Responses to the referee report on “Infinite Gammoids: Minors and duality” by S. H. Afzazli Borujeni, H.-F. Law and M. Müller

The authors thank the referee for carefully reading the manuscript, which was significantly improved by the referee’s suggestions.

0 Introduction

- Page 1, second paragraph: I was a bit confused at the beginning with the meaning of *path*, as it was not clear to me whether a path was allowed to be infinite (particularly because a ray is defined as an infinite path). It should be clear that paths are always finite, and maybe avoid the term path in the definition of ray.

Answer: Inserted “analogue of” in the definition of ray to indicate that a path is not infinite.

- Page 2, first paragraph: say that the definition of a matroid will be given later, and maybe emphasize that the construction does not always give a matroid.

Answer: A reference to the definition of a matroid is given at the end of the first paragraph. The old definition of strict gammoid was designed with the aim to not distinguish the case when the dimaze defines a matroid or not. It is clearer now.

- Page 2, paragraph after Theorem 2: the notation (D, B_0) for dimazes has not been introduced yet.

Answer: Moved the whole definition of “topologically linkabe” (including the notation (D, B_0)) from the introduction to preliminaries.

- Page 2, same paragraph: I was wondering if the outgoing combs and linking fans of the topological linkages need also be disjoint (the definition seems to allow that the spikes of different combs may intersect, say). I guess it will be no problem if they intersect, but please clarify.

Answer: Made a comment after the definition.

- Page 2, line 12: remove the second *the*

Answer: Done.

- Page 3, line 6: saying *it might still be possible* suggests that the answer is not known, which is not the case. One could instead say “Although these examples do not rule out the possibility that infinite gammoids are closed under duality, we will see etc.”.

Answer: Implemented.

- The two paragraphs before Theorem 4 are too detailed, and not easy to follow just at the beginning of the paper, when one has not had time to absorb all what is going on yet. Also, one expects that we will continue hearing about duality, but suddenly there is a long paragraph about strict gammoids without alternating combs. My advice is just to say the minimum necessary to undestand the statement of Theorem 4, and leave the rest for Sections 1.2 and 3.

Answer: The paragraphs are rewritten.

1 Preliminaries

- Section 1.1 is not easy for readers not familiar with infinite matroid theory. It is true that it refers to the bibliography for undefined terms, but it is confusing that the easiest terms are given and the more difficult ones are not. I think it would clarify things if the definitions of dual, deletion and contraction were stated, and said explicitly that they are matroids. (Also, in this section and througout the paper, there are many terms that need to be emphasized when first defined.)

Answer: The basics of matroid theory are now given at a slower pace.

- Page 4: I would wait until page 18, the only place where it is needed, to introduce the notation $\mathcal{C}(M)$.

Answer: A part of the proof was rewritten so that we don't need $\mathcal{C}(M)$ anymore.

- Lemma 1.2 is Lemma 3.3.2 in Oxley's book (2nd ed). I think it is perfectly safe to say that the proof there extends literally to the infinite setting (plese double-check this), so there is no need to give a proof. This saves some little space in a long paper (plus the current proof has some undefined terms as spans and $\mathcal{B}$).

Answer: You are right. We checked the new edition and the proof works for infinite matroids as well.

- The titles of Sections 1.2 and 1.3 should be plural.

Answer: Done.

- Section 1.2, first paragraph: on the last two lines, I think the B should be B_0. Also, remove the parentheses around *vertex disjoint*.

Answer: The latter comment is done. Regarding the former, although we indeed mean B, the old formulation of the definition of "linkability" was a bit confusing. The definition is better written now.

- About Thm. 1.4: as an introduction to the result for readers not familiar with it, it may be worth saying that it is trivial that maximally linkable sets are always linkable onto the exits. After the statement, remark that there are strict gammoids given by dimazes that are not C^A-free. Also, given that $C^A = R^A$, I do not see why not just say R^A-free (actually, on page 27 both C^A and R^A are used just one line apart).

Answer: Added the stated remarks and also a hint that C^A and R^A differ in their exits after their definition.

- Last line of Section 1.2: write *it* instead of *which*, or say *and its restrictions are called topological gammoids.*

Answer: Reformulated.

2 Minor

- The title of the section should be plural.

Answer: Corrected.

- Line 5 of the section: write *duals of transversal matroids.*

Answer: This line was lost in the reorganization of the paragraph.

- The first two paragraphs of the section are confusing, mostly because of the continuos references to Section 3, that the reader has not read yet. In particular, where it says *construction with any given base* there is no way to know what is meant without reading Section 3; and somewhere below, where it says *the construction sketched*, it is not that the construction has been sketched, it has been just mentioned. So the authors should try to rewrite this part minimizing references to Section 3, and probably being less detailed. Also, in the introduction there is a mention to a construction due to Ardila and Ruiz, but this is not mentioned again, so we do not know how it is related to the methods of this section.

Answer: "the construction sketched above" was meant to refer to the construction "go to the dual $\to$ delete there $\to$ dualize again", not to the dualization construction (converting). The introduction to this section was rewritten. Also a "history"-paragraph was added after the definition of $\vec{Q}$. Ardila and Ruiz were moved to this paragraph from the introduction.

- Topological gammoids are defined, or an intuition is given for them, several times in the text: Section 0, Section 1.2, in the third paragraph of Section 2 and in the first of Section 2.2. Wouldn't it be more natural to group all the definitions and intuitions in one, at most two, places? For instance, it is not until Section 2 that we learn that they were introduced to answer a question of Diestel. This information would be most suitable for an introduction/preliminaries section. Also, the useful fact "it is not possible to separate a vertex on a topological path..." appears first in the proof of Lemma 2.4. So my advise is to define topological gammoids once and for all, rather than scattering the information throughout the paper.

Answer: Now we have only one topological gammoid paragraph in preliminaries.

- Section 1.2, first line: is it really proved that all contractions of strict gammoids are strict gammoids? I thought it was just for C^O-free ones. Or is it that it has to be understood that the aim would be to prove it for all of them, but the proof only works for C^O-free ones? Please clarify.

Answer: Right, “aim” means “dream” here. It should be clearer now.

- Before Lemma 2.1: I think it is not by Lemma 1.2 that one can assume that S is independent. This lemma says that M/S is $M/S' \setminus R'$, with S' independent and R' coindependent. The reason why one can assume S independent is that otherwise the contraction has loops, and adding loops does not take us outside of the class of strict gammoids.

Answer: A minor of a gammoid (on the vertex set E) has the form $(M_L(D, B_0)|E)/S' \setminus R'$, which is $M/C \setminus D$ for $M = M_L(D, B_0)$, $C = S'$ and $D = (V \setminus E) \cup R'$. And now we use Lemma 1.2 to find an *independent* S (and an R) such that our minor of the gammoid equals $M/S \setminus R$. Oxley's formulation of Lemma 1.2 is clearly what we want.

- Proof of Lemma 2.1: in the third $\Longleftrightarrow$ equivalence, one needs that $\mathcal{Q}$ contains no vertex of S (otherwise removing S could break some linking paths).

Answer: $Ter(\mathcal{Q}) \cap S$ is empty, which is by definition of dimaze (exits are sinks) equivalent to the needed disjointness of $\mathcal{Q}$ and S.

- Middle of page 9: say *Define a bijection* $\vec{\mathcal{Q}}$. Also, is it assumed that the paths/rays in $\mathcal{Q}$ are all non-trivial? Otherwise it is not clear what is the image of the vertices that constitute the trivial paths. I guess it all goes well if those vertices are mapped to themselves, but it should be checked.

Answer: Clarified the definition of the bijection. The trivial paths in $\mathcal{Q}$ are in $S \cap T$, so we don't need an image for them.

- It took me a while to understand $\vec{\mathcal{Q}}(W)$, because I was expecting it would have vertices and edges; it would be useful to add it is just a sequence of vertices.

Answer: Added a hint that we want to investigate the relation between $\mathcal{Q}$-alternating walks and paths/rays. Also added explicitly that we delete all *the edges* e_i so that it is clear that there are no edges in the sequence.

- Lemma 2.2: at this point I had forgotten the definition of disjoint $\mathcal{Q}$-alternating walks, and was confused for a while. Maybe the definition could be given here instead of in page 6, or just add a reminder *recall the definition of disjoint $\mathcal{Q}$-alternating walks from page 6.*

Answer: Added a reminder before the Lemma.

- Lemma 2.6, line -3 of proof: throughout the section and the proof T has been used instead of $\mathrm{Ter}(\mathcal{Q})$.

Answer: Done.

- Lemma 2.7, line -2 of proof: write *would meet* instead of *meet*.

Answer: Done.

- Corollary 2.9: when it says *as well*, it means that it is C^O-free too, right? I think it is best to spell out $\{F^\infty, C^O\}$-free, so that the statement is context-independent.

Answer: It was supposed to mean "Assume the situation of Lemma 2.8 is given, then if additionally (D, B_0) is F^∞-free, then so is (D_1, B_1)". Actually we just need that (D, B_0) is F^∞-free, nothing else. So the Corollary is a context-independent Lemma now.

- Proof of Theorem 2.11: where it says *extend S in B_0 to a base B_1*, does it mean that we add elements of B_0 to S to complete to a base? Saying S *in* B_0 sounds odd as S need not be a subset of B_0, so it would be better to rephrase it. Also, if the interpretation is not as I said, then T would be just a subset of $B_0 \setminus B_1$, not equal to it.

Answer: Your interpretation was right. The sentence is rephrased.

- Before Lemma 2.12: say which linkage theorem is meant here.

Answer: Added "Pym's".

- Lemma 2.12: the following are comments on the proof, which I found rather hard to follow.
  - In the statement, write $B_1 = B_0 \setminus T \cup S$ (just to remind the reader) and put a comma after *base.*
  - In general, try to tell the reader what is happening. For instance, the second and following paragraphs do not go directly into constructing the $\mathcal{Q}$-alternating walks claimed on the first paragraph; so it could help to say "As a first step, we obtain an auxilliary linkage $\mathcal{Q}^\infty$.". Also later something like "We next construct a family $\mathcal{W}$ of walks that will prove useful later", and before using Lemma 2.6, "We finally are in position to construct a set of disjoint $\mathcal{Q}$-alternating walks."
  - It is not clear in a first read whether the linkage theorem is proved or just used. Maybe say something like "The linkage theorem asserts that there is a linkage $\mathcal{Q}^\infty$ from ... We first review how this is constructed."
  - The notation about paths $f_x^{i-1}P_x$, $f_x^{i-1}P_x\mathring{v}$ and so on is not explained anywhere. I was highly confused, until I realized it is from Diestel's book. Although the preliminaries section points to this book, it is hard to remember this in the middle of a proof eleven pages after (and even then it is not so easy to find). A little reminder would help many readers.
  - Page 14, line -8: *inside* B_1 means *using elements of* B_1?
  - Just a bit afterwards, before citing Lemma 3.7 from [10], add *as* $|B_2 \setminus B_1|$ *is finite*, so that readers can spot more easily where the hypotheses is used.

Answer: Implemented all the suggestions.

- Move the paragraph from before to after Theorem 2.13 (we do not know before the statement what *the following* is).

Answer: Done.

- Proof of Theorem 2.13: write $M = N/S \setminus R$.

Answer: Done.

- Section 2.2, second paragraph: omit the last sentence (again we do not know what is *the following*), and start with it the proof of the lemma. Or, say something like *to prove that strict topological gammoids are strict gammoids it suffices...*

Answer: Moved the sentence.

- Lemma 2.14, line 3 of proof: remove *from* D'.

Answer: Done.

3 Duality

- The first paragraph of the section and the first of Section 3.1 are somewhat similar, maybe it is better to merge them in one and go directly to bimazes in Section 3.1.

Answer: These paragraphs are rewritten.

- Section 3, line 5: plural *duals*.

Answer: Done.

- Section 3, second paragraph: here it says C^I but afterwards R^I is used.

Answer: It says C^A now and refers to the example we mean.

- Page 18, line -1: the words *identity matching* are not used anywhere else.

Answer: It is deleted now.

- Page 19, line 1: maybe point also to Section 2.4 of Oxley's book in case readers need a more accessible source.

Answer: Done.

- Page 19, line -9: rather than *let...be* write *if... is*.

Answer: Done.

- Proof of Lemma 3.4: in the *Conversely* paragraph, add *in* (D, B_0) after *from* B *onto* B_0. Also in that paragraph, shouldn't the *walks* be *paths*?

Answer: Added "in (D, B_0)". You are right in that the alternating walks here are in fact paths. We decided to call the whole concept "m-alternating *walk*" because we wanted to capture the finite and infinite case in one word rather than distinguishing "alternating paths" and "alternating rays" (which also have a different meaning already).

- Proof of Proposition 3.5: I quite did not see that infinite components intersect $V \setminus B$ (it could be that $B = V$, for instance), but anyway I think that what is needed is that "the first vertex in any one-sided ray of $m \cup m_0$ is in B", and this is guaranteed by B being a basis. Also, the replacement of the last sentence maybe should be made more explicit: replace each m-edge by the m_0-edge with which it shares a vertex.

Answer: Inserted the forgotten "not".

- Pagraph before Lemma 3.6: I would start a new paragraph with the second sentence, as it is completely unrelated to the first one.

Answer: Done.

- Maybe I am missing something, but it seems to me that the forward direction of ‡ follows immediately from Lemma 3.6; thus, one could save one third of the proof of Lemma 3.7 (the argument there seems just the same as that of Lemma 3.6).

Answer: You are right!

- Paragraph after Question 3.12: I guess that the fundamental circuit C is with respect to the basis matched by m_0. Also, more importantly, it took me a while to figure out why there is an m_0-alternating path from u ending in v, maybe some more explanation could be given.

Answer: Done.

- Example 3.13: the dimaze is not very well described and I eventually had to go to the paper [2] to see what was really going on; maybe it would be clearer if first it was said who is B_0 and then that edges are directed towards B_0 (if we do not know that every edge has one endpoint in B_0, the phrase *directed towards* B_0 is not very clear).

Answer: The description is rephrased.

- Several comments on the proof of Ex. 3.13:
  - The first cite of [2] should be [2, Corollary 3.5].
  - Change the name of the tree to $\mathcal{T}$, to avoid confusion with the T involved in the construction of $\vec{\mathcal{Q}}$.
  - Does the set U_i contain u_i?

  Answer: Yes. U_i is defined as vertices separated from the exits by u_i. So it can be interpreted as those vertices, from which any path to the exit passes u_i. Now, any path from u_i to the exits passes u_i, so u_i is there.

  - When speaking of the fundamental circuit of s_i in M^*, I think it is with respect to the complement of B_1, as B_1 need not be a basis of M^*.

Answer: Implemented the suggestions.

- Figure 3: something slightly confusing about this figure is that by definition an alternating ray has no exits, whereas the dimaze on the left does have them.

Answer: The caption has been corrected.

- Lemma 3.16, last line: say *element* instead of *vertex*.

Answer: Done.

- Just after Lemma 3.16: start by recalling that we only consider presentations where W is covered by a matching (I had completely forgotten this assumption, and it is needed in what follows). Now, with this assumption Bondy's result does not need the coloop-free hypotheses: a finite transversal matroid of rank r has a unique maximal presentation with r sets (equivalently, $|W| = r$). It was a bit confusing that Bondy's result had the coloop-free assumption and Proposition 3.17, which includes Bondy's result, had not.

Answer: We now recall what we mean by a presentation. Moreover, we omit the coloop-free assumption from Bondy's result.

- Same paragraph as above, line 6: after the semicolon, write *here* rather than *where*.

Answer: Done.

- Proof of Proposition 3.17:
  - Page 24, line -4: shouldn't it be $v \in C_{vw}$ and $w \notin N_G(C_{vw})$?
  - To apply Bondy's result in Claim 1, one should check that $G[F' \cup N_G(F')]$ is finite, which it is, but maybe it helps the reader to give the reason (for instance, that every element is in a finite circuit).
  - End of proof of Claim 1: I think it would help if the observation "no non-edge between F etc" appeared before.
  - The finitary hypotheses is used again in finding A with finite g and h, so probably it is worth mentioning it.
  - Next paragraph: the extension of B_1 to a basis that admits a matching should be made more carefully, as it could be that the matching does not contain a basis of $M \backslash A$ (for instance, if the edges of the graph are $1a, 2b, 3c, 4a, 4b, 4c, 4d, 5d, 6d$, with $A = \{1, 2, 3, 4\}$, a basis of $M.A$ could be $\{1, 2, 3\}$ and by adding 4 it is a basis of M that contains no basis of $M \setminus A = \{5, 6\}$). I think it would be simpler just to add to B_1 a basis B_2 of $M \setminus A$.

    Also about this part, I was wondering whether it is obvious that in a finitary transversal matroid a basis must be matched onto W (the general case seems to be dealt with in "Brualdi and Mason, Transversal matroids and Hall's theorem, Pacific J. Math. 41, 1972, Theorem 3.2").

  Answer: We do not see how being finitary makes the fact obvious, as a proof makes use of alternating walks (which are already "left locally finite").

  - Somewhat after, add G and H after *in both graphs*, just to clarify.

Answer: Implemented the suggestions.

- Page 25, line -7: write A_i instead of $N(w_i)$.

Answer: According to the labeling it is $N(A_i)$.

- Figure 4: the dotted edges are confusing, just have them as straight lines. Also, try to enlarge the drawing to include A_4, which is the first vertex whose adjacencies follow the general pattern.

Answer: Done. The dotted lines were supposed to indicate which edges can be added/deleted without changing the matroid.

- Before Conjecture 3.23: maybe give more detail about the *It is also possible etc.* Is it with similar techniques? Is it done somewhere else?

Answer: Done.

4 Open problems / References

- If there is an open problems section, I think it is better to state all such problems and conjectures there, rather than scattered. Actually, as almost all problems are from Section 3, an option is to have them as Section 3.4.

Answer: The problems are collected into Section 4

- Reference [5]: *planted.*

Answer: Done.