ergodic properties of non-homogeneous finite Markov chains. *Mathematical Proceedings of the Cambridge Philosophical Society*, 52:67–77, 1956.

[20] J. Hajnal. On products of non-negative matrices. *Mathematical Proceedings of the Cambridge Philosophical Society*, 79:521–530, 1976.

[21] D. J. Hartfiel. Nonhomogeneous Matrix Products. World Scientific, River Edge, NJ, 2002.

[22] C. C. Heyde and J. E. Cohen. Confidence intervals for demographic projections based on products of random matrices. *Theoretical Population Biology*, 27:120–153, 1985.

[23] J. P. S. Kung, G.-C. Rota and C. H. Yan. Combinatorics: The Rota Way. Cambridge University Press, 2009.

[24] J. C. Lagarias. The Ultimate Challenge: The $3x + 1$ Problem. AMS, 2010.

[25] M. Lewin and Y. Vitek. A system of gaps in the exponent set of primitive matrices. *Illinois Journal of Mathematics*, 25:87–98, 1981.

[26] J. D. Louck and M. L. Stein. The $(1 + 1)$-Nonlinear Universe Of The Parabolic Map And Combinatorics. World Scientific, 2015.

[27] A. A. Melkman. Subdivision schemes with non-negative masks converge always – unless they obviously cannot? *Annals of Numerical Mathematics*, 4:451–460, 1997.

[28] G. Merlet, T. Nowak, H. Schneider and S. Sergeev. Generalizations of bounds on the index of convergence to weighted digraphs. *Discrete Applied Mathematics*, 178:121–134, 2014.

[29] C. A. Micchelli and H. Prautzsch. Uniform refinement of curves. *Linear Algebra and its Applications*, 114/115:841–970, 1989.

[30] D. D. Olesky, B. Shader and P. van den Driessche. Exponents of tuples of nonnegative matrices. *Linear Algebra and its Applications*, 356:123–134, 2002.

[31] A. Paz. Introduction to Probabilistic Automata. Academic, New York, 1971.

[32] V. Yu. Protasov. Semigroups of non-negative matrices. *Communications of the Moscow Mathematical Society*, 65:1186–1188, 2010.

[33] V. Yu. Protasov. Classification of k-primitive sets of matrices. *SIAM Journal on Matrix Analysis and Applications*, 34:1174–1188, 2013.

[34] V. Yu. Protasov and A. S. Voynov. Sets of nonnegative matrices without positive products. *Linear Algebra and its Applications*, 437:749–765, 2012.

[35] P. Santesso and M. E. Valcher. Monomial reachability and zero controllability of discrete-time positive switched systems. *Systems & Control Letters*, 57:340–347, 2008.

[36] J.-Y. Shao. On the properties of nonnegative primitive matrices, irreducible matrices and thier associated directed graphs. Ph.D. thesis, University of Wisconsin-Madison, 1984.

[37] J.-Y. Shao. On a conjecture about the exponent set of primitive matrices. *Linear Algebra and its Applications*, 65:91–123, 1985.

[38] M. V. Volkov. Synchronizing automata and the Černý conjecture. *Lecture Notes in Computer Science*, 5196:11–27, 2008.

[39] A. Voynov. Shortest positive products of nonnegative matrices. *Linear Algebra and its Applications*, 439:1627–1634, 2013.

[40] X. Wang, Y. Wu and Z. Xiang. Wielandt-type matrices and primitive matrix set. manuscript, 2015.

[41] Y. Wang. Subdivision schemes and refinement equations with nonnegative masks. Journal of Approximation Theory, 113:207–220, 2001.

[42] H. Wielandt. Unzerlegbare, nicht negative Matrizen. *Mathematische Zeitschrift*, 52:642–648, 1959.

[43] Y. Wu and J. Shen. Opinion dynamics with stubborn vertices. *Electronic Journal of Linear Algebra*, 23:790–800, 2012.

[44] Y. Wu, Z. Xu and Y. Zhu. Expansion property of Boolean linear maps. submitted. available at http://math.sjtu.edu.cn/faculty/ykwu/data/Paper/20150815.pdf.

[45] Y. Wu, Z. Xu and Y. Zhu. Reachability in higher-order Markov chains. manuscript, 2015.

[46] K. M. Zhang. On Lewin and Vitek's conjecture about the exponent set of primitive matrices. *Linear Algebra and its Applications*, 96:101–108, 1987.

[47] X. Zhou. Positivity of refinable functions defined by nonnegative finite masks. *Applied and Computational Harmonic Analysis*, 27:133–156, 2009.